

1 Vectors

Learning Outcome:

1. Understand vectors in two- and three-space, lines and planes in three-space, and be able to perform associated computations.

1.1 Scalar and Vector Quantities

Performance Criterion:

- Determine whether a quantity is a scalar quantity or a vector quantity.

For the working scientist or engineer, mathematics is the language used in describing and working with physical quantities. In this class we will work extensively with two types of physical quantities, **scalar** quantities and **vector** quantities. Scalar quantities are ones that can be represented with a single number; examples of such quantities are things like length, weight, temperature, pressure, and so on. There are some physical quantities that cannot simply be described with numbers. For example, force and velocity have *both amount and direction*. To describe such quantities we use mathematical objects called vectors, that allow us to specify both amount and direction with one object. We call the amount the **magnitude** or **length** of the vector. (It gets a little tiresome to continually say scalar *quantity* and vector *quantity*, so from here on we will simply use the terms scalar and vector.) Let's reiterate:

- A scalar is a quantity having only amount - some examples are temperature, mass, pressure, length.
- A vector is a quantity having both magnitude and direction - examples are velocity, acceleration, force.
- An important distinction that we will make in this class is between **speed** and **velocity**. Speed is a scalar telling us how fast something is moving. Velocity is a vector, telling us both how fast something is moving and the direction it is moving as well. *Speed is the magnitude of velocity.*
- Another distinction we'll make is between distance and displacement. Distance is simply "how far," and is a scalar. Displacement is how far something has moved as well as the direction it moved - displacement is a vector.

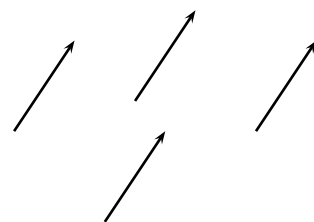
Unknown scalars and vectors are represented with letters. When typeset, unknown scalars are generally lower case italicized letters like x , v , t and so on; in this class we will occasionally use capital letters as well. *You are expected to use the correct case for all unknowns.* In textbooks, unknown vectors are denoted with boldface lower case letters, like \mathbf{i} , \mathbf{v} , and \mathbf{r} . When writing scalars by hand we simply write their letter. To distinguish vectors from scalars when handwritten, we put a small arrow pointing to the right above a letter representing a vector. *I will be insistent on this practice in this class!* In these notes I'll use boldface with an arrow for vectors, like $\vec{\mathbf{i}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{r}}$.

1.2 Geometric Representation of Vectors, Vector Operations

Performance Criterion:

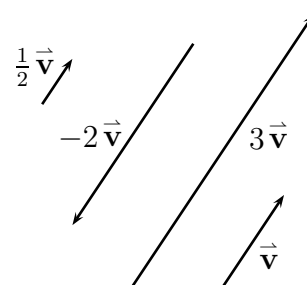
- (a) Draw the scalar multiple of a drawn vector, the sum or difference of two drawn vectors, or a linear combination of two vectors. Illustrate both the parallelogram method and tip-to-tail method for adding two vectors.

Geometrically, a vector is represented by an arrow; the direction of the arrow indicates the direction of the vector, and the length of the arrow indicates the magnitude of the vector. Vectors that have the same magnitude and direction but are at different locations are called **equivalent vectors**. Several equivalent vectors are shown in the picture to the right.

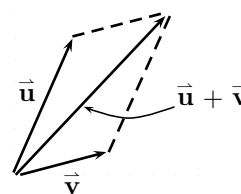
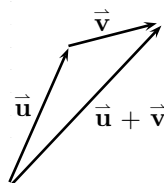
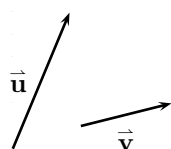


We will have a variety of operations we will perform with vectors or with vectors and scalars together, but the two main ones are multiplying a vector by a scalar and adding vectors.

- Multiplying a vector by a scalar:* Suppose a force is acting on an object in a particular direction and the force doubles, with no change in direction. Then the arrow representing the new force vector points in the same direction as the arrow representing the original force, but it is twice as long. We have then multiplied the original vector by the scalar two. If we multiply a vector by a negative scalar, the direction of the new vector is in the opposite direction as the original vector. See the diagram to the right, for some multiples of a vector \vec{v} .



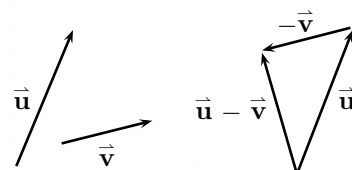
- Adding two vectors:* There are two methods for adding two vectors geometrically, and I want you to know both. The first I will call the **tip-to-tail method**. To add $\vec{u} + \vec{v}$, draw \vec{v} with its tail at the tip of \vec{u} . The vector $\vec{u} + \vec{v}$ goes from the tail of \vec{u} to the tip of \vec{v} . See the diagram in the center below, for the two vectors \vec{u} and \vec{v} shown below and to the left.



