Ordinary Differential Equations

for Engineers and Scientists

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Contents

1 Functions and Derivatives, Variables and Parameters

Learning Outcomes:

1. Understand functions and their derivatives, variables and parameters. Understand differential equations, initial and boundary value problems, and the nature of their solutions.

   Performance Criteria:

   (a) Determine the independent and dependent variables for functions modeling physical and biological situations. Give the domain(s) of the independent variable(s).

   (b) For a given physical or biological situation, sketch a graph showing the qualitative behavior of the dependent variable over the domain (or part of the domain, in the case of time) of the independent variable.

   (c) Interpret derivatives in physical situations.

   (d) Find functions whose derivatives are given constant multiples of the original functions.

   (e) Identify parameters and variables in functions or differential equations.

   (f) Identify initial value problems and boundary value problems. Determine initial and boundary conditions.

   (g) Determine the independent and dependent variables for a given differential equation.

   (h) Determine whether a function is a solution to an ordinary differential equation (ODE); determine values of constants for which a function is a solution to an ODE.

   (i) Classify differential equations as ordinary or partial; classify ordinary differential equations as linear or non-linear. Give the order of a differential equation.

   (j) Identify the functions $a_0(x)$, $a_1(x)$, ..., $a_n(x)$ and $f(x)$ for a linear ordinary differential equation. Classify linear ordinary differential equations as homogenous or non-homogeneous.

   (k) Write a first order ordinary differential equation in the form $\dfrac{dy}{dx} = F(x,y)$ and identify the function F. Classify first-order ordinary differential equations as separable or autonomous.

   (l) Determine whether a function satisfies an initial value problem (IVP) or boundary value problem (BVP); determine values of constants for which a function satisfies an IVP or BVP.

Much of science and engineering is concerned with understanding the relationships between measurable, changing quantities that we call **variables**. Whenever possible we try to make these relationships precise and compact by expressing them as equations relating variables; often such equations define *functions*. In this chapter we begin by looking at ideas you should be familiar with (functions and derivatives), but hopefully you will now see them in a deeper and more illuminating way than you did in your algebra, trigonometry and calculus courses.

We then go on to introduce the idea of a differential equation, and we will see what we mean by a solution to a differential equation, initial value problem, or boundary value problem. We will also learn various classifications of differential equations. This is important in that the method used to solve a differential equation depends on what type of equation it is.

It is valuable to understand these fundamental concepts before moving on to learning techniques for solving differential equations, which are addressed in the remainder of the text.

1.1 Functions and Variables

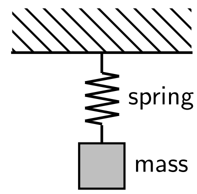
Performance Criteria:

1. (a) Determine the independent and dependent variables for functions modeling physical and biological situations. Give the domain of the independent variable(s).

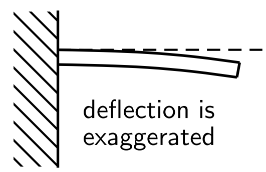
   (b) For a given physical or biological situation, sketch a graph showing the qualitative behavior of the dependent variable over the domain (or part of the domain, in the case of time) of the independent variable.

As scientists and engineers, we are interested in relationships between measurable physical quantities, like position, time, temperature, numbers or amounts of things, etc. The physical quantities of interest are usually changing, so are called **variables**. When one physical quantity (variable) depends on one or more other quantities (variables), the first quantity is said to be a **function** of the other variable(s).

⋄ **Example 1.1(a):** Suppose that a mass is hanging on a spring that is attached to a ceiling, as shown to the right. If we lift the mass, or pull it down, and let it go, it will begin to oscillate up and down. Its height (relative to some fixed reference, like its height before we lifted it or pulled it down) varies as time goes on from when we start it in motion. We say that *height is a function of time*.



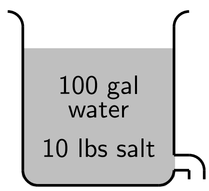
⋄ **Example 1.1(b):** Consider a beam that extends horizontally ten feet out from the side of a building, as shown to the right. The beam will deflect (sag) some, with the distance below horizontal being greater the farther out on the beam one looks. *The amount of deflection at a point on the beam is a function of how far the point is from the wall the beam is embedded in.*

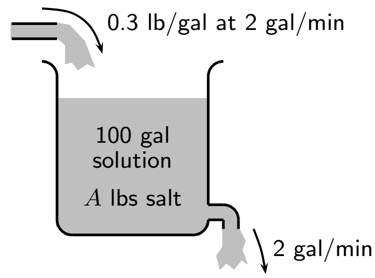


⋄ **Example 1.1(c):** Consider the equation $y = \dfrac{12}{x^2}$. For any number other than zero that we select for x, there is a corresponding value of y that can be determined by substituting the x value and computing the resulting value of y. y depends on x, or y *is a function of* x

⋄ **Example 1.1(d):** A drumhead with a radius of 5 inches is struck by a drumstick. The drum head vibrates up and down, with the height of the drumhead at a point determined by the location of that point on the drumhead and how long it has been since the drumhead was struck. *The height of the drumhead is a function of the two-dimensional location on the drumhead and time.*

◇ **Example 1.1(e):** Suppose that we have a tank containing 100 gallons of water with 10 pounds of salt dissolved in the water, as shown to the left below. At some time we begin pumping a 0.3 pounds salt per gallon (of water) solution into the tank at two gallons per minute, mixing it thoroughly with the solution in the tank. At the same time the solution in the tank is being drained out at two gallons per minute as well. See the diagram to the right below.





Because the rates of flow in and out of the tank are the same, the volume in the tank remains constant at 100 gallons. The initial concentration of salt in the tank is 10 pounds/100 gallons = 0.1 pounds per gallon. Because the incoming solution has a different concentration, the amount of salt in the tank will change as time goes on. (The amount will increase, since the concentration of the incoming solution is higher than the concentration of the solution in the tank.) We can say that *the amount of salt in the tank is a function of time*.

◇ **Example 1.1(f):** Different points on the surface of a cube of metal one foot on a side are exposed to different temperatures, with the temperature at each surface point held constant. The cube eventually attains a temperature **equilibrium**, where each point on the interior of the cube reaches some constant temperature. *The temperature at any point in the cube is a function of the three-dimensional location of the point.*

In each of the above examples, one quantity (variable) is dependent on (is a function of) one or more other quantities (variables). The variable that depends on the other variable(s) is called the **dependent variable**, and the variable(s) that its value depends on is (are) called the **independent variable(s)**.

◇ **Example 1.1(g):** Give the dependent and independent variable(s) for each of Examples 1.1(a) - (f).

Solution: Example 1.1(a): The dependent variable is the height of the mass, and the independent variable is time.

Example 1.1(b): The dependent variable is the deflection of the beam at each point, and the independent variable is the distance of each point from the wall in which the beam is embedded.

Example 1.1(c): The dependent variable is y, and the independent variable is x.

Example 1.1(d): The dependent variable is the height at each point on the drumhead, and the independent variables are the location (in two-dimensional coordinates) of the point on the drum head, and time. Thus there are *three* independent variables.

Example 1.1(e): The dependent variable is the amount of salt in the tank, and the independent variable is time.

Example 1.1(f): The dependent variable is the temperature at each point in the cube, and the independent variables are the three coordinates giving the position of the point, in three dimensions.

When studying phenomena like those given in Examples 1.1(a), (b), (d), (e) and (f), the first thing we do after determining the variables is establish coordinate systems for the variables. The purpose for this is to be able to attach a number (or ordered set of numbers) to each point in the domain, and for different positions or states of the dependent variable:

- When position is an independent variable, we must establish a one (for the spring), two (for the drumhead) or three (for the cube of metal) dimensional coordinate system. This coordinate system will have an origin (zero point) at some convenient location, indication of which direction(s) is(are) positive, and a scale on each axis. (The two space variables for the drumhead in Example 1.1(d) would most likely be given using polar coordinates, since the head of the drum is circular.)

- If time is an independent variable, we must establish a "time coordinate system" by determining when time zero is. (Of course all times after that are considered positive.) We must also decide what the time units will be, providing a "scale" for time.

- It may not be clear that there is a coordinate system for the temperature in the cube of metal, or the amount of salt in the tank. For the temperature, the decision whether to measure it in degrees Fahrenheit or degrees Celsius is actually the establishing of a coordinate system, with a zero point and a scale (both of which differ depending on which temperature scale is used).

