## 2 Parametric Motion

## Learning Outcome:

2. Understand vector-valued functions of one variable and their derivatives, perform associated computations, and apply understanding and computations to solve problems.

### 2.1 Introduction

Parametric motion refers the movement of an object in two- or three-space. Positions in these spaces are represented via the variables $x$ and $y$, for two-space, and $x, y$ and $z$ for three-space. When an object is moving, each of these variables can be expressed in terms of a third or fourth variable called a parameter. The two most commonly used parameters are time and distance traveled along the path of motion.

Parametrization with respect to time is the easier of these two for most of us to conceptualize. In this case, The position $(x, y, z)$ of a moving object at any time $t$ is given by three functions of $t$ like, for example,

$$
x=5 \cos \pi t, \quad y=5 \sin \pi t, \quad z=3 t .
$$

In general, we will refer to any three such functions as

$$
x=x(t), \quad y=y(t), \quad z=z(t)
$$

(I will do most things in three dimensions, but everything works the same in two dimensions except that there is no $z$.)

When we are studying the motion of an object, there are a number of things we will want to know:

- The exact path of the object, and its positions on that path at various times.
- The velocity, speed and direction of travel of the object along the path.
- The acceleration at any time, and its effect on speed/velocity and the path.

In addition to answering those questions, we'll look at some applications of parametric motion.

### 2.2 Rectangular Equations of Paths

## Performance Criterion:

2. (a) Find the rectangular equation of the path for parametric motion in two dimensions and identify its "shape" (line, circle, ellipse, etc.).

Suppose that we have an object moving in two-space, with its position at any time $t \geq 0$ given by

$$
x=-4+2 t, \quad y=3-t .
$$

Previously we learned that these are the parametric equations of a line in two dimensions, but suppose that we did not know that. To determine the path of the object we could calculate the position of the object at a few times and plot the locations on a graph, as shown to the left below. The positions of the object at
times $t=0,1,2,3,4$ are indicated, with each labeled with the time that the object is at that point. Based on those points, we would probably assume that the object is moving in a straight line, as shown in the middle graph below. However, given only the points at which we've determined the position of the object, it is possible that the path might be the one shown below and to the right!




What we'd like to do here is verify that the path is indeed a line. To do this, we combine the two equations $x=-4+2 t$ and $y=1-t$ in such a way as to eliminate the parameter $t$. We can solve the second equation for $t$, getting $t=1-y$. Substituting this into the first equation and simplifying gives $x=-2-2 y$, which you may recognize as the equation of a line. We will call any equation relating $x$ and $y$ directly the rectangular equation corresponding to the parametric equations. In this case we can take it a bit farther by solving for $y$ : $y=-\frac{1}{2} x-1$. From this slope-intercept form we can see that this is in fact the line drawn in the middle graph above, with $y$-intercept -1 and slope $-\frac{1}{2}$.

Consider the parametric equations

$$
x=4 \cos t, \quad y=4 \sin t
$$

describing the motion of an object. On the graph to the right we can see the locations of the object at times $t=0,1,2,3,4,5$. One might guess that the path is a circle, which in fact it is. This can be show using a clever method of combining the two parametric equations. We first square both sides of each equation to get

$$
x^{2}=16 \cos ^{2} t, \quad y^{2}=16 \sin ^{2} t
$$

From this we can see that


$$
x^{2}+y^{2}=16 \cos ^{2} t+16 \sin ^{2} t=16\left(\cos ^{2} t+\sin ^{2} t\right)=16
$$

The rectangular equation is then $x^{2}+y^{2}=16$, the equation of a circle with radius four, centered at the origin.

### 2.3 Velocity, Speed and Acceleration

## Performance Criterion:

2. (b) Find velocity, speed and acceleration for parametric motion in two or three dimensions.