The other method for adding two vectors is the **parallelogram method**. To add \vec{u} and \vec{v} , place the tails of the two vectors together. They then form two adjacent sides of a parallelogram; complete the parallelogram. Then $\vec{u} + \vec{v}$ has its tail at the tails of the original two vectors and its tip at the opposite vertex of the parallelogram. See the diagram at the bottom right of the previous page.

We can use the above two ideas to define division of a vector by a scalar and subtraction of one vector from another. To divide a vector by a scalar (which we will do quite a bit), simply multiply by the reciprocal of the scalar. For example, dividing a scalar by two is the same as multiplying it by $\frac{1}{2}$. By definition,

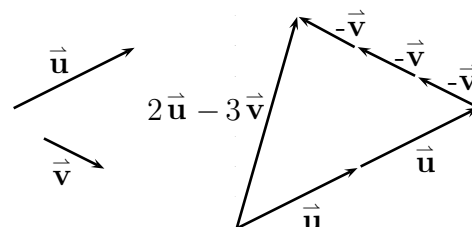
$$\vec{u} - \vec{v} = \vec{u} + (-\vec{v}),$$



so we find $\vec{u} - \vec{v}$ as shown to the right.

A concept that is related to addition and multiplication by scalars is what we call a **linear combination** of vectors. A linear combination of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_n$ results when each of these vectors is multiplied by a scalar (we'll refer to the respective scalars as $c_1, c_2, c_3, \dots, c_n$) and the results are all added together:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 + \dots + c_n \vec{v}_n$$



Of course, if any of the scalars are negative the corresponding additions can be looked at as subtractions. In the diagram above and to the right you can see two vectors \vec{u} and \vec{v} and the linear combination $2\vec{u} - 3\vec{v}$.

Two vectors are said to be **parallel** if they point in the same direction or in opposite directions. It should be clear that *two vectors are parallel if, and only if, they are scalar multiples of each other*. See the picture on the previous page for scalar multiplication to see four vectors that are all parallel to each other.

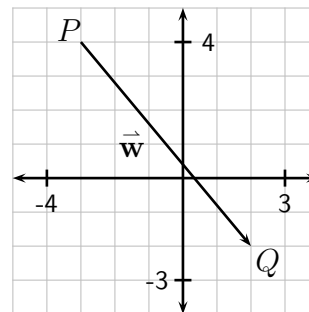
1.3 Vectors in a Coordinate System

Performance Criteria:

1. (b) Find the vector from one point to another, give the initial or terminal point of a vector.
- (c) Perform algebraic operations with vectors.
 - Determine whether two vectors are equivalent. Determine whether two vectors are parallel. Determine whether two vectors point in the same direction.

Working with vectors by just drawing pictures like the ones you've seen so far is tedious and imprecise. It is much better to work with vectors by placing them in a coordinate system like the familiar xy -plane, or the three-dimensional equivalent, which we'll call **three-space**. (Sometimes we may refer to the xy -plane as **two-space**.) When we do that we can then perform the operations of addition and multiplication by scalars by simply adding and multiplying numbers.

It should be clear what the tip and tail of a vector are. When we have a vector in the coordinate plane or in three-space, the location of its tail (point P) is called its **initial point**, and the location of its tip (point Q) is called its **terminal point**. The vector \vec{w} shown to the right has the point $(-3, 4)$ as its initial point, and $(2, -2)$ as its terminal point. We will describe the vector itself by $\vec{w} = \langle 5, -6 \rangle$, which indicates that the vector goes seven units to the right and five units down. Note that we use parentheses for a point and “angle brackets” for a vector. Sometimes we refer to a vector by its initial point followed by its terminal point, as letters, with an arrow over them. In this case, $\vec{w} = \overrightarrow{PQ}$.



The numbers 7 and -5 are called the **components** of the vector \vec{w} . More specifically, 7 is the x -component of \vec{w} and -5 is the y -component. Symbolically it is convenient to refer to them as w_1 and w_2 ; this notation indicates clearly that *the components of a vector are scalars*. Using this language, we can easily describe how to find a vector \overrightarrow{PQ} from a point P to another point Q : the first component of \overrightarrow{PQ} is found by subtracting the x -coordinate of P from the x -coordinate of Q . The second component of \overrightarrow{PQ} is found by subtracting the y -coordinates in the same order.