- The choice of zero for the amount of salt in the tank will be the same regardless of how it is measured, but the scale can change, depending on the units of measurement.

Once we've established the coordinate system(s) for the variable(s), we should determine the **domain** of our function, which means the values of the independent variable(s) for which the dependent variable will have values. The domain is usually given using inequalities or interval notation. Let's look at some examples.

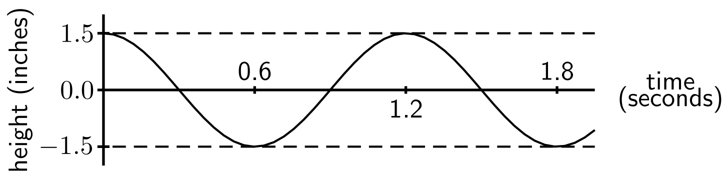
◇ **Example 1.1(h):** For Example 1.1(a), suppose that we pull the mass down and then let it go at a time we call time zero (the origin of our time coordinate system). Time t is the independent variable, and the values of it for which we are considering the height of the mass are $t \geq 0$ or, using interval notation, $[0, \infty)$.

◇ **Example 1.1(i):** For Example 1.1(b), we will use a position coordinate system consisting of a horizontal number line at the top of where the beam emerges from the wall (so along the dashed line in the picture), with origin at the wall and positive values (in feet) in the direction of the beam. Letting x represent the position along the beam, the domain is $[0, 10]$ feet.

The four functions described in Examples 1.1(a), (b), (c) and (e) are **functions of a single variable**; the functions in Examples 1.1(d) and (f) are examples of **functions of more than one variable**. The differential equations associated with functions of one variable are called **ordinary differential equations**, and the differential equations associated with functions of more than one variable are (out of necessity) **partial differential equations**. In this class we will study primarily ordinary differential equations.

The function in Example 1.1(c) is a *mathematical* function, whereas all of the other functions from Example 1.1 are not. (We might call them "physical functions.") In your previous courses you have studied a variety of types of mathematical functions, including polynomial, rational, exponential, logarithmic, and trigonometric functions. The main reason that scientists and engineers are interested in mathematics is that many physical situations can be **mathematically modeled** with mathematical functions or equations. This means that we can find a mathematical function that reasonably well describes the relationship between physical quantities. For a mass on a spring (Example 1.1(a)), if we let y represent the height of the mass, then the equation that models the situation is $y = A\cos(bt)$, where A and b are constants that depend on the spring and how far the mass is lifted or pulled down before releasing it. We will see that the deflection of the beam in Example 1.1(b) can be modeled with a fourth degree polynomial function, and the amount of salt in the tank of Example 1.1(c) can be modeled with an exponential function.

Of course one tool we use to better understand a function is its graph. Suppose for Example 1.1(a) we started the mass in motion by lifting it 1.5 inches and releasing it (with no upward or downward force). Then the equation giving the height y at any time t would be of the form $y = 1.5\cos(bt)$, where b depends on the spring and the mass. Suppose that $b = 5.2$ (with appropriate units). Then the graph would look like this:

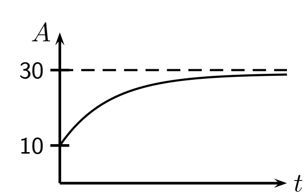


When graphing functions of one variable *we always put the independent variable (often it will be time) on the horizontal axis, and the dependent variable on the vertical axis.* We can see from the graph that the mass starts at a height of 1.5 inches above its equilibrium position $(y = 0)$. It then moves downward for the first 0.6 seconds of its motion, then back upward. It is back at its starting position every 1.2 seconds, the **period** of its motion. This periodic up-and-down motion can be seen from the graph. (Remember that the period T is the time at which $bT = 2\pi$, so $T = \frac{2\pi}{b}$.) Such behavior is called **simple harmonic motion**, and will be examined in detail later in the course because of its importance in science and engineering.

Note that even if we didn't know the value of b in the equation $y = 1.5\cos(bt)$ we could still create the given graph, we just wouldn't be able to put a scale on the time (horizontal) axis. In fact, we could even create the graph without an equation, using our intuition of what we would expect to happen. Let's do that for the situation from Example 1.1(e).

◇ **Example 1.1(j):** A tank contains 100 gallons of water with 10 pounds of salt dissolved in it. At time zero a 0.3 pounds per gallon solution begins flowing into the tank at 2 gallons per minute and, at the same time, thoroughly mixed solution is pumped out at 2 gallons per minute. (See Example 1.1(e).) Sketch a graph of the amount of salt in the tank as a function of time.

Solution: The initial amount of salt in the tank is 10 pounds. We know that as time goes on the *concentration* of salt in the tank will approach that of the incoming solution, 0.3 pounds per gallon. This means that the *amount* of salt in the tank will approach 0.3 lbs/gal × 100 gal = 30 pounds, resulting in the graph shown to the right, where A represents the amount of salt, in pounds, and t represents time, in minutes.



NOTE: We have been using the notation $\sin(bt)$ or $\cos(bt)$ to indicate the sine or cosine of the quantity bt. It gets to be a bit tiresome writing in the parentheses every time we have such an expression, so we will just write $\sin bt$ or $\cos bt$ instead.

Section 1.1 Exercises To Solutions

1. Some material contains a radioactive substance that decays over time, so the amount of the radioactive substance is decreasing. (It doesn't just go away - it turns into another substance that is not radioactive, in a series of steps. For example, uranium eventually turns into lead when it decays.)

   (a) Give the dependent and independent variables.

   (b) Sketch a graph of the amount A of radioactive substance versus time t. *Label each axis with its variable - this will be expected for all graphs.*

2. A student holds a one foot plastic ruler flat on the top of a table, with half of the ruler sticking out and the other half pinned to the table by pressure from their hand. They then "tweak" the end of the ruler, causing it to vibrate up and down. (This is roughly a combination of Examples 1.1(a) and (b).)

   (a) Give the dependent and independent variables. (**Hint:** There are *two* independent variables.)

   (b) Give the domains of the independent variables.

3. Consider the drumhead described in Example 1.1(e). Suppose that the position of any point on the drumhead is given in polar coordinates (r, θ), with r measured in inches and θ in radians. Suppose also that time is measured in seconds, with time zero being when the head of the drum is struck by a drumstick. Give the domains of each of these three independent variables.

4. Consider the cube of metal described in Example 1.1(f). Suppose that we position the cube in the first octant (where each of x, y and z is positive), with one vertex (corner) of the cube at the origin and each edge from that vertex aligned with one of the three coordinate axes. Each point in the cube then has some coordinates (x, y, z). Give the domains of each of these three independent variables.

5. Consider again the scenario from Section 0.2, in which a rock is fired straight upward with a velocity of 60 feet per second, from a height of 20 feet off the ground. In that section we derived the equation
$$h = -16t^2 + 60t + 20$$
for the height h (in feet) of the rock at any time t (in seconds) after it was fired. Using the equation, determine the domain of the independent variable time.

6. When a solid object with some initial temperature T_0 is placed in a medium (like air or water) with a constant temperature T_m, the object will get cooler or warmer (depending on whether T_0 is greater or less than T_m), with its temperature T approaching T_m. The rate at which the temperature of the object changes is proportional to the difference between its temperature T and the temperature T_m of the medium, so it cools or warms rapidly while its temperature is far from T_m, but then the cooling or warming slows as the temperature of the object approaches T_m.

   (a) Suppose that an object with initial temperature $T_0 = 80°$ F is place in a water bath that is held at $T_m = 40°$ F. Sketch a graph of the temperature as a function of time. You should be able to indicate two important values on the vertical axis.

   (b) Repeat (a) for $T_m = 40°$F and $T_0 = 30°$F.

   (c) Repeat (a) for $T_m = 40°$F and $T_0 = 40°$F.

7. (a) Suppose that a mass on a spring hangs motionless in its equilibrium position. At some time zero it is set in motion by giving it a sharp blow downward, and there is no resistance after that. Sketch the graph of the height of the mass as a function of time.

   (b) Suppose now that the mass is set in motion by pulling it downward and simply releasing it, and suppose also that the mass is hanging in an oil bath that resists its motion. Sketch the graph of the height of the mass as a function of time.