Suppose that an object is moving in two dimensions with parametric equations of motion

$$
x=x(t), \quad y=y(t)
$$

Then, for example, at time 2 the object is at the point $(x(2), y(2))$. Recall that a vector from the origin to a point is called the position vector for that point. We commonly use the name $\overrightarrow{\mathbf{r}}(t)$ for the position vector for a moving object at any time $t$. Thus the point $(x(2), y(2))$ corresponds with the position vector $\overrightarrow{\mathbf{r}}(2)$. In general,

$$
\begin{equation*}
\stackrel{\rightharpoonup}{\mathbf{r}}(t)=\langle x(t), y(t), z(t)\rangle=x(t) \stackrel{\rightharpoonup}{\mathbf{i}}+y(t) \stackrel{\rightharpoonup}{\mathbf{j}}+z(t) \stackrel{\rightharpoonup}{\mathbf{k}} \tag{1}
\end{equation*}
$$

for an object moving along a path in three-space. This indicates that the position vector is a vector function of time $t$. That is, for a moving object whose parametric equations are known, the position function is a function that "takes in" a time $t$ and "gives out" the position vector $\overrightarrow{\mathbf{r}}(t)$ for the object's position at that time.

Why take the seemingly simple concept of parametric motion and complicate it so? Recall from first term calculus that the derivative of position with respect to time is velocity, and the derivative of velocity (second derivative of position) with respect to time is acceleration. For our object with vector position function (1) we define

$$
\stackrel{\rightharpoonup}{\mathbf{v}}(t)=\overrightarrow{\mathbf{r}}^{\prime}(t)=\left\langle x^{\prime}(t), y^{\prime}(t), z^{\prime}(t)\right\rangle=x^{\prime}(t) \overrightarrow{\mathbf{i}}+y^{\prime}(t) \overrightarrow{\mathbf{j}}+z^{\prime}(t) \overrightarrow{\mathbf{k}}
$$

and

$$
\overrightarrow{\mathbf{a}}(t)=\overrightarrow{\mathbf{v}}^{\prime}(t)=\overrightarrow{\mathbf{r}}^{\prime \prime}(t)=\left\langle x^{\prime \prime}(t), y^{\prime \prime}(t), z^{\prime \prime}(t)\right\rangle=x^{\prime \prime}(t) \overrightarrow{\mathbf{i}}+y^{\prime \prime}(t) \overrightarrow{\mathbf{j}}+z^{\prime \prime}(t) \stackrel{\rightharpoonup}{\mathbf{k}}
$$

By making the position into a vector, taking the derivative gives a velocity as a vector, which we know intuitively that it should be. It may not be as clear that acceleration is a vector quantity as well, but it will become clearer later that it is.

Let's look at an example, the object with position given by the parametric equations

$$
x=5 \cos \pi t, \quad y=5 \sin \pi t, \quad z=3 t
$$

The position vector function for this object is

$$
\begin{equation*}
\stackrel{\rightharpoonup}{\mathbf{r}}(t)=\langle 5 \cos \pi t, 5 \sin \pi t, 3 t\rangle \tag{2}
\end{equation*}
$$

and the velocity vector is

$$
\begin{equation*}
\overrightarrow{\mathbf{v}}(t)=\langle-5 \pi \sin \pi t, 5 \pi \cos \pi t, 3\rangle \tag{3}
\end{equation*}
$$

Note the use of the chain rule in the derivatives of the cosine and sine functions. Finally, the acceleration function is

$$
\begin{equation*}
\stackrel{\rightharpoonup}{\mathbf{a}}(t)=\left\langle-5 \pi^{2} \cos \pi t,-5 \pi^{2} \sin \pi t, 0\right\rangle \tag{4}
\end{equation*}
$$

Remember the difference between velocity and speed: velocity is a vector indicating how fast an object is moving, and the direction it is moving as well; speed is a scalar telling how fast the object is going. Speed is found by taking the magnitude of velocity, so

$$
\begin{equation*}
\text { speed }=\|\overrightarrow{\mathbf{v}}(t)\|=\sqrt{\left[x^{\prime}(t)\right]^{2}+\left[y^{\prime}(t)\right]^{2}+\left[z^{\prime}(t)\right]^{2}} \tag{5}
\end{equation*}
$$