Suppose now that we have two vectors $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$, and scalar c . Then

$$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle, \quad \vec{u} - \vec{v} = \langle u_1 - v_1, u_2 - v_2 \rangle, \quad c\vec{u} = \langle cu_1, cu_2 \rangle$$

These operations go the same way in three dimensions.

1.4 Magnitudes of Vectors

Performance Criterion:

1. (d) Find the magnitude (length) of a vector. Find a vector satisfying given direction and magnitude criteria.

Recall that a vector quantity has both amount (magnitude) and direction. Geometrically, the magnitude of a vector is indicated by its length. When a vector is placed in the coordinate plane or three-space, its length is easily found by using the Pythagorean theorem. We denote the magnitude of the vector $\vec{v} = \langle v_1, v_2 \rangle$ (or $\vec{u} = \langle u_1, u_2, u_3 \rangle$) by $\|\vec{v}\|$ ($\|\vec{u}\|$), and it is computed by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2} \quad \left(\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2} \right)$$

A vector with length one is called a **unit vector**. Often we wish to create a unit vector in the direction of a given vector \vec{v} ; To do this we simply divide \vec{v} by its length. The process of doing this is sometimes called **normalizing** the vector.

Suppose that we wish to find a vector of a given length and in the same, or opposite, direction of another given vector. To do this we simply normalize the “direction vector” and then multiply the result by whatever length we are given. If we wish for the new vector to point in the opposite direction, we multiply by the negative of the desired length.

One last thing of note concerning magnitudes is that if c is a scalar, then $\|c\vec{v}\| = |c|\|\vec{v}\|$. This is occasionally handy.

1.5 Standard Basis Vectors

Suppose now that we have a vector $\vec{u} = \langle u_1, u_2 \rangle$ in the xy -plane, which we often refer to as \mathbb{R}^2 (spoken as “R-two”). Then we can see that

$$\vec{u} = \langle u_1, u_2 \rangle = \langle u_1, 0 \rangle + \langle 0, u_2 \rangle = u_1 \langle 1, 0 \rangle + u_2 \langle 0, 1 \rangle$$

The vectors $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$ are unit vectors, and they point in the direction of the positive ends of each of the two (three) coordinate axes. We call these the **standard basis vectors** of \mathbb{R}^2 and we name them \vec{i} and \vec{j} , respectively. Using this notation, we see that the vector $\vec{u} = \langle u_1, u_2 \rangle$ can also be written as $\vec{u} = u_1 \vec{i} + u_2 \vec{j}$.

When working in three-dimensional space, which we sometimes refer to as \mathbb{R}^3 , the standard basis vectors are $\vec{i} = \langle 1, 0, 0 \rangle$, $\vec{j} = \langle 0, 1, 0 \rangle$ and $\vec{k} = \langle 0, 0, 1 \rangle$ and the vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ can be written $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$. These show that an arbitrary vector in \mathbb{R}^2 or \mathbb{R}^3 can be represented as a linear combination of the standard basis vectors.

1.6 “Multiplying” Vectors; The Dot Product

Performance Criterion:

1. (e) Find the dot product of two vectors, and know that it is a scalar. Determine whether two vectors are orthogonal (perpendicular). Find a vector orthogonal to a given vector.

There is no true multiplication of vectors, but there are two products of vectors that we will use. The first is the **dot product**; if we have two vectors $\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$ and $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$, their dot product is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 .$$

A similar formula is used for the dot product of vectors in \mathbb{R}^2 or in higher dimensions. Note that *the dot product of two vectors is a scalar* - for this reason it is sometimes called the **scalar product**. The dot product has three very important uses for us:

- 1) If $\vec{u} \cdot \vec{v} = 0$, then \vec{u} and \vec{v} are perpendicular. We sometimes use the term **orthogonal** synonymously with perpendicular. (Additionally, $\vec{u} \cdot \vec{v} > 0$ if the angle between \vec{u} and \vec{v} is less than 90° and $\vec{u} \cdot \vec{v} < 0$ if the angle between \vec{u} and \vec{v} is greater than 90° .)
- 2) Another way to compute the dot product of two vectors is $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$, where θ is the angle between the vectors \vec{u} and \vec{v} . From this we get a formula for the cosine of the angle between two vectors - it can be found in your book or easily derived from the formula just given. We will rarely use this formula, but you should be aware that it exists.
- 3) The dot product is used to find something called the **projection** of one vector on another vector.

