Math 253N INTRODUCTION TO SERIES EXERCISES, PART ONE

You will use the exercises from APEX Calculus Section 8.2 for all of the following.

1. (a) Write the summation for the series of Exercise 8, followed by the first five terms of the series and $\cdots$.
   (b) Give the first five partial sums of the series. Does the series converge? If so, to what?
   (c) The series is actually geometric, of the form $a\sum_{n=0}^{\infty} r^n$. What are a and r?
   (d) Does Theorem 60 support your conclusion as to whether the series converges?

2. In Exercises 6 through 28 there are five other geometric series that can be written in the form

$$a\sum_{n=0}^{\infty} r^n = a(1 + r + r^2 + r^3 + \cdots)\,.$$

   Tell which ones they are, and give the values of a and r for each. Then determine whether each converges, or diverges. For those that converge, determine the sum of the series.

3. In Exercises 6 through 29 there are six p-series or generalized p-series, with one of them disguised a bit. Attempt to find them and write each in one of the forms given in Theorem 33. (In the case of perhaps the hardest one to spot, you must factor a number out of the series.) Then use Theorem 61 to determine whether each converges.

4. Do Exercise 33. You will need to do a variation of what we did in class on 4/10 for $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n}$.

5. Do Exercise 32, giving two separate formulas, one for the odd partial sums and one for the even partial sums. It is possible to give just one formula - we may consider that later.

6. Try Exercise 30. I haven't been able to figure it out yet...

7. An interesting and commonly encountered sequence is the sequence of **triangular numbers**:

$$1, 3, 6, 10, 15, 21, ...$$

   Note that we can write a sort of mixed recursive/explicit definition of this sequence:

$$a_1 = 1, \quad a_{n+1} = a_n + n + 1.$$

   (a) Write the sequence whose terms are each twice the corresponding triangular numbers, and find an explicit formula for it.
   (b) Use your answer to (a) to give an explicit formula for the sequence of triangular numbers.
   (c) Do Exercise 31.

Suppose that we have two functions f and g and we write

$$0 < f(x) \ll g(x) \quad \text{as} \quad x \to \infty. \tag{1}$$

What this means is that g grows much faster than f as x goes to infinity. This implies that as $x \to \infty$

$$\frac{f(x)}{g(x)} \to 0\,, \qquad \frac{g(x)}{f(x)} \to \infty\,, \qquad g(x) \pm f(x) \to g(x). \tag{2}$$

In a previous assignment you saw that $\lim\limits_{x\to\infty} x\,e^{-x} = \lim\limits_{x\to\infty} \frac{x}{e^x} = 0$ by L'Hopital's rule. In fact,

$$x^p \ll e^x \quad \text{for} \quad p > 0 \quad \text{as} \quad x \to \infty,$$

which says that x to any positive power grows slower than e^x as x goes to infinity. Thus, by the first part of (2),

$$\lim_{x\to\infty} x^p\,e^{-x} = \lim_{x\to\infty} \frac{x^p}{e^x} = 0$$

for any fixed value of p, no matter how large!

Multiplying by positive constants does not change (1). That is, for any positive constants c and d,

$$0 < cf(x) \ll dg(x) \quad \text{as} \quad x \to \infty. \tag{3}$$

We also have that, for $m < n$, $x^m \ll x^n$ as $x \to \infty$. We can use this, along with (2) and (3) above, to do limits like

$$\lim_{x\to\infty} \frac{x^2 - 5x + 2}{4x^3 + 7x^2 + 3x - 5} = \lim_{x\to\infty} \frac{x^2}{4x^3} = 0.$$

For sequences $\{a_n\}$ and $\{b_n\}$, if we have that $0 < a_n \ll b_n$ as $n \to \infty$, the equivalent statements to (2) above are

$$\frac{a_n}{b_n} \to 0\,, \qquad \frac{b_n}{a_n} \to \infty\,, \qquad b_n \pm a_n \to b_n. \tag{4}$$

as $n \to \infty$. Here are some useful such comparisons for commonly encountered sequences:

$$n^p \ll n^{p+s} \ll b^n \ll n! \quad \text{for} \quad s, p > 0,\ b > 1\,, \quad \text{as} \quad n \to \infty. \tag{5}$$

8. Use the above ideas and methods to do Exercises 14 - 18 from Section 8.2 of the APEX Calculus.