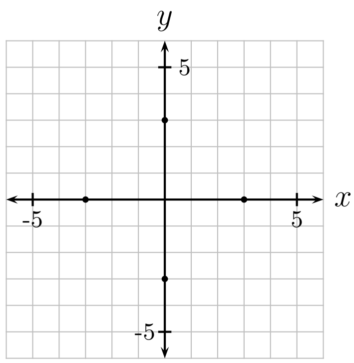
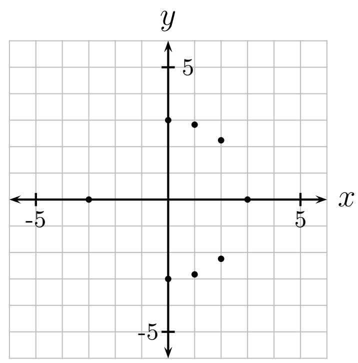
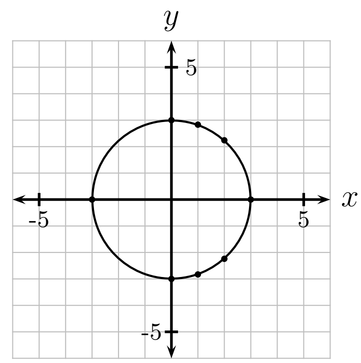
Graphing Circles

◇ **Example 1:** Graph the solution set of $x^2 + y^2 = 9$

If we let $x = 0$ we get the equation $y^2 = 9$, which is true for $y = \pm 3$. Therefore our graph contains the points $(0, 3)$ and $(0, -3)$. Letting $y = 0$ gives us the points $(3, 0)$ and $(-3, 0)$. These four points are plotted on the grid below and to the left, indicating that our graph is likely some sort of object centered at the origin. To get further information we can let $x = 1$, which leads to $y = \pm\sqrt{8} \approx 2.83$. Letting $x = 2$ gives the two additional pairs $(2, \sqrt{5})$ and $(2, -\sqrt{5})$. The four new points are added on the grid in the middle below. At this point we can see that the graph is the circle shown below and to the right. ♠

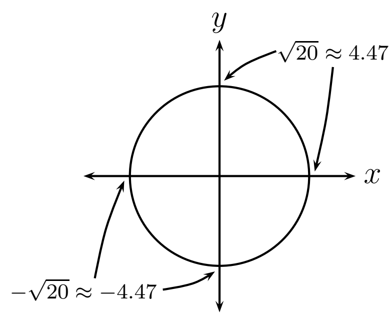






◇ **Example 2:** Graph the solution set of $x^2 + y^2 = 20$

We can write this equation in the form $x^2 + y^2 = (\sqrt{20})^2$. Noting that the equation $x^2 + y^2 = 3^2$ in the previous example resulted in a circle centered at the origin with radius 3 units, we conclude that the graph of $x^2 + y^2 = 20$ is a circle, centered at the origin and with radius $\sqrt{20} \approx 4.47$. The graph is shown to the right. ♠

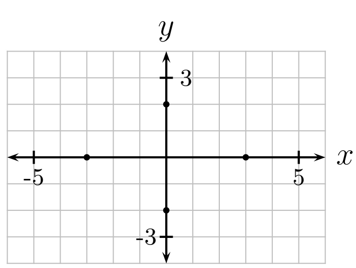


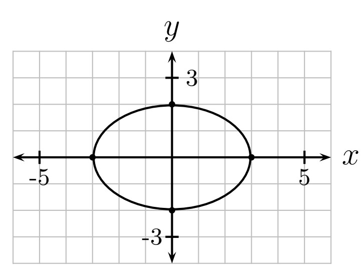
Graphing Ellipses

Note that if we were to divide both sides of the equation $x^2 + y^2 = 9$ we would obtain $\dfrac{x^2}{9} + \dfrac{y^2}{9} = 1$. What would happen if the denominators of the two fractions were different, like

$$\frac{x^2}{9} + \frac{y^2}{4} = 1?$$

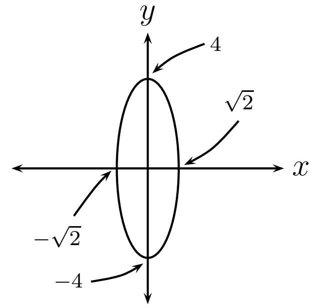
In this case, if $x = 0$ we must then have $y = \pm 2$ and when $y = 0$ we obtain $x = \pm 3$. Plotting the four points thus obtained gives us the graph below and to the left. This suggests that the graph is a "stretched circle," more formally called an ellipse; the graph of the ellipse is shown below and to the right.