It will be useful in interpreting something you will see soon to note that $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

1.7 Projections, Orthogonal Components of a Vector

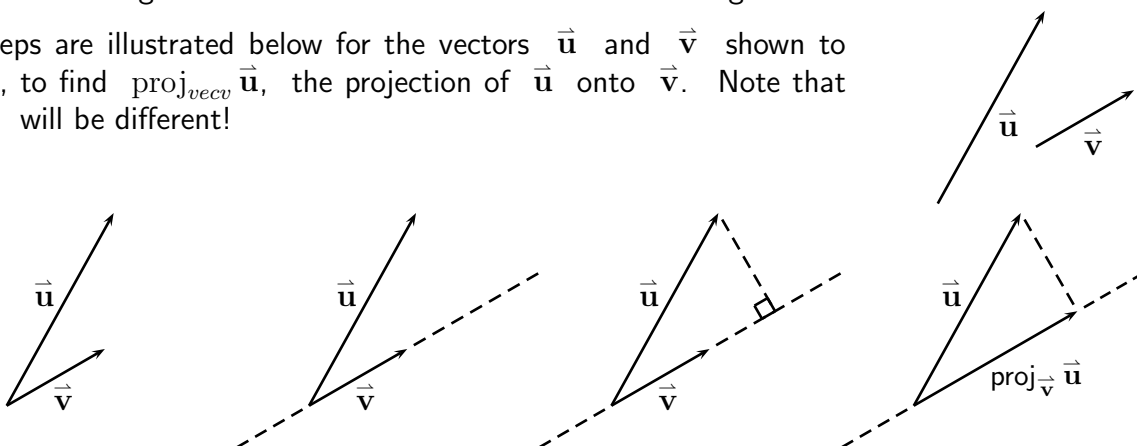
Performance Criteria:

1. (j) Draw the projection of one drawn vector on another drawn vector. Find the projection of one vector onto another vector algebraically.
- (k) Draw the components of a vector \mathbf{v} that are parallel and perpendicular to a vector \mathbf{b} . Find the components algebraically.

Given two vectors \vec{u} and \vec{v} , we can create a new vector called the **projection of \vec{u} on \vec{v}** , denoted by $\text{proj}_{\vec{v}} \vec{u}$. This is a very useful idea, in many ways. Graphically, we can find $\text{proj}_{\vec{v}} \vec{u}$ as follows:

- Bring \vec{u} and \vec{v} together tail-to-tail.
- Sketch in the line containing \vec{v} , as a dashed line.
- Sketch in a dashed line segment from the tip of \vec{u} to the dashed line containing \vec{v} , *perpendicular to that line*.
- Draw the vector $\text{proj}_{\vec{v}} \vec{u}$ from the point at the tails of \vec{u} and \vec{v} to the point where the dashed line segment meets \vec{v} or the dashed line containing \vec{v} .

These steps are illustrated below for the vectors \vec{u} and \vec{v} shown to the right, to find $\text{proj}_{\vec{v}} \vec{u}$, the projection of \vec{u} onto \vec{v} . Note that $\text{proj}_{\vec{u}} \vec{v}$ will be different!



We will also want to know how to find projections algebraically; for two vectors \vec{u} and \vec{v} , the vector $\text{proj}_{\vec{v}} \vec{u}$ is given by

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \frac{\vec{v}}{\|\vec{v}\|}$$

Why three versions of the same thing? The first version is the simplest computationally, and the middle version reminds us that $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$. The last version shows that the projection is the scalar $\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$ times the unit vector $\frac{\vec{v}}{\|\vec{v}\|}$. The scalar is positive if the angle between \vec{u} and \vec{v} is less than 90° , and it is negative if the angle is greater than 90° .

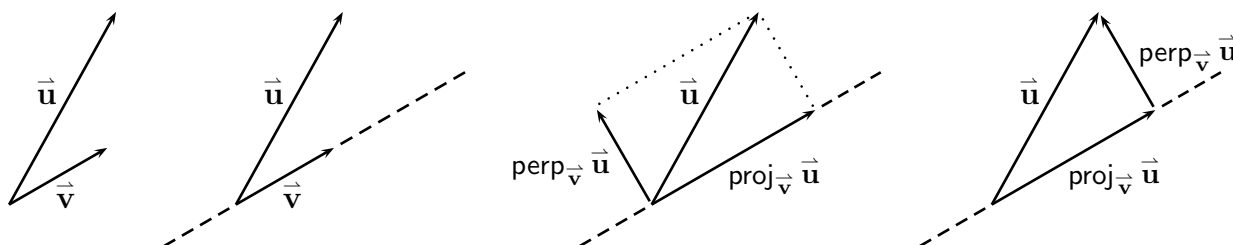
One of the main uses of projections is to break a given vector \vec{u} down into two vectors such that

- one of the vectors is parallel to another given vector \vec{v} and the other is perpendicular to \vec{v} , and
- the sum of the two vectors is \vec{u} .