For our example above, the speed is

$$
\begin{aligned}
\|\overrightarrow{\mathbf{v}}(t)\| & =\sqrt{(-5 \pi \sin \pi t)^{2}+(5 \pi \cos \pi t)^{2}+3^{2}} \\
& =\sqrt{25 \pi^{2} \sin ^{2} \pi t+25 \pi^{2} \cos ^{2} \pi t+9} \\
& =\sqrt{25 \pi^{2}\left(\sin ^{2} \pi t+\cos ^{2} \pi t\right)+9} \\
& =\sqrt{25 \pi^{2}+9}
\end{aligned}
$$

because $\sin ^{2} \pi t+\cos ^{2} \pi t=1$. The speed is then a constant, indicating that the object is moving at a constant speed, neither speeding up nor slowing down. This does not mean, however, that there is no acceleration; equation (4) above shows otherwise.

### 2.4 Determining Equations of Motion - Circular Paths

## Performance Criterion:

2. (c) Determine the parametric equations of motion for an object traveling on a circular or helical path.

Consider again the moving object with vector equation of motion

$$
\begin{equation*}
\overrightarrow{\mathbf{r}}(t)=\langle 5 \cos \pi t, 5 \sin \pi t, 3 t\rangle . \tag{6}
\end{equation*}
$$

If we were looking down from above on the object as it moved, we would not be able to see the vertical movement, given by the component $3 t$. From that perspective, we would be considering an object moving in the $x y$-plane with equations

$$
\begin{equation*}
\stackrel{\rightharpoonup}{\mathbf{r}}(t)=\langle 5 \cos \pi t, 5 \sin \pi t\rangle \quad \text { and } \quad \overrightarrow{\mathbf{v}}(t)=-5 \pi \sin \pi t, 5 \pi \cos \pi t\rangle \tag{7}
\end{equation*}
$$

Using the method shown at the end of Section 2.2, we find that the rectangular equation of motion is $x^{2}+y^{2}=25$, so the object is moving on a circular path, centered at the origin, with radius five.

What we don't know at this point is where the object is at time zero, and whether it is traveling clockwise or counterclockwise on the circle (still looking down from above). We can easily determine both from the above equations (7). At time zero we see that $\overrightarrow{\mathbf{r}}(0)=\langle 5,0\rangle$ and $\overrightarrow{\mathbf{v}}(0)=\langle 0,5 \pi\rangle$. These show that the object is at $(5,0)$ at time zero, and it is moving in the positive $y$ direction at that time. Therefore the object is moving counter-clockwise around the circle.

Going back to the original equation of motion (6), what about the vertical component $3 t$ ? From equation (3) on the previous page we can see that the vertical component of velocity is constant, with value 3. This indicates that the object is rising at a constant rate of three units of distance per unit of time. Putting together this vertical information with what we determined about the motion viewed from above, we now know that our object is climbing upward at a constant rate on a helical (spiral) path, traveling counter-clockwise, as seen from above, on that path.

The equations of motion for an object traveling in a circular path with radius $a$ around the origin in $\mathbb{R}^{2}$ will have equations

$$
x= \pm a \cos b t, \quad y= \pm a \sin b t \quad \text { OR } \quad x= \pm a \sin b t, \quad y= \pm a \cos b t
$$

The choice of whether to use sine or cosine for each variable depends on which axis the object starts on, and the choice of $\pm$ depends on the desired direction of travel. The constant $b$ determines the period, the time it takes to complete one full revolution. We must have $b t=2 \pi$ at the time $t$ corresponding to the period, because $2 \pi$ is the period of both the sine and cosine functions.

### 2.5 Displacement and Distance Traveled

## Performance Criterion:

2. (d) Find displacement and distance traveled for parametric motion in two or three dimensions.