◇ **Example 3:** Graph the solution set of $8x^2 + y^2 = 16$

This equation would be the equation of a circle with radius four if it weren't for the coefficient 8 of x^2. To get this into a recognizable form, we divide both sides by 16 to get one on the right side of the equation: $\dfrac{x^2}{2} + \dfrac{y^2}{16} = 1$. We can also write the denominator of each fraction as a square, to get $\dfrac{x^2}{(\sqrt{2})^2} + \dfrac{y^2}{4^2} = 1$. We now recognize the equation of an ellipse, with x-intercepts of $\pm\sqrt{2} \approx \pm 1.41$ and y-intercepts of ± 4. The graph is shown to the right. ♠



Graphing Hyperbolas

The third variety of conic section we will work with are hyperbolas. Let's begin by looking at the equation

$$\frac{x^2}{25} - \frac{y^2}{9} = 1. \tag{1}$$

We first note that if $y = 0$ then $x = \pm 5$. Now $\frac{y^2}{9} \geq 0$ and $-\frac{y^2}{9} \leq 0$ regardless of the value of y, so there is no value of y corresponding to $x = 0$. So at this point all we know is that the points $(-5, 0)$ and $(5, 0)$ are on the graph. If we solve (1) for $\frac{y^2}{9}$ we get

$$\frac{y^2}{9} = \frac{x^2}{25} - 1, \tag{2}$$

so we can obtain values of y as long as $\frac{x^2}{25} \geq 1$ (because $\frac{y^2}{9} \geq 0$), which occurs for $x \geq 5$ and $x \leq -5$. So now we know two points on the graph, and we know that the graph must lie in the two regions shaded gray in the graph to the left below.

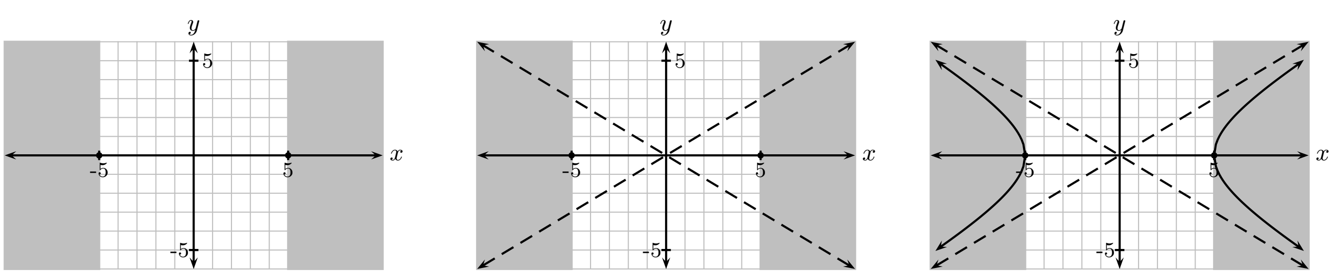
If we multiply both sides of (2) by 9 and then take the square root of both sides we get

$$y = \pm\sqrt{\frac{9}{25}x^2 - 9}. \tag{3}$$

For large values of $|x|$ the -9 under the root becomes insignificant relative to the $\frac{9}{25}x^2$ term, so we have

$$y \approx \pm\sqrt{\frac{9}{25}x^2} = \pm\frac{3}{5}x \tag{4}$$

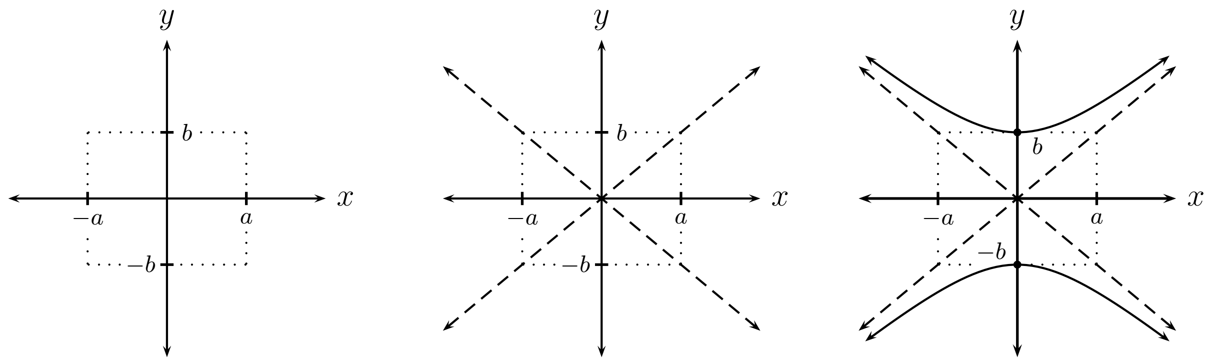
when x is a large positive or negative value. The equations $y = \frac{3}{5}x$ and $y = -\frac{3}{5}x$ are lines through the origin with slopes $\frac{2}{5}$ and $-\frac{2}{5}$, respectively. In the middle graph below we have added these as dashed lines. They are asymptotes, with the graph approaching them as $x \to \infty$ and $x \to -\infty$. Therefore the final graph is as shown to the right below.