The vector parallel to \vec{v} is $\text{proj}_{\vec{v}} \vec{u}$, and the other vector is sometimes called $\text{perp}_{\vec{v}} \vec{u}$. From the second bullet above we have

$$\text{proj}_{\vec{v}} \vec{u} + \text{perp}_{\vec{v}} \vec{u} = \vec{u}, \quad \text{so} \quad \text{perp}_{\vec{v}} \vec{u} = \vec{u} - \text{proj}_{\vec{v}} \vec{u}$$

The two vectors $\text{proj}_{\vec{v}} \vec{u}$ and $\text{perp}_{\vec{v}} \vec{u}$ are called the **orthogonal components** of \vec{u} with respect to \vec{v} . This concept is illustrated below, showing the two logical positions for $\text{perp}_{\vec{v}} \vec{u}$. Note that the first indicates \vec{u} as the sum of $\text{proj}_{\vec{v}} \vec{u}$ and $\text{perp}_{\vec{v}} \vec{u}$ by the parallelogram method, and the second by the tip-to-tail method.



1.8 “Multiplying” Vectors; The Cross Product

Performance Criterion:

1. (f) Find the cross product of two vectors in 3-space. Know that the cross product of two vectors in 3-space is a vector that is orthogonal to both original vectors.

The other vector product is called the **cross product**. It has several important features:

- 1) The cross product is only valid for vectors in three dimensions.
- 2) The cross product of two vectors is itself a vector, and *it is perpendicular to the two original vectors*.

The cross product of two vectors can be computed in several ways, some which you can find in your textbook or online. Although it is a bit complicated to describe in writing, I’ll try to explain the method that most students seem to prefer. Suppose that we are trying to find the cross product of the two vectors $\vec{u} = u_1 \vec{i} + u_2 \vec{j} + u_3 \vec{k}$ and $\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$. Here is the process for finding the cross product:

- 1) First we create something called a *determinant* whose first row is the three basis vectors \vec{i} , \vec{j} and \vec{k} in \mathbb{R}^3 , and whose second and third rows are the components of \vec{u} and \vec{v} , respectively and in order. This is shown below and to the left.

- 2) We repeat the first two *columns* of the matrix after the third column, as shown below and to the right. Then we multiply diagonally downward beginning at each of first three objects in the top row, and diagonally upward beginning at each of the first three objects in the bottom row. The diagonals are indicated by the dashed arrows below, with the result of each multiplication shown at the tip of each arrow.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \Rightarrow \begin{array}{ccccc} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ & u_1 & u_2 & u_3 & u_1 & u_2 \\ v_1 & v_2 & v_3 & v_1 & v_2 & v_3 \\ & & & & u_2 v_3 \vec{i} & u_3 v_1 \vec{j} \\ & & & & & u_1 v_2 \vec{k} \end{array}$$

- 3) We get the cross product by adding up each of the results of the downward multiplications and then subtracting each of the results of the upward multiplications. This is shown below.

$$\begin{aligned} \vec{u} \times \vec{v} &= u_2 v_3 \vec{i} + u_3 v_1 \vec{j} + u_1 v_2 \vec{k} - v_1 u_2 \vec{k} - v_2 u_3 \vec{i} - v_3 u_1 \vec{j} \\ &= (u_2 v_3 - u_3 v_2) \vec{i} + (u_3 v_1 - u_1 v_3) \vec{j} + (u_1 v_2 - u_2 v_1) \vec{k} \end{aligned}$$

If you find all that hard to look at, I don't blame you! Let's do a specific example, with say $\vec{u} = \langle 3, 1, 6 \rangle$ and $\vec{v} = \langle -5, 2, 4 \rangle$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 6 \\ -5 & 2 & 4 \end{vmatrix} \Rightarrow \begin{array}{ccccc} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ & 3 & 1 & 6 & 3 & 1 \\ -5 & 2 & 4 & -5 & 2 & 4 \\ & & & & 4 \vec{i} & -30 \vec{j} \\ & & & & & -5 \vec{k} \\ & & & & & & 12 \vec{i} & 12 \vec{j} \end{array}$$

$$\begin{aligned} \vec{u} \times \vec{v} &= 4 \vec{i} + (-30) \vec{j} + 6 \vec{k} - (-5) \vec{k} - 12 \vec{i} - 12 \vec{j} \\ &= -8 \vec{i} - 42 \vec{j} + 11 \vec{k} \end{aligned}$$

Now that you have seen both products of vectors, let's summarize a few things about them. Some of these things are new, some are not. Note the *compare and contrast* nature of the following.