Suppose that an object is traveling on a path, and the position of the object at two times $t_{1}$ and a later time $t_{2}$ are given by $\overrightarrow{\mathbf{r}}\left(t_{1}\right)$ and $\overrightarrow{\mathbf{r}}\left(t_{2}\right)$. We then define the following:

- The displacement from time $t_{1}$ to $t_{2}$ is the vector from the position of the point at time $t_{1}$ to the position at time $t_{2}$. This vector indicates the net change of the object's position, both in direction and distance in a straight line from the initial point to the terminal point. It is denoted by $\Delta \overrightarrow{\mathbf{r}}$ and is computed by $\Delta \overrightarrow{\mathbf{r}}=\overrightarrow{\mathbf{r}}\left(t_{2}\right)-\overrightarrow{\mathbf{r}}\left(t_{1}\right)$.
- The distance traveled from time $t_{1}$ to $t_{2}$ is a scalar that gives the actual distance that the object travels along its path. Assuming that the particle does not change direction along its path, the distance traveled is given by

$$
\text { distance traveled }=\int_{t_{1}}^{t_{2}}\|\overrightarrow{\mathbf{v}}(t)\| d t
$$

Note that $\|\overrightarrow{\mathbf{v}}(t)\|$ is the speed of the object, as a function of time $t$. the above integral is simply computing "distance equals rate (speed) times time ( $d t$ )."

Let's look at a simple example that illustrates the difference between these two concepts. Suppose that an object travels the semi-circle shown to the right, in the direction shown by the arrowheads. Then the displacement $\Delta \overrightarrow{\mathbf{r}}$ is the vector from where the object starts to where it ends, so $\Delta \overrightarrow{\mathbf{r}}=\langle 8,0\rangle$. The distance traveled is the length of the semicircle. It could be found using an integral, as described above, but in this case it is simpler to realize that the circumference of a circle with radius four is $C=2 \pi(4)=8 \pi$, then just take half of that to get
 $4 \pi$ for the distance traveled.

It is not always necessary to compute an integral in order to find distance traveled:

- If we know that an object is traveling on a line and has not reversed its direction, the distance traveled is simply the distance between the initial point and the final point.
- If an object is moving at a constant speed, the distance traveled is simply the speed times the time traveled.


### 2.6 Initial Value Problems

## Performance Criterion:

2. (e) Solve an initial value problem for parametric motion in two or three dimensions.

If we have a vector equation describing the position of an object at any time $t$, we have seen that we can differentiate each component to get the velocity equation. For example,

$$
\text { if } \quad \overrightarrow{\mathbf{r}}(t)=\left\langle t^{2}-5 t, 4 e^{3 t}+7,7 \sin (\pi t)\right\rangle \text {, then } \quad \overrightarrow{\mathbf{v}}(t)=\left\langle 2 t-5,12 e^{3 t}, 7 \pi \cos (\pi t)\right\rangle
$$

It should be clear that if we knew the velocity function for a particle, but not the position function, that the position function could be obtained by finding the antiderivative of each component of the velocity. For example, suppose we had

$$
\stackrel{\rightharpoonup}{\mathbf{v}}(t)=\left\langle 7-2 e^{-2 t}, 3 \sin (3 t)\right\rangle
$$

describing the velocity of an object in $\mathbb{R}^{2}$. If we could find two functions $x(t)$ and $y(t)$ such that $x^{\prime}(t)=7-2 e^{-2 t}$ and $y^{\prime}(t)=\sin (3 t)$, we would have our position function $\overrightarrow{\mathbf{r}}(t)$. Remembering that the derivative of a constant is zero, we can see that

$$
\begin{equation*}
\stackrel{\rightharpoonup}{\mathbf{r}}(t)=\left\langle 7 t+e^{-2 t}+C_{1},-\cos (3 t)+C_{2}\right\rangle, \tag{1}
\end{equation*}
$$