8. As you are probably aware, populations (of people, rabbits, bacteria, etc.) tend to grow exponentially *when there are no other factors that might impeded that growth.*

   (a) The variables in such a situation are time and the number of individuals in the population. Which variable is independent, and which is dependent?

   (b) Using t for time and N for the number of individuals, sketch (and label, of course) a graph showing growth of such a population.

   (c) Often there are environmental conditions that lead to a **carrying capacity** for a given population, meaning an upper limit to how many individuals can exist. Suppose that 500 fish are stocked in a sterile lake (no fish in it) that has a carrying capacity of 3000 fish. When a population like this starts at well below the carrying capacity, it experiences "almost exponential" growth for a while, then the growth levels off as the population approaches the carrying capacity. Sketch a graph of the fish population versus time. This sort of growth is called **logistic growth**.

   (d) Sketch a graph showing how you would expect the population of fish to behave if 5000 fish were introduced into the same lake having a carrying capacity of 3000 fish.

In this course we will often be interested in just a few "families" of functions, like

$$y(t) = A\sin\omega t + B\cos\omega t, \qquad y = a + be^{-rt}, \qquad y(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$

where each of A, B, ω, a, b, r, a_0, a_1, a_2, a_3, a_4 are constants that we will call **parameters** (more on this in Section 1.3) and x, t, and y are variables. (ω is the Greek letter *omega*.) We will often omit showing the dependence of y on x or t, as done for the second function above. The behavior of each of the above functions remains roughly the same, but varies somewhat depending on the values of the parameters. The point of the following exercises is to see what the graph of each type of function looks like in general, and how the values of the parameters affect various aspects of the graph.

9. We begin with the graph of $y = A\sin\omega t + B\cos\omega t$.

   (a) Sketch what you expect the graph to look like when $A = 1$, $B = 0$ and $\omega = 1$. Sketch a separate graph for $A = 0$, $B = 1$ and $\omega = 1$. These two graphs should be familiar to you. Do you have any idea what the graph would look like if $A = B = \omega = 1$?

   (b) Enter the function into *Desmos*, using w in place of ω. It will ask you if you want sliders for A, w, B and t. select the first three, but not t. Check your answers to part (a) by setting the sliders for the appropriate values.

   (c) Recall that trigonometric graphs have three important characteristics:

   - **Amplitude** - The maximum distance the function (y value) gets from the horizontal axis. (We will assume no vertical shifting of the graph, which we don't need in this course.)
   - **Period** - The distance along the horizontal axis from any point on the graph to where the graph first repeats itself.
   - **Phase** - This refers to the horizontal point on the graph where it crosses the vertical axis. For example does the graph cross the vertical axis at a peak, trough, near the top of an "upslope," etc.?

   Set $B = 0$. Now use the slider to change A. Which of the above characteristics does changing A seem to affect? Which characteristics does changing w affect? Which characteristic is not affected by changing A or w when $B = 0$?

   (d) Now set $B = 1$ and change A. Which characteristic or characteristics does this affect? What does changing w affect? How about changing B?

   Note: We'll see later how to take a function of the form $y = A\sin\omega t + B\cos\omega t$ and turn it into just a sine function of the form $y = C\sin(\omega t + \phi)$, which is a little easier to work with.

10. The type of function we looked at in the previous exercise models a mass on a spring, as described in Example 1.1(a), as well as certain electric circuits. We will see that ω is determined by the mass and the spring, but A and B are determined by how the mass is set in motion. We assume there is nothing resisting the motion of the mass, a situation we refer to as **undamped**. In applications we often have some sort of resisting force called damping. It shows up mathematically as an exponential function times a function of the sort we saw in Exercise 9:

    $$y = e^{-rt}(A\sin\omega t + B\cos\omega t), \qquad r > 0.$$

    Speculate, based on either this function equation or the physical situation, what the graph of such a function would look like. Check your conjecture using *Desmos*.

11. Now we consider the function $y = a + be^{-rt}$, where $r > 0$.

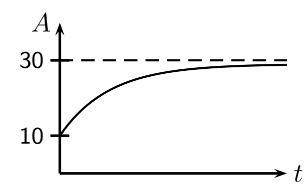
   (a) What do you expect the graph of this function to look like when $a = 0$ and $b = r = 1$? Enter the function in *Desmos*, setting sliders for a, b and r and check your conjecture. When working with this kind of function we are really just interested in its behavior for $t \geq 0$. You can restrict the graph in *Desmos* by entering $\{t > 0\}$ after the function - do that now.

   (b) For graphs of this sort of function, we are interested in three things:

      - What the y-intercept is.
      - What y value the graph tends toward as time goes on. This is the **horizontal asymptote** of the graph, and it can be expressed in the language of calculus as the limit of y as t goes to infinity: $\lim_{t\to\infty} y(t)$
      - How rapidly the function approaches the aymptote.

   How do the parameters a, b and r affect each of the above? Be as specific as possible. Remember that we are only interested in positive values of r. For the applications we will be interested in we will usually have only positive values of a, but b will sometimes be positive, sometimes negative.

12. To the right is the graph from Example 1.1(j), for a tank containing 100 galloms of water with 10 pounds of salt dissolved in it. The horizontal axis is time, in minutes, and the vertical axis is the amount of salt dissolved in the water, in pounds.



   (a) Determine values of a, b and r for which the graph of $A = a + be^{-rt}$ is the graph shown. Use *Desmos* to check your answer.

   (b) In part (a) you should have found that you cannot determine the value of r from the graph given. Graph your answer to (a) with *Desmos*, for $r = 0.1$. Sketch what you see, using a dotted line for the graph. Sketch in, as a dashed line and a solid line, what you think the graph would look like for $r = 0.05$ and $r = 0.5$. Check your answers with *Desmos*.

13. You cook a potato in a microwave oven, and when you take it out, its temperature is $160°$F. It is too hot to eat, so you decide to let it cool. In the meantime you start playing a video game, and completely forget about the potato for several hours. The temperature in your house is $70°$F.

   (a) Sketch a graph of what you expect the temperature T of the potato to be as a function of time, t.

   (b) Give values of a and b for which the graph of $T = a + be^{-rt}$ has the appearance of your graph from part (a).

14. There are two facts that are helpful in understanding the appearance of the graph of $y = a + be^{-rt}$, where we emphasize again that we are only interested in $r > 0$. Those two facts are

$$e^0 = 1 \qquad \text{and} \qquad \lim_{t\to\infty} e^{-rt} = 0 \text{ for any fixed } r > 0.$$

(a) What is the value of t at the y-intercept of the graph of $y = a + be^{-rt}$? Given that and the above, what is the y-intercept of $y = a + be^{-rt}$?

(b) Based on the above, what is the limit of y as t goes to infinity? What does that tell us about the graph of $y = a + be^{-rt}$?

15. Use what you discovered in the previous exercise to sketch the graph of each of the following functions for some value of $r > 0$. Use *Desmos* to check your answers.

(a) $y = 100 + 50e^{-rt}$ (b) $y = 100 - 50e^{-rt}$ (c) $y = 200e^{-rt}$

16. The functions that model how a horizontal beam deflects (a fancy way of saying "sags") under its own weight are always of the form

$$y = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0, \tag{1}$$

where x is the distance along the beam from one end (usually the left end as we look at it from the side), y is the amount of deflection, and c_4, c_3, c_2, c_1 and c_0 are constants determined by properties of the beam and how it is supported at its ends. We will consider the following ways of supporting a beam at its ends:

- **Embedded** - The end of the beam is held at a fixed angle (which we will always take to be horizontal) coming out of a wall.
- **Pinned** - Sometimes called **simply supported**. The end of the beam is held up on the end by a hinged joint that allows it to pivot at that point.
- **Free** - The end of the beam is not supported at all. In this case the other end must be embedded.

Give the Roman numeral of the form of equation (1) given below that models a ten foot horizontal beam with the given left-right end conditions. Graph each function using *Desmos* to help you do this, and enter $\{0 < x < 10\}$ after each equation to restrict the graph to zero to ten feet.