- The dot product of two vectors is a scalar, the cross product of two vectors is a vector.
- The dot product is commutative. That is, the result is the same regardless of the order of the two vectors. *The cross product is not commutative*; changing the order of the vectors will give the negative of the vector obtained in the original order.
- If two vectors are perpendicular, their dot product is zero.
- The cross product of two vectors is a new vector that is *perpendicular to BOTH of the original two vectors*.

1.9 Lines in Space

Performance Criterion:

1. (h) Find the parametric equations or vector equation for a line
 - i. through a given point and parallel to a given vector,
 - ii. through two given points in 2 or 3-space.

The set of points in \mathbb{R}^2 satisfying an equation of the form $y = mx + b$ is a line; any such equation can be rearranged into the form $ax + by = c$. (The values of b in the two equations are clearly not the same.) But if we add one more term to get $ax + by + cz = d$, with the (x, y, z) representing the coordinates of a point in \mathbb{R}^3 , we get the equation of a plane, not a line! The object of this section is to show how we can represent lines using vectors, in both \mathbb{R}^2 and \mathbb{R}^3 . We will come back to planes in the next section.

To begin, let's just work in the plane. Blah, blah, all the stuff I said in class.

- ◇ **Example 1.9(a):** Give the parametric equations of the line in two-space containing the points $P(-2, 3)$ and $Q(1, 4)$.

Solution: The parametric equations of a line in two-space look like

$$x = a + ct, \quad y = b + dt,$$

where (a, b) is *either* of the two points, and c and d are the differences between the x and y coordinates, respectively, of the two points. We'll take (a, b) to be the first point $(-2, 3)$ and

$$c = 1 - (-2) = 3 \quad \text{and} \quad d = 4 - 3 = 1.$$

The parametric equations of the line are then

$$x = -2 + 3t, \quad y = 3 + t.$$

It is a little misleading to say that the equations found above are "the" parametric equations of the line - there are many possibilities. We could use the coordinates of the point Q as a and b to get

$$x = 1 + 3t, \quad y = 4 + t.$$

If we were to subtract the coordinates of the points in the opposite order to get c and d , we would get either

$$x = -2 - 3t, \quad y = 3 - t$$

or

$$x = 1 - 3t, \quad y = 4 - t.$$

The process for getting the parametric equations of a line in three dimensions is the same:

- ◇ **Example 1.9(b):** Give the parametric equations of the line in two-space containing the points $P(4, -1, 3)$ and $Q(1, 5, -2)$.

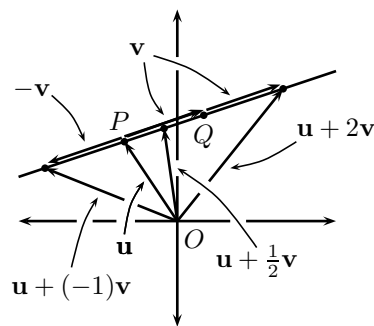
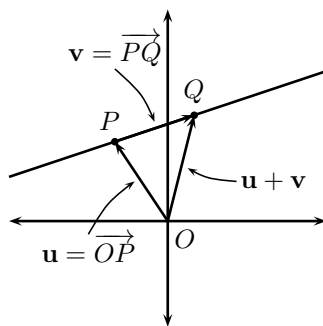
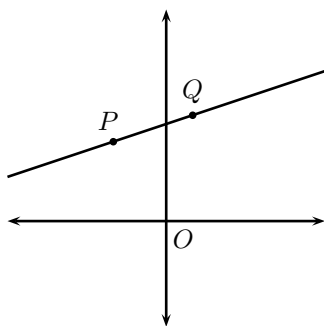
Solution: In this case the parametric equations of a line in three-space look like

$$x = a + dt, \quad y = b + et, \quad z = c + ft$$

where (a, b, c) is either of the two points, and d, e and f are the differences between the coordinates. One possibility is then

$$x = 4 - 3t, \quad y = -1 + 6t, \quad z = 3 - 5t.$$

We now see how to obtain what is called the vector equation of a line, which is closely related to the parametric equations. Consider the line shown below and to the left, containing the points P and Q . If we let $\vec{u} = \overrightarrow{OP}$ and $\vec{v} = \overrightarrow{PQ}$, then the points P and Q correspond to the vectors \vec{u} and $\vec{u} + \vec{v}$ (in standard position, which you should assume we mean from here on), as shown in the second picture. From this you should be able to see that if we consider all the vectors \vec{x} defined by $\vec{x} = \vec{u} + t\vec{v}$ as t ranges over all real numbers, the resulting set of points is our line! This is shown in the third picture, where t is given the values $-1, \frac{1}{2}$ and 2 .