where $C_{1}$ and $C_{2}$ are arbitrary constants (meaning they can each have any value). If we had the additional information consisting of the value of $\overrightarrow{\mathbf{r}}(0)$, we could determine the constants $C_{1}$ and $C_{2}$. Suppose that we know $\overrightarrow{\mathbf{r}}(0)=\langle 1,3\rangle$. From (1) above we also have $\overrightarrow{\mathbf{r}}(0)=\left\langle 1+C_{1},-1+C_{2}\right\rangle$. (Remember that $e^{0}=1$.) These two expressions for $\overrightarrow{\mathbf{r}}(0)$ must be equal, so it must be the case that $C_{1}=0$ and $C_{2}=4$. Therefore

$$
\stackrel{\rightharpoonup}{\mathbf{r}}(t)=\left\langle 7 t+e^{-2 t}, 4-\cos (3 t)\right\rangle
$$

When we are given the derivative of a function and the value of the function (not the derivative) at zero (or some other value), we can carry out a process like this. The combination of a derivative function and an initial value of the original function is called an initial value problem, and such problems are common in science and engineering. The initial value problems that we will be interested in right now will usually consist of an acceleration function, and the values of both the velocity and position at time zero. We need both of those because acceleration is the second derivative of position, so we must repeat the above process twice. The entire process goes like this:
(1) Find the antiderivatives of the components of acceleration $\overrightarrow{\mathbf{a}}(t)$, each of which will contain an arbitrary constant. This gives us the general form of the velocity $\overrightarrow{\mathbf{v}}(t)$, containing constants $C_{1}$, $C_{2}$ and $C_{3}$ (or just $C_{1}$ and $C_{2}$ in two dimensions).
(2) Evaluate the function obtained in (1) for $t=0$, and set each component of the resulting vector equal to the components of the given vector $\overrightarrow{\mathbf{v}}(0)$. Determine the values of the unknown constants $C_{1}, C_{2}$ and $C_{3}$.
(3) Put the constants you found in (2) into the general form of the velocity that you found in (1) to obtain the final equation for the velocity $\overrightarrow{\mathbf{v}}(t)$.
(4) Find the antiderivatives of the components of velocity $\overrightarrow{\mathbf{v}}(t)$ to obtain the general form of the position function $\overrightarrow{\mathbf{r}}(t)$, which will contain new constants $C_{4}, C_{5}$ and $C_{6}$.
(5) Use the given initial position to determine the values of the constants, and put them into the general form of the position function to obtain the final form of the position function $\overrightarrow{\mathbf{r}}(t)$. You're done!

### 2.7 Projectile Motion

## Performance Criterion:

2. (f) Solve projectile motion problems.

This topic consists of applying the previous topics to "real life" applications. We'll go over this in class, and it is addressed thoroughly in both your book and on the web. Here are some basic principles that are useful:

- The maximum height of a projectile occurs when the vertical component of the velocity is zero.
- A projectile hits the ground when the vertical component of the position is zero (assuming that $y=0$ is ground level).
- The maximum height occurs halfway through the flight only if the the vertical components of the launch site and landing site are the same.


### 2.8 Finding Tangential and Normal Components of Acceleration

## Performance Criterion:

2. (g) Find the tangential and normal vector components $\overrightarrow{\mathbf{a}}_{T}$ and $\overrightarrow{\mathbf{a}}_{N}$ of the acceleration vector $\overrightarrow{\mathbf{a}}$. Write the acceleration vector at some time in the form $\mathbf{a}=\overrightarrow{\mathbf{a}}_{T}+\overrightarrow{\mathbf{a}}_{N}$.

Suppose that an object is moving to the right along the curve shown to the right, and that at time $t=3$ it is at the point indicated by the point $P$. The velocity vector $\overrightarrow{\mathbf{v}}(3)$ points in the direction of travel, so it is tangent to the curve as indicated in the diagram. The vector $\overrightarrow{\mathbf{a}}(3)$ is the acceleration vector at that time.