(a) pinned-pinned (b) embedded-embedded (c) embedded-free

I. $y = -0.001x^4 + 0.02x^3 - 0.1x^2$ II. $y = -0.0001x^4 + 0.004x^3 - 0.06x^2$

III. $y = -0.001x^4 + 0.02x^3 - x$

1.2 Derivatives and Differential Equations

Performance Criteria:

1. (c) Interpret derivatives in physical situations.
   (d) Find functions whose derivatives are given constant multiples of the original functions.

Scientists and engineers are usually concerned with the behavior of a **system**, which is a collection of physical objects. Some examples of physical systems that we saw in the previous section and will see again later include

- a mass on a spring, hanging from a "ceiling"
- a horizontal beam, supported somehow on one or both ends
- a tank or reservoir of liquid, with liquid added and removed over time

More complex examples of systems are electrical components and devices (including computers), heating and cooling systems, mechanical systems, and constructed things like roads, building structures, bridges and so on. (We will focus on systems of interest in mechanical, civil and electrical engineering, but things like biological and sociological systems can be modeled using differential equations as well!)

In the previous section's exercises you graphed the behaviors of some physical systems. Those graphs are *models* of the systems' behaviors; that is, they are human constructed descriptions of how the systems behave. Such graphical models are good for giving us an overall *qualitative* idea of the behavior of a system, but are generally inadequate if we would like to know precise values of the dependent variable based on a value (or values in the case of more than one) of the independent variable(s). When we desire such *quantitative* information, we attempt to develop an *analytical* model, which usually consists of an equation of a function.

Analytical models for physical situations can often be developed from various principles and laws of physics. The physical principles do not usually lead us directly to the functions that model physical situations, but to equations involving derivatives of those functions. This is because what we usually know is how our variables are changing in relationship with each other, and such change is described with derivatives. Equations containing derivatives are called **differential equations**. In this section we will review the concept of a derivative and see an example of a simple differential equation, along with how it arises.

When you hear the word "derivative," you may think of a process you learned in a first term calculus class. Throughout this course it will be important that you can carry out the process of "finding a derivative"; *if you need review or practice, see Appendix B*. In this section our concern is not the mechanics of finding derivatives, but instead we wish to recall what derivatives are and what they mean.

To reiterate what was said in the previous section, a function is just a quantity that depends on one or more other quantities, one in most cases that we will consider. Again, we refer to the first quantity (the function) as the dependent variable and the second quantity, that it depends on, is the independent variable. If we were to call the independent variable x and the dependent variable y, then you should recall the **Leibniz notation** $\dfrac{dy}{dx}$ for the derivative. This notation can be loosely interpreted as *change in y per unit of change in x*. Technically speaking, any derivative of a function is really the derivative of the dependent variable (which *IS* the function) *with respect to the the independent variable*. We sometimes use the notation y' instead of $\frac{dy}{dx}$. Obviously it is easier to write y', but that notation does not indicate what the independent variable is and it does not suggest a ratio, or rate.

Let's consider a couple examples of the meaning of the derivative in physical situations.

◇ **Example 1.2(a):** Suppose again that we take a mass hanging from a ceiling on a spring, lift it and let it go, and suppose the equation of motion is $y = 1.5\cos 5.2t$. The derivative of this function is $\frac{dy}{dt} = -7.8\sin 5.2t$, a new function of the independent variable. This function's value at any time t can be interpreted as how fast the the height of the mass is changing with respect to time, *at that particular time*. If the height units are inches and the time units seconds, then the units of the derivative are $\frac{\text{inches}}{\text{seconds}}$ = inches per second, indicating that the derivative of the function y at a given time is the velocity of the mass at that time. For example, the derivative at time 0.5 seconds is

$$\left.\frac{dy}{dt}\right|_{t=0.5} = y'(0.5) = -7.8\sin[(5.2)(0.5)] = -4.02 \text{ in/sec},$$

telling us that the mass is moving downward (indicated by the negative sign) at about four inches per second at one half second after being set in motion. **NOTE:** *Your calculator will need to be set in radians for all trigonometric computations in this course!*

◇ **Example 1.2(b):** Now recall the beam of Example 1.1(b), sticking out from a wall that it is embedded in. If x represents a horizontal position along the beam and y represents the deflection ("sag") of the beam at that horizontal position, then the derivative $\frac{dy}{dx}$ is the change in deflection per unit of horizontal change, which is just the slope of the beam at that particular point.

We'll now take a break from actual physical situations to ask some questions about derivatives, in a mathematical sense. After doing so, we'll see that such questions relate directly to certain "real-life" situations.

◇ **Example 1.2(c):** Find a function whose derivative is seven times the function itself.

Solution: Note that the derivative of $y = e^{kt}$, where k is a constant, is $y' = ke^{kt}$. This shows that exponential functions are essentially their own derivatives, with perhaps a constant multiplier. If k was seven, the original function would be $y = e^{7t}$ and the derivative would be $y' = 7e^{7t} = 7y$, seven times the original function y.

◇ **Example 1.2(d):** Find a function whose second derivative is sixteen times the function itself.

Solution: Here we should again be expecting an exponential function, but it will get multiplied twice because of the chain rule. Note that if $y = e^{4t}$, then $y' = 4e^{4t}$ and $y'' = 16e^{4t} = 16y$, so $y = e^{4t}$ is the function we are looking for. But in fact it is not *the* function, but only one such function. The function $y = e^{-4t}$ is another such function, as is $y = 5e^{-4t}$. (You should verify this last claim for yourself.) In fact, $y = Ce^{-4t}$ is a solution *for any value of* C. We will see later why this is, and what we do about it.

◇ **Example 1.2(e):** Find a function whose derivative is -16 times the function itself.

Solution: The previous example shows that the desired function is not an exponential function, as the only likely candidates were shown to have second derivatives that are *positive* sixteen times the original function. What we want to note here is that if we take the derivative of sine or cosine twice, we end up back at sine or cosine, respectively, but with opposite sign. However, each time we take the derivative of a sine or cosine of kx, the chain rule gives us a factor of k on the "outside" of the trig function. Thus we see that

$$y = \sin 4x \qquad \Longrightarrow \qquad y' = 4\cos 4x \qquad \Longrightarrow \qquad y'' = -16\sin 4x = -16y$$

$$y = \cos 4x \qquad \Longrightarrow \qquad y' = -4\sin 4x \qquad \Longrightarrow \qquad y'' = -16\cos 4x = -16y$$

This shows that $y = \sin 4x$ and $y = \cos 4x$ are functions whose derivatives are -16 times the original functions themselves.

Examples 1.2(c) and (d) show the importance of exponential functions in the study of derivatives and differential equations. Regarding Example 1.2(e), we can see that if $y = e^{4ix}$ where $i^2 = -1$,

$$y' = 4ie^{4ix} \qquad \Longrightarrow \qquad y'' = (4i)^2 e^{4ix} = 16i^2 e^{4ix} = -16e^{4ix} = -16y. \tag{1}$$

The same sort of computation would show that the second derivative of $y = e^{-4ix}$ would also be $-16y$. Later we will see that these two functions are "equivalent" to the sine and cosine, in some sense. The point, for now, is that the functions we are looking for are again exponential functions.

Consider Example 1.2(e) above. The words "the second derivative is -16 times the original function" can be written symbolically as

$$\frac{d^2y}{dx^2} = -16y, \tag{2}$$

since *the function is the dependent variable* y. This is a **differential equation**, an equation containing a derivative. (Differential equations can contain derivatives of any order. The **order of a differential equation** is the highest order derivative occurring in the differential equation, so this is a second order differential equation.) Any function that makes such an equation true is a **solution to the differential equation**, so Example 1.2(e) shows that both $y = \sin 4x$ and $y = \cos 4x$ are both solutions to the differential equation (2), and (1) shows that $y = e^{4ix}$ is as well. We will find later that every solution to (2) is a function of the form

$$y = C_1 \sin 4x + C_2 \cos 4x,$$

where C_1 and C_2 are constants that can be any numbers (and that must be able to be complex numbers to account for the fact that $y = e^{4ix}$ is a solution).

We now show that Example 1.2(e) and equation (2) are not just an exercise in understanding derivatives or a mathematical curiosity, but can arise from a physical situation. Suppose one end of a spring is attached to a ceiling, as shown in Figure 1.2(a) at the top of the next page. We then hang an object with mass m (we will refer to both the object itself *and* its mass as the "mass" - one must note from the context which we are talking about) on the spring, extending it by a length l to where the mass hangs in equilibrium. See Figure 1.2(b). There are two forces acting on the mass, a downward force of mg, where g is the acceleration due to gravity, and an upward force of kl, where k is the **spring constant**, a measure of how hard the spring "pulls back" when stretched. The spring constant is a property of the particular spring. When the mass hangs in equilibrium these two force are equal in magnitude to each other, but in opposite directions. This is expressed by $mg = kl$.