Now this may seem like an overly complicated way to describe a line, but with a little thought you should see that the idea translates directly to three (and more!) dimensions, as shown in the picture to the right. Let's look at a specific example of how this works. Suppose we wish to describe the line in \mathbb{R}^3 through the points $P(3, -1, 2)$ and $Q(-4, 5, 1)$. In this case $\vec{u} = \overrightarrow{OP} = \langle 3, -1, 2 \rangle$ and

$$\vec{v} = \overrightarrow{PQ} = \langle -4 - 3, 5 - (-1), 1 - 2 \rangle = \langle -7, 6, -1 \rangle.$$

The vector equation of the line through P and Q is then

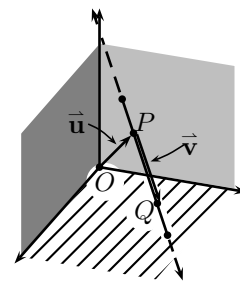
$$\langle x, y, z \rangle = \langle 3, -1, 2 \rangle + t\langle -7, 6, -1 \rangle.$$

We can multiply by the scalar t and add the two vectors to get

$$\langle x, y, z \rangle = \langle 3 - 7t, -1 + 6t, 2 - t \rangle,$$

and from this we can get the three **parametric equations** describing the line:

$$x = 3 - 7t, \quad y = -1 + 6t, \quad z = 2 - t$$



These equations are not unique! With a bit of thought you should be able to come up with more parametric equations describing the same line. Note that when $t = 0$ we get the point P , and when $t = 1$ we get Q .

We can also work backward from parametric equations to obtain a vector equation, as shown in the following examples.

- ◇ **Example 1.9(c):** Give a vector equation of the line in two-space containing the points $P(-2, 3)$ and $Q(1, 4)$.

Solution: In example 1.9(a) we determine the parametric equations

$$x = -2 + 3t, \quad y = 3 + t \quad (1)$$

for the line. From this we can get

$$\vec{x} = \langle x, y \rangle = \langle -2 + 3t, 3 + t \rangle = \langle -2, 3 \rangle + \langle 3t, t \rangle = \langle -2, 3 \rangle + t\langle 3, 1 \rangle.$$

In one statement,

$$\vec{x} = \langle -2, 3 \rangle + t\langle 3, 1 \rangle.$$

Note the relationship between this and the parametric equations (1).

- ◇ **Example 1.9(d):** Give a vector equation of the line in three-space containing the points $P(4, -1, 3)$ and $Q(1, 5, -2)$.

Solution: We found parametric equations in Example 1.9(b), but let's proceed as if we didn't have those equations to work from. The vector equation of the line is simply

$$\vec{x} = \vec{OP} + t\vec{PQ} = \langle 4, -1, 3 \rangle + t\langle 1 - 4, 5 - (-1), -2 - 3 \rangle = \langle 4, -1, 3 \rangle + t\langle -3, 6, -5 \rangle.$$

In one statement, $\vec{x} = \langle 4, -1, 3 \rangle + t\langle -3, 6, -5 \rangle$.

1.10 Planes in Space

Performance Criteria:

1. (i) Find the equation of a plane, given
 - i. a point on the plane and a normal vector to the plane,
 - ii. three points on the plane.

As stated at the beginning of the previous section, the set of points (x, y, z) satisfying an equation of the form $ax + by + cz = d$, with a, b, c and d being constants, is a plane. A plane in space is a set of points (x, y, z) that make up an infinite "sheet" (which is of course "flat"). To know which particular plane we are talking about, we must determine two things:

the “tilt” of the plane, and the location of the plane. The tilt can be specified by giving a vector that is **normal** (perpendicular) to the plane. *If we know the tilt of the plane and a point that the plane contains, then we know exactly what plane we are talking about.*

All the points (x, y, z) on a plane that has normal vector $\vec{n} = \langle a, b, c \rangle$ (where a , b , and c are constants) satisfy an equation of the form

$$ax + by + cz = d,$$

where d is another constant. In a sense, a , b and c determine the orientation of the plane and d determines where the plane is located. The following example shows how this is done.

- ◇ **Example 1.10(a):** Give the equation of the plane with normal vector $\vec{n} = \langle 4, 2, -5 \rangle$ and containing the point $(3, -1, 7)$.

Solution: From the above we know that the equation has the form $4x + 2y - 5z = d$. But since the point $(3, -1, 7)$ lies on the plane, it must satisfy the equation of the plane. Substituting 3, -1 and 7 for x , y and z and solving for d gives us $d = -25$, so the equation of the plane is $4x + 2y - 5z = -25$.