In general, the direction of an acceleration cannot be predicted when
 viewing the path of an object. To properly interpret the acceleration vector we need to break it into two components, one tangent to the curve and one normal (perpendicular) to the curve. We'll refer to these two vectors as $\overrightarrow{\mathbf{a}}_{T}$ and $\overrightarrow{\mathbf{a}}_{N}$, the tangential and normal components of acceleration. From the diagram above and to the right you should be able to see that $\overrightarrow{\mathbf{a}}_{T}$ is simply the projection of the acceleration vector onto the velocity vector, and $\overrightarrow{\mathbf{a}}_{N}=\overrightarrow{\mathbf{a}}-\overrightarrow{\mathbf{a}}_{T}$. The diagram immediately to the right shows the original acceleration vector at time $t=3$, along with the two
 components $\overrightarrow{\mathbf{a}}_{T}(3)$ and $\overrightarrow{\mathbf{a}}_{N}(3)$.

You should recognize the tangential component $\overrightarrow{\mathbf{a}}_{T}$ of the acceleration as the projection of the acceleration onto the velocity, and the normal component $\overrightarrow{\mathbf{a}}_{N}$ as the component of $\overrightarrow{\mathbf{a}}$ that is perpendicular to the velocity. That is,

$$
\stackrel{\rightharpoonup}{\mathbf{a}}_{T}=\operatorname{proj}_{\stackrel{\rightharpoonup}{\mathbf{v}}} \stackrel{\rightharpoonup}{\mathbf{a}} \quad \text { and } \quad \overrightarrow{\mathbf{a}}_{N}=\operatorname{perp}_{\stackrel{\rightharpoonup}{\mathbf{v}}} \stackrel{\rightharpoonup}{\mathbf{a}}
$$

### 2.9 Interpreting Tangential and Normal Components of Acceleration

## Performance Criterion:

2. (h) Given the path and direction of motion of a particle and information about whether it is speeding up, slowing down, or moving at a constant speed at a point, sketch possible velocity and acceleration vectors at that point. Sketch possible tangential and normal components of the acceleration at that point.
(i) Given the velocity and normal and tangential components of acceleration for a particle, determine whether the particle is (a) speeding up, slowing down or moving at constant speed and (b) whether the path of the particle is straight or curved.

The reason for breaking the acceleration $\overrightarrow{\mathbf{a}}$ into the two components $\overrightarrow{\mathbf{a}}_{T}$ and $\overrightarrow{\mathbf{a}}_{N}$ is as follows.

- The tangential component $\overrightarrow{\mathbf{a}}_{T}$ always acts in the direction of velocity or opposite to it. Its effect is to change the speed of the object; if $\overrightarrow{\mathbf{a}}_{T}$ points in the direction of $\overrightarrow{\mathbf{v}}$, the object is speeding up, and if $\overrightarrow{\mathbf{a}}_{T}$ points in the direction opposite to $\overrightarrow{\mathbf{v}}$, the object is slowing down.
- The normal component $\overrightarrow{\mathbf{a}}_{N}$ of acceleration has no effect on speed. It's significance is that it is the cause for the path of the object to curve, and it always points in the direction that the object is curving.

It should be clear that if the path of an object is curving, the acceleration vector points generally to the inside of the curve, although possibly "forward" or "backward" of an inward normal vector.

Like the distinction between velocity and speed, we also want to make sure we understand the relationship between the terms acceleration, speeding up, and slowing down:

- Acceleration is change in velocity, so there will be acceleration any time that the velocity vector vector is not constant. There will then be acceleration whenever the velocity changes in either direction or magnitude.
- Speeding up and slowing down relate to speed of an object, not its direction. If the speed of an object remains the same regardless of its direction, then it is neither speeding up nor slowing down.

Because of these differences some counter-intuitive things can happen. For example an object can be accelerating but maintaining a constant speed (neither speeding up nor slowing down). Hmmmm...!