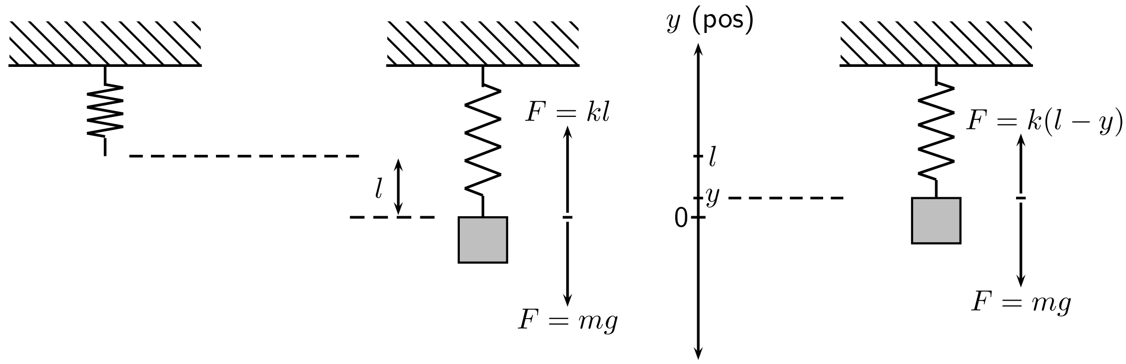


Figure 1.2(a) Figure 1.2(b) Figure 1.2(c)

We will put a coordinate system (with a scale in appropriate length units, like inches) beside the mass, with the zero at the point even with the top of the mass at rest and *with the positive direction being up*. If we then lift the mass up to a position y_0, where $y_0 < l$, and release it, it will oscillate up and down. If we assume (for now) that there is no resistance, it will oscillate between y_0 and $-y_0$ forever; as noted in the previous section, this is called **simple harmonic motion**. Consider the mass when it is at some position y in this oscillation, as shown in Figure 1.2(c) above. There will be an upward force of $k(l - y)$ due to the spring and a downward force of $-mg$ (the negative indicating downward) due to gravity. Remembering that force is mass times acceleration and that acceleration is the second derivative of position with respect to time, the net force is then

$$F = ma = m\frac{d^2y}{dt^2} = k(l - y) - mg = kl - ky - mg = -ky,$$

since $kl = mg$.

Extracting the equation $m\frac{d^2y}{dt^2} = -ky$ from the above and dividing both sides by m gives $\frac{d^2y}{dt^2} = -\frac{k}{m}y$. If the values of k and m are such that $\frac{k}{m} = 16$, this becomes $\frac{d^2y}{dt^2} = -16y$, the equation describing the situation from Example 1.2(e) (with the variable x replaced with t). Based on the discussion from the previous page, the equation that models the motion of the mass is

$$y = C_1 \sin 4t + C_2 \cos 4t. \tag{3}$$

This is to be expected, as we know the mass will oscillate up and down. The values C_1 and C_2 will depend on how the mass is set in motion (more on that in Section 1.4), but as long as $\frac{k}{m} = 16$ we will have a solution of the form (3). All of this shows that what seems like a whimsical mathematical question about derivatives (posed in Example 1.2(e)) is actually very relevant for a practical application.

Section 1.2 Exercises To Solutions

1. Find the derivative of each function without using your calculator. You *MAY* use the course formula sheet. Give your answers using correct derivative notation.

   (a) $y = 2\sin 3x$ (b) $y = 4e^{-0.5t}$ (c) $x = t^2 + 5t - 4$

   (d) $y = 3.4\cos(1.3t - 0.9)$ (e) $y = te^{-3t}$ (f) $x = 4e^{-2t}\sin(3t + 5)$

2. Find the second derivatives of the functions from parts (a) - (c) of Exercise 1. Give your answers using correct derivative notation.

3. The temperature T of an object (in degrees Fahrenheit) depends on time t, measured in minutes, and $\dfrac{dT}{dt} = 2.7$ when $t = 7$. $\left(\text{We sometimes write this as } \left.\dfrac{dT}{dt}\right|_{t=7} = 2.7\right)$ Interpret the derivative in a sentence, using either *increasing* or *decreasing*.

4. The amount A of salt in a tank depends on the time t. If A is measured in pounds and t is measured in minutes, interpret the fact that $\left.\dfrac{dA}{dt}\right|_{t=12.5} = -1.3$. Again, use *increasing* or *decreasing*.

5. The height of a mass on a spring at time t is given by y, where t is in seconds and y is in inches.

   (a) Interpret the fact that $\dfrac{dy}{dt} = -5$ when $t = 2$.

   (b) Interpret the fact that $\dfrac{d^2y}{dt^2} = 3$ when $t = 2$.

   (c) Based on the values of these two derivatives, is the mass speeding up or slowing down at time $t = 2$? Explain.

6. The number of bacteria in a test dish is denoted by N, and time t is measured in hours. Write a sentence interpreting the fact that $\dfrac{dN}{dt}$ is 430 when $t = 5.4$. Include one of the words *increasing* or *decreasing* in your answer.

7. For this exercise, consider the beam of Examples 1.1(b) and 1.2(b). *Note that deflection downward is generally considered positive for this situation!*

   (a) Will the value of the derivative $\dfrac{dy}{dx}$ be positive, or negative, for points x with $x > 0$?

   (b) Suppose that $0 \le x_1 < x_2 \le 10$. Which is greater, the *absolute value* of the derivative at x_1, or the absolute value of the derivative at x_2?

8. (a) Find a function $y(x)$ whose derivative is -3 times the original function. Is there more than one such function? If so, give another.

   (b) Find a function $y(t)$ whose second derivative is -9 times the original function. Is there more than one such function? If so, give another.

   (c) Find a function $x(t)$ whose second derivative is 9 times the original function. Is there more than one such function? If so, give another.

   (d) Find a function $y(x)$ whose second derivative is -5 times the original function. Is there more than one such function? If so, give another.

9. For each of the situations in Exercise 8, write a differential equation whose solution is the desired function. (See the second paragraph after Example 1.2(e).) *Use the given independent and dependent variables, and give your answers using Leibniz notation.*

1.3 Parameters and Variables

Performance Criterion:

1. (e) Identify parameters and variables in functions or differential equations.

If you have not recently read the explanation of the spring-mass system at the end of the last section, you should probably skim over it again before reading this section. Recall that for the spring-mass system, the independent variable is time and the dependent variable is the height of the mass. Assuming no resistance, once the mass is set in motion, it will exhibit periodic oscillation (simple harmonic motion). It should be intuitively clear that changing either the amount of the mass or the stiffness of the spring (expressed by the spring constant k) will change the period of oscillation. The mass m and the spring constant k are what we call **parameters**, and they should not be confused with the variables, which are time and the height of the mass. When working with real world mathematical models of physical systems, parameters will show up in three places:

- As characteristics of the physical systems themselves, quantified by numerical values.
- As constants within differential equations.
- As constants in the solutions to differential equations.

Let's illustrate these three manifestations of parameters using our spring-mass system. As mentioned above, the two physical parameters are the mass of the object hanging on the spring, and the stiffness of the spring, given by the spring constant. If the mass was $m = 0.5$ kg and the spring constant was $k = 8$ N/m (Newtons per meter) the differential equation would be

$$0.5\frac{d^2y}{dt^2} = -8y.$$

Here we see the two physical parameters, characteristics of the physical system, showing up in the differential equation. If we multiply both sides by two and subtract the right side from both sides we obtain

$$\frac{d^2y}{dt^2} + 16y = 0,$$

where the 16 is the new parameter $\frac{k}{m}$, which we often rename as ω^2. In this case $\omega^2 = 16\frac{1}{\sec^2}$. So the physical parameters k and m give us the parameter ω^2 in the differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0. \tag{1}$$

As stated in the previous section, the most general solution to this equation is $y = C_1 \sin\omega t + C_2\cos\omega t$. The variables are t and y, and C_1, C_2 and ω are parameters. ω determines the period of oscillation and C_1 and C_2 determine the amplitude and phase shift. The parameter ω depends on the mass and spring constant $\left(\omega = \sqrt{\dfrac{k}{m}}\right)$ and the parameters C_1 and C_2 depend on how the mass is set in motion, by what we will call **initial conditions**. In Sections 1.4 and 1.7 we will see what initial conditions are and how they are used to determine C_1 and C_2.