Although I would hope that you understand the reasoning above, there is a fairly quick and simple method for graphing hyperbolas whose equations have the one of the forms

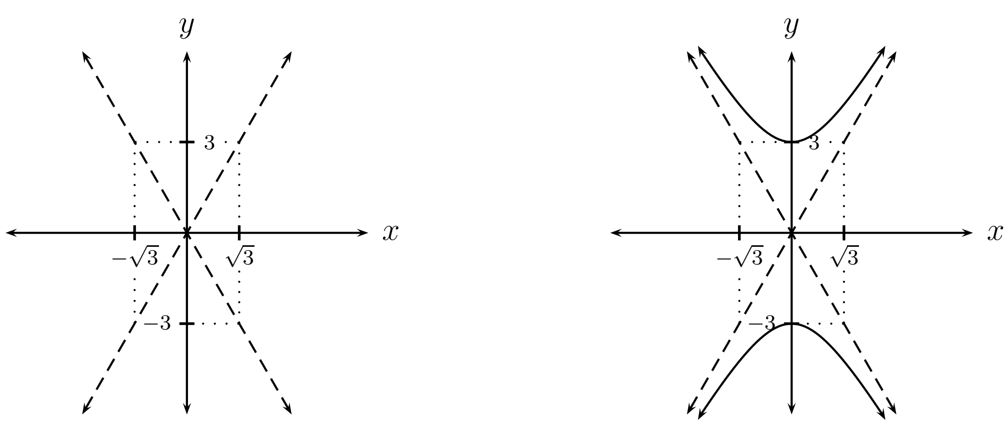
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{or} \qquad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

We begin by making a rectangle centered at the origin, with its left and right sides at $x = \pm a$ and its top and bottom at $y = \pm b$, as shown in the graph below and to the left. We then draw in the diagonals of the rectangle and extend them to get two lines that will be the asymptotes of the hyperbola. This is shown in the middle below. Finally, we plot the two points $(a, 0)$ and $(-a, 0)$ or $(0, b)$ and $(0, -b)$, depending on whether the equation is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ or $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$, respectively, and draw in the hyperbola. In the case of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ the graph will have the appearance of the one on the previous page. For $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$, the graph will look like the one to the right below. We will say that the hyperbola at the bottom of the previous page opens to the left and right, and the one below opens up and down.



◇ **Example 4:** Graph the solution set of $\dfrac{y^2}{9} - \dfrac{x^2}{3} = 1$.

We begin by writing the equation as $\dfrac{y^2}{3^2} - \dfrac{x^2}{(\sqrt{3})^2} = 1$. Using this, we create the rectangle and asymptotes shown below and to the left. We then see that the points $(0, 3)$ and $(0, -3)$ are on the hyperbola, so it opens up and down. Thus the graph is as shown to the right below. ♠



Graphing Parabolas

The equations of parabolas that we will be particularly interested in are ones of the form

$$z = ax^2 + b \qquad\qquad z = ay^2 + b.$$

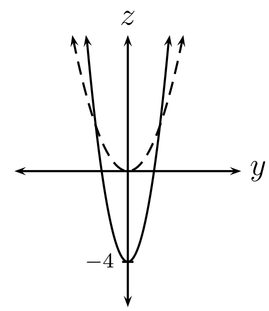
The graphs are identical in both cases, with the horizontal axis being one of the variables x or y, and the vertical axis being the variable z (you'll soon see where that comes from). The graph has the following characteristics:

- The vertex is at $z = b$ on the vertical axis.
- The parabola opens up if $a > 0$, down if $a < 0$.
- The parabola is "sharper" than $y = x^2$ is $|a| > 1$, "flatter" than $y = x^2$ if $|a| < 1$.

Let's look at a couple examples.

◇ **Example 5:** Graph the solution set of $z = 3y^2 - 4$.

The vertex of the parabola is at $(0, -4)$ and the parabola opens upward because $3 > 0$. The parabola is also sharper than $z = y^2$ because $|3| > 1$. The graph to the right shows $z = y^2$ as the dashed curve and $z = 3y^2 - 4$ as the solid curve. ♠



◇ **Example 6:** Graph the solution set of $z = -\dfrac{x^2}{4} + 1$.

In this case we can rewrite the equation as $z = -\frac{1}{4}x^2 + 1$. The vertex of the parabola is at $(0, 1)$ and the parabola opens downward because $-\frac{1}{4} < 0$. The parabola is also flatter than $z = x^2$ because $|-\frac{1}{4}| < 1$. The graph to the right shows $z = x^2$ as the dashed curve and $z = -\dfrac{x^2}{4} + 1$ as the solid curve. ♠