We can also specify a plane by giving *three* points on the plane instead of a normal vector and one point. (The three points can't lie on a line - can you see why?) How do we find the equation of the plane in this sort of situation? If we call the three points P , Q and R , we can form three vectors lying in the plane. We really only need two, though; if we find two like \vec{PQ} and \vec{PR} , then find their cross product, we have a normal vector for the plane. We can then repeat the process we used in Example 1.10(a) to get the equation of the plane.

- ◇ **Example 1.10(b):** Give the equation of the plane containing the points $P(4, 1, -3)$, $Q(-2, 2, 1)$ and $R(2, 5, 6)$.

Solution: First we find the vectors $\vec{PQ} = \langle -6, 1, 4 \rangle$ and $\vec{PR} = \langle -2, 4, 9 \rangle$. Then the vector $\vec{n} = \vec{PQ} \times \vec{PR} = \langle -7, 42, -22 \rangle$ is perpendicular to both \vec{PQ} and \vec{PR} , so it is normal to the plane whose equation we are seeking. From the above we know that the equation has the form $-7x + 42y - 22z = d$. But since the point $(4, 1, -3)$ lies on the plane, it must satisfy the equation of the plane. Substituting 4, 1 and -3 for x , y and z and solving for d gives us $d = 80$, so the equation of the plane is $-7x + 42y - 22z = 80$.

1.11 More Lines and Planes

Performance Criteria:

1. (l) Determine whether two planes are parallel, perpendicular, or neither. Determine whether a line and a plane are parallel, perpendicular, or neither.
- (m) Find the point of intersection of two lines; find the point of intersection of a line and a plane.

1.12 Position and Displacement Vectors

Performance Criterion:

1. (n) Find vectors representing displacement, average velocity, change in velocity, and average acceleration.

Technically speaking, a vector has no position. However, when we put the tail of a vector at the origin of a coordinate system, it is called a **position vector**. We will also say that the vector is in **standard position**. Its tip then represents a point in 2- or 3-space. For example, the vector $3\vec{i} - 2\vec{j}$ represents the point $(3, -2)$. We will want to represent points with vectors because we can do arithmetic with vectors that we can't do with points.

Suppose that we have two points P_1 and P_2 in \mathbb{R}^3 . If we let the letter O represent the origin in \mathbb{R}^3 , then the position vectors for the two points are $\vec{r}_1 = \vec{OP}_1$ and $\vec{r}_2 = \vec{OP}_2$. (It is common to use the letter \vec{r} for a position vector.) If an object moves from point P_1 to point P_2 , we call the vector $\vec{P_1P_2}$ the **displacement vector**. Sometimes we will refer to it as $\Delta\vec{r}$, where the symbol Δ ("delta") means "change." Since \vec{r} represents position, $\Delta\vec{r}$ represents "change in position" and is a vector, found by $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$. Its magnitude tells how far the object moved, and its direction gives the direction the object moved. *We will see later that the magnitude of the displacement does not necessarily indicate how far an object has actually traveled; it is simply the straight line distance from where it begins traveling to where it stops.*

1.13 Average Velocity and Average Acceleration

Performance Criterion:

1. (o) Find vectors representing displacement, average velocity, change in velocity, and average acceleration.

It should be clear that if an automobile travels 300 miles in 5 hours, it has had an average *speed* of $300 \div 5 = 60$ miles per hour. This indicates that speed is distance traveled divided by time. In 2- or 3-space, *velocity is always a vector*, and it is displacement divided by time. If an object has a displacement vector of $\Delta\vec{r}$ over a time period of length Δt , then its average velocity over that time period is

$$\vec{v}_{\text{ave}} = \frac{\Delta\vec{r}}{\Delta t}$$

Note that $\Delta\vec{r}$ is a vector and Δt is a scalar, so the quantity $\frac{\Delta\vec{r}}{\Delta t}$ makes sense in terms of vector operations, and the result is a vector. From this point on you must realize that velocity is a vector quantity when working in \mathbb{R}^2 or \mathbb{R}^3 . The relationship between speed and velocity is this:

Speed is simply how fast an object is moving, and does not indicate direction. Velocity is a vector that indicates both speed and direction of motion; speed is the magnitude of velocity.

The notation $\vec{v}_{\text{ave}} = \frac{\Delta \vec{r}}{\Delta t}$ indicates that *velocity is the rate of change of position with respect to time*. In turn, *acceleration is the rate of change of velocity with respect to time*. That is,

$$\text{acceleration} = \vec{a} = \frac{\Delta \vec{v}}{\Delta t}$$

Note that $\Delta \vec{v}$ is a vector and Δt is a scalar, so acceleration is also a vector. We will study the velocity and acceleration of a moving object in much greater detail soon.