We now consider the horizontal beam of Example 1.1(b). One might guess that some parameters that determine the amount of deflection of the beam would be the material the beam is made of, the thickness and cross-sectional shape of the beam (square, "I-beam," etc.), the length of the beam, and perhaps other things.

◇ **Example 1.3(a):** The differential equation, and its solution, for the beam of Example 1.1(b) are

$$EI\frac{d^4y}{dx^4} = w \qquad \text{and} \qquad y = \frac{w}{24EI}x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

where E is Young's modulus of elasticity of the material the beam is made of, I is the cross-sectional moment of inertia of the beam about the "neutral axis," and w is the weight per unit of length. Give the variables and parameters for both the differential equation and the solution.

Solution: We can see from the derivative in the differential equation that the independent variable is x and the dependent variable is y. The remaining letters all represent parameters: the modulus of elasticity E, the cross-sectional moment of inertia I, and the weight per unit of length w. In the solution we see these parameters again, combined as the single parameter $\frac{w}{24EI}$, along with four others, c_0, c_1, c_2 and c_3.

The last four parameters c_0, c_1, c_2, and c_3 in the solution will depend on the length of the beam and how it is supported, in this case by being embedded in the wall at its left end and having no support at the right end. These things are what are called **boundary conditions**. We'll discuss them more in the next section and Section 1.7, and look at specific applications involving boundary conditions in Chapter 5.

In summary, the physical parameters are variables that can change from situation to situation, but once the situation is determined the values of those parameters are constant. At that point, the only things that change are the variables. The physical parameters then show up alone, or with each other, as parameters in the differential equation modeling the physical situation. Finally, the solution to the differential equation will be some familiar type of function like an exponential function, trigonometric function or polynomial function, with its exact behavior determined by parameters that are dependent on the parameters in the differential equation and the initial or boundary conditions.

NOTE: *In this course we will never again refer to parameters as variables, and we will consider them distinct from the variables of interest.*

Section 1.3 Exercises

To Solutions

1. As mentioned previously, when a solid object with some initial temperature T_0 is placed in a medium (like air or water) with a constant temperature T_m, the object's temperature T will approach T_m as time goes on. The rate at which the temperature of the object changes is proportional to the difference between its temperature T and the temperature T_m of the medium, giving us the differential equation

$$\frac{dT}{dt} = k(T_m - T),$$

where k is a constant dependent on the material the object is made from.

   (a) Keeping in mind that parameters are quantities that vary from situation to situation but do not change once the situation is fixed, give all of the parameters.

   (b) Give the independent variable(s).

   (c) Give the dependent variable.

2. Suppose that a mass on a spring hangs motionless in its equilibrium position. At some time zero it is set in motion by pulling it downward and simply releasing it, and suppose also that the mass is hanging in an oil bath that resists its motion. The independent variable is time, and the dependent variable is the height of the mass. Give as many physical parameters as you can think of for this situation - there are three or four that occur to me.

3. When dealing with certain electrical circuits we obtain the differential equation and solution

$$L\frac{di}{dt} + Ri = E \qquad \text{and} \qquad i = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right)e^{-\frac{R}{L}t}.$$

Give the independent variable, dependent variable, and all the parameters.

4. At some time a guitar string is plucked, and the dependent variable that we are interested in is the displacement of the string from its initial position.

   (a) What is(are) the independent variable(s)?

   (b) What are some physical parameters of importance?

5. Recall the situation of Example 1.1(d): A tank containing 100 gallons of water with 10 pounds of salt dissolved in the water. At some time we begin pumping a 0.3 pounds salt per gallon (of water) solution into the tank at two gallons per minute, mixing it thoroughly with the solution in the tank. At the same time the solution in the tank is being drained out at two gallons per minute as well. Our interest is the amount, in pounds, of salt in the tank at any time.

   (a) What are the independent and dependent variables, in that order?

   (b) What are the parameters?

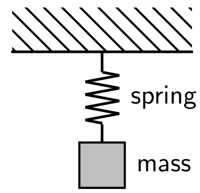
1.4 Initial Conditions and Boundary Conditions

Performance Criterion:

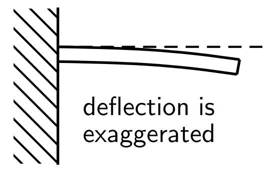
1. (f) Identify initial value problems and boundary value problems. Determine initial or boundary conditions.

Recall Examples 1.1(a) and 1.1(b):

◇ **Example 1.1(a):** Suppose that a mass is hanging on a spring that is attached to a ceiling, as shown to the right. If we lift the mass, or pull it down, and let it go, it will begin to oscillate up and down. The height y of the mass (relative to some fixed reference, like its height before we lifted it or pulled it down) is a function of the time t that has elapsed since we set the mass in motion.



◇ **Example 1.1(b):** Consider a beam that extends horizontally ten feet out from the side of a building, as shown to the right. The beam will deflect (sag) some, with the distance below horizontal being greater the farther out on the beam one looks.



The differential equations modeling these two situations are

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \qquad \text{and} \qquad EI\frac{d^4y}{dx^4} = w,$$

where k, m, E, I and w are physical parameters, as described in the previous section. We've already seen in Example 1.2(d) that a differential equation can have infinitely many different solutions, all of which are obtained by varying one (or more) constants. In this case, the most general solutions to the above two differential equations are

$$y = C_1 \sin\omega t + C_2 \cos\omega t \qquad \text{and} \qquad y = \frac{w}{24EI}x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

where C_1, C_2, c_3, c_2, c_1 and c_0 are arbitrary (meaning they can have any values) constants *differing from, and not depending on, the parameters* k, m, E, I *and* w. (Remember that we are case sensitive in mathematics, science and engineering, so C_1 and c_1 are not necessarily the same value.) *Note that the number of such arbitrary constants in the solution of a differential equation is equal to the order of the differential equation.* (Again, the **order of a differential equation** is the highest order derivative in the differential equation - more on this in Section 1.6.)

The height of the mass at any time t depends on the amount of the mass and the "stiffness" of the spring (given quantitatively by the spring constant k), but it also depends on how we set the mass in motion. It can be lifted or pulled down and let go, or given a blow, or some combination of these things. The combination that sets it in motion are what are called **initial conditions**. Suppose that we set the mass in motion by simply pulling it down by two units and then letting it go. Calling the moment we let it go time zero, we would have that

$$y = -2 \qquad \text{and} \qquad \frac{dy}{dt} = 0 \qquad \text{when} \qquad t = 0.$$

(Remember that the derivative is the velocity, so the second statement says that the mass has zero velocity at the moment we let it go.) Using function notation and the fact that the derivative is the function y', this is usually expressed by

$$y(0) = -2, \quad y'(0) = 0.$$

The two numbers -2 and 0 are called **initial values**, a term we will use interchangeably with initial conditions, even though the concepts are slightly different. We will see later how these two pieces of information can be used to determine the values of the constants C_1 and C_2 in the solution $y = C_1 \sin \omega t + C_2 \cos \omega t$.

Let's now think about the horizontal beam. The independent variable is x, the horizontal distance along the beam, measured from the wall. Time is not a variable at all; the beam deflects immediately when put into place, then retains its displacement from then on. The deflection, though, is dependent on what is going on at the two ends of the beam. At the left end the beam is what we call **embedded**. The effect of this is two things: the displacement of the beam is zero at that point, and the slope of the beam is zero right where it leaves the wall. We can express these two things by

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0,$$

which are **boundary conditions**. The right end of the beam is "free," which is described mathematically by the boundary conditions

$$y''(10) = 0 \qquad \text{and} \qquad y'''(10) = 0.$$

We'll discuss the origin of these two conditions a bit more in Chapter 5. Altogether we have *four* boundary conditions

$$y(0) = 0, \quad y'(0) = 0, \quad y''(10) = 0, \quad y'''(10) = 0$$

which allow us to determine the four constants c_3, c_2, c_1 and c_0 in the solution

$$y = \frac{w}{24EI}x^4 + c_3x^3 + c_2x^2 + c_1x + c_0.$$

The numerical values of zero for all these derivatives at $x = 0$ and $x = 10$ are **boundary values**. As with initial values/initial conditions, we will blur the distinction between boundary values and boundary conditions. Let's now look at some more examples of initial and boundary conditions.

◇ **Example 1.4(a):** Consider the mass on the spring, set in motion by lifting it one inch and letting it go. Give the height and velocity of the mass at the time it is let go, using function notation.

Solution: Taking up to be positive, at time zero (the moment we set the mass in motion) the height of the mass is one inch, so we write $y(0) = 1$. Since we simply release the mass at time zero, the velocity at time zero is zero. Recalling that velocity is the first derivative of position, we can describe this by $y'(0) = 0$. The initial conditions are then $y(0) = 1$, $y'(0) = 0$.

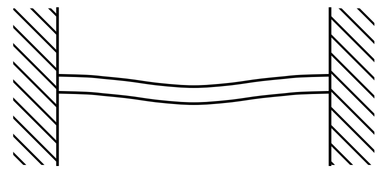
◇ **Example 1.4(b):** Consider the mass on the spring, this time setting it in motion by hitting it downward at three inches per second from its position at rest. Give the initial conditions for the height function y.

Solution: Because we are forcing the mass from its position at rest, its initial height is zero. This is given using function notation by $y(0) = 0$. The fact that it has downward velocity of three inches per second at time zero gives us the initial condition $y'(0) = -3$.

◇ **Example 1.4(c):** Suppose that the mass is set in motion by pulling it down two inches, then giving it an upward velocity of five inches per second to begin. Give the initial conditions for the height function y.

Solution: The initial conditions are $y(0) = -2$ and $y'(0) = 5$.

◇ **Example 1.4(d):** Consider a twenty foot beam that is embedded in walls at both ends, as shown to the right. The beam will deflect downward some in the middle; the deflection is exaggerated in the picture. Give the boundary conditions for the beam.



Solution: The boundary conditions are $y(0) = 0$, $y'(0) = 0$, $y(20) = 0$ and $y'(20) = 0$.

This last example warrants a bit more thought. The independent variable is the distance along the beam (most likely from the left wall) and the dependent variable is the amount of deflection downward. (Again, standard convention for this sort of problem is that down is positive.) The shape the beam takes, given by the deflections at all points, depends on the material the beam is made of, the design (cross section) of the beam, and the way that the beam is supported at the ends. One might then think that the type of support would be a parameter, but it is not. The type of support can be expressed as values of the dependent variable and some of its derivatives, and that is what distinguishes the support (boundary conditions or values) from the parameters. Similarly, the behavior of a mass on a spring is dictated in part by how it is set in motion, but that can be described by values of the dependent variable and its first derivative. Thus the way the mass is set in motion is given by initial conditions/values rather than parameters.

We conclude this section with the following remarks:

- Situations in which time is the independent variable will have initial conditions.
- Situations in which position along a line is the independent variable will have boundary conditions.
- Situations where a function depends on both position and time will have both initial conditions *and* boundary conditions. We will not see these, because they are described by **partial differential equations**.
- Partial differential equations are also required when working with boundary conditions only, when the function of interest is a function of more than one space variable. Such functions would arise when dealing with sheets or solids, rather than beams, which can be thought of as one-dimensional lines.

We will work primarily with initial conditions, but you will see boundary conditions later in the course (Section 1.7 and Chapter 5).

Section 1.4 Exercises

To Solutions

1. In each of the following, the independent variable is given for a situation (the dependent variable should be clear), along with initial or boundary conditions, in function form. For each, give every initial or boundary condition in the form "$variable = number$ when $variable = number$."

   (a) independent variable x, $\quad y(0) = 7, \quad y'(0) = -3$

   (b) independent variable t, $\quad x(0) = 1, \quad x'(0) = 5$

   (c) independent variable x, $\quad y(0) = 0, \quad y''(0) = 0, \quad y(15) = 0, \; y'(0) = 0.$

2. For each of the following, give the initial conditions for a mass on a spring that is set in motion in the way described. *Give the conditions using function notation, as done in Examples 1.4(a), (b) and (c).* Let the dependent variable in each case be y.

   (a) The mass is pulled down five units and let go with no initial velocity.

   (b) The mass is not displaced, but it is given an downward velocity of two units per second.

   (c) The mass is lifted by one unit and given an upward velocity of two units per second.

   (d) The mass is pulled down by three units and given an upward velocity of one unit per second.

3. If a mass on a spring is set in motion are there is no resistance to its vibration, it will oscillate in the same manner forever. (Resistance to its motion we will call **damping**, and we'll study its effect in Chapters 3 and 4.) Assuming such conditions, sketch the graph of the displacement of the mass at any time t for each of the sets of initial conditions listed in Exercise 2. Extend your graph far enough to show at least two full periods. You will not be able to label a scale on the horizontal axis, but *for three of the cases you should be able to label the vertical axis with a scale. Take care to make sure that the graph has the correct slope where it leaves the vertical axis.*

4. There is one other condition (besides embedded or free) we'll see at the end of a beam, called **simply supported** or **pinned**. This means that the end is supported but allowed to pivot freely, as shown in the diagram below and to the right. In that case the deflection at the end is zero, and the *second* derivative of deflection is zero there as well. For each of the following scenarios, give the boundary conditions for the beam, assuming a dependent variable of y.

   (a) A 20 foot beam that is simply supported at its left end and embedded at its right end.

   (b) A 12 foot beam that is simply supported at both ends.

5. Suppose that we have a 70 centimeter metal rod that is perfectly insulated along the length of the rod, so that no heat can enter or leave along its length, but heat *CAN* enter or leave at its ends. We then put the rod horizontally in front of us and consider a coordinate system that puts zero at the left end of the rod and 70 cm at the right end, and we let $u(x)$ represent the temperature at any point x along the length of the rod, using our coordinate system.

   Suppose also that we hold an ice cube (temperature 32° Fahrenheit at the left end and a hair dryer blowing 115° F air on the right end. Because the independent variable is a space variable x, this situation has boundary conditions. Give them, using function notation.

1.5 Differential Equations and Their Solutions

Performance Criteria:

1. (g) Determine the independent and dependent variables for a given differential equation.

   (h) Determine whether a function is a solution to an ordinary differential equation (ODE); determine values of constants for which a function is a solution to an ODE.

An equation that contains one or more derivatives is called a **differential equation**. Here are some examples that we will be considering:

Equation 1: $\dfrac{dy}{dx} + 3y = 0$ **Equation 2:** $y'' + 3y' + 2y = 0$

Equation 3: $y'' + 9y = 26e^{-2t}$ **Equation 4:** $15.3\dfrac{d^4y}{dx^4} = 1.4$

Equation 5: $\dfrac{dy}{dx} = \dfrac{x}{y}$ **Equation 6:** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial u}{\partial t}$

Note that equations 1, 2, 3 and 5 contain not only derivatives of the function y, but the function itself as well. (We can really think of the function as the "zeroth" derivative.)

The first five of these equations are all **ordinary differential equations**, meaning that they contain "ordinary" derivatives, which are appropriate when there is only one independent variable. The last one contains partial derivatives (which are written with the symbol ∂ instead of d) and is called a **partial differential equation**. (Some of you may have not yet taken a course in which you learn about partial derivatives.) We often use the abbreviations ODE for ordinary differential equation and PDE for partial differential equation. Video Discussion, 0:00 to 2:00

The **order** of a differential equation is the order of the highest derivative in the equation. Equations 1 and 5 above are first order, Equations 2, 3 and 6 are second order, and Equation 4 is fourth order. In this course we will focus almost entirely on ordinary differential equations, and most of the equations we will work with will be first or second order. Video Discussion, 2:00 to 3:40

When looking at a differential equation, it is often possible to determine the independent and dependent variables of interest. Derivatives are always of the dependent variable, and with respect to the independent variable (or one of the independent variables in the case of a function of more than one variable). So for Equation 1, the dependent variable is y and the independent variable is x.

◇ **Example 1.5(a):** Give the dependent and independent variables for the rest of the equations.

Solution: For Equations 4 and 5 the dependent variable is y and the independent variable is x. For equation 3 the dependent variable is y, and since the derivative is an ordinary derivative there must be only one independent variable, and it has to be t, the only other variable visible in the equation. The dependent variable in Equation 2 is y, and it is not possible to determine the independent variable in that case. Lastly, u is the dependent variable in Equation 6. There are three independent variables, x, y and t, which is why partial derivatives are required. *Any situation with more than one independent variable will result in a partial differential equation.*

Is $x = 5$ a solution to $4x - 2 = 10$? One way to answer this question is to substitute five for x in the left hand side of the equation and see if it simplifies to become the right hand side. If it does, then five is a solution to the equation:

$$4(5) - 2 = 20 - 2 = 18 \neq 10, \text{ so } x = 5 \text{ is not a solution}$$

On the other hand, $x = 3$ is a solution to $4x - 2 = 10$:

$$4(3) - 2 = 12 - 2 = 10, \text{ so } x = 3 \text{ is a solution}$$

What the above shows us is that a solution to an *algebraic* equation is a number that, when substituted for the unknown value, makes the equation true. We should recall that some equations have more than one solution. For example, both 3 and -3 are solutions to the equation $x^2 - 9 = 0$.

In the case of a differential equation, a solution to the equation is *NOT* a number, it is a function.

> **Solution to a Differential Equation**
>
> A **solution to a differential equation** is a function for which the function and its relevant derivatives can be substituted into the equation to obtain a true statement.

There are some differential equations whose solutions are *relations* rather than functions; we'll solve a few of those, but for all of the applications we will consider, the solutions to the ODEs modeling the situations will be functions.

When asked to verify that, or determine whether, a function is a solution to an ODE, you need to show some work supporting whatever your conclusion is. The following example shows one way to do this.

◇ **Example 1.5(b):** Show that $y = 5\cos 4t$ is a solution to $\dfrac{d^2y}{dt^2} = -16y$.

Solution: We compute the left hand side (LHS) and right hand side (RHS) separately:

$$\frac{dy}{dt} = -20\sin 4t \quad \Longrightarrow \quad \text{LHS} = \frac{d^2y}{dt^2} = -80\cos 4t$$

$$\text{RHS} = -16(5\cos 4t) = -80\cos 4t$$

Because LHS = RHS, $y = 5\cos 4t$ is a solution to $\dfrac{d^2y}{dt^2} = -16y$.

For the above example the left hand side was just one derivative. When the left hand side is more complicated, a standard method of verifying a solution is to first calculate any derivatives that appear on the left hand side of the equation, then substitute them into the left hand side. If the right hand side is fairly simple, we might be able to simplify the left side directly to the right hand side, as done in the next example.

◇ **Example 1.5(c):** Determine whether $y = Ce^{-2t}$, where C is any constant, is a solution to the differential equation $y'' + 3y' + 2y = 0$. Another Example

Solution: First we see that $y' = Ce^{-2t}(-2) = -2Ce^{-2t}$ and $y'' = -2Ce^{-2t}(-2) = 4Ce^{-2t}$, so

$$\text{LHS} = 4Ce^{-2t} + 3(-2Ce^{-2t}) + 2(Ce^{-2t}) = 4Ce^{-2t} - 6Ce^{-2t} + 2Ce^{-2t} = 0 = \text{RHS}.$$

Therefore $y = Ce^{-2t}$ is a solution to $y'' + 3y' + 2y = 0$.

This last example shows that a differential equation can have an infinite number of solutions (since C can be any real number), and we'll see the same thing in the next example as well.

◇ **Example 1.5(d):** Verify that $y = C_1 \sin 3t + C_2 \cos 3t$, where C_1 and C_2 are any constants, is a solution to $y'' + 9y = 0$.

Solution: First we see that $y' = 3C_1 \cos 3t - 3C_2 \sin 3t$ and $y'' = -9C_1 \sin 3t - 9C_2 \cos 3t$. Therefore

$$\text{LHS} = (-9C_1 \sin 3t - 9C_2 \cos 3t) + 9(C_1 \sin 3t + C_2 \cos 3t) = 0 = \text{RHS},$$

so $y = C_1 \sin 3t + C_2 \cos 3t$ is a solution to $y'' + 9y = 0$.

In this last example the function $y = C_1 \sin 3t + C_2 \cos 3t$ is a solution regardless of the values of the parameters C_1 and C_2. Because C_1 and C_2 can take any values, we say they are *arbitrary* constants. We will often use the lower case c and upper case C for arbitrary constants, sometimes with subscripts like above. We call all the functions obtained by letting the constants take different values a **family of solutions** for the differential equation. The solution to every first order equation will contain a constant that can take on infinitely many values, and solutions to second order equations contain two arbitrary constants, as in the above example. This may seem to contradict the result of Example 1.5(c), but the most general solution in that case is $y = C_1 e^{-2t} + C_2 e^{-t}$; the solution verified in that example is for the case in which $C_2 = 0$. The fact that C_1 and C_2 are subscripted differently means that they are probably, but not necessarily, different constants. Other letters will occasionally be used as constants.

In this next example you will see a situation where a function is a solution only when the parameter takes a certain value; in this case the constant (parameter) is *NOT* arbitrary.

◇ **Example 1.5(e):** Determine any values of C for which $y = Ce^{-2t}$ is a solution to the differential equation $y'' + 9y = 26e^{-2t}$.

Solution: The derivatives of the given function are $y' = -2Ce^{-2t}$ and $y'' = 4Ce^{-2t}$. Substituting the second derivative into the left hand side of the ODE gives

$$\text{LHS} = 4Ce^{-2t} + 9Ce^{-2t} = 13Ce^{-2t}.$$

$y = Ce^{-2t}$ is a solution only if LHS = RHS, which requires that $13C = 26$. Therefore $y = Ce^{-2t}$ is a solution to the differential equation $y'' + 9y = 26e^{-2t}$ only when $C = 2$.

The equation $y'' + 9y = 0$ is what we will call the **homogenous equation** associated with the equation $y'' + 9y = 26e^{-2t}$. (More on this later.) $y'' + 9y = 0$ is a second order homogenous equation, and Example 1.5(d) shows that the solution to the second order homogenous equation has not one, but two, arbitrary constants. The function $y = 2e^{-3t}$ is what we call a **particular solution** to the **non-homogenous equation** $y'' + 9y = 26e^{-2t}$. A particular solution is one for which the values of constants *are not* arbitrary: The constant in this case *must* be two.

A family of solutions that has one arbitrary constant, like the family from Example 1.5(c), is often referred to as a **one-parameter family of solutions**. The parameter is the constant C. The family $y = C_1 \sin 3t + C_2 \cos 3t$ from Example 1.5(d) is a **two-parameter family of solutions**, with the parameters being C_1 and C_2. Solutions containing all possible arbitrary constants will be called **general solutions**.

This section has contained a lot of information! Let's summarize the important points:

- A **solution to a differential equation** is a function for which the function and its relevant derivatives can be substituted into the equation to obtain a true statement.
- Solutions to first order differential equations contain one arbitrary constant, and solutions to second order differential equations contain two arbitrary constants. All the solutions obtained by letting constants take all possible values are called **families of solutions**.
- A solution to a differential equation that contains constants that are not arbitrary is called a **particular solution** to the differential equation.
- A family that encompasses all possible solutions of a differential equation is called a **general solution** to the differential equation.
- General solutions to first order differential equations contain one arbitrary constant, and general solutions to second order differential equations contain two arbitrary constants. All the solutions obtained by letting constants take all possible values are called **families of solutions**.

In Section 1.7 we will see that if we have initial or boundary conditions along with a differential equation, the values of the arbitrary constants can be determined.

Section 1.5 Exercises

To Solutions

1. For each of the following differential equations, determine the independent and dependent variables *when possible*. (You should always be able to identify the dependent variable.)

   (a) $\dfrac{dy}{dx} - 2y = 0$ (b) $y'' - y = 0$ (c) $\dfrac{\partial^2 u}{\partial t^2} = 3\left(\dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \dfrac{\partial^2 u}{\partial x_3^2}\right)$

   (d) $y'' + 9y = 26e^{-2t}$ (e) $\dfrac{\partial u}{\partial t} = 0.5\dfrac{\partial^2 u}{\partial x^2}$ (f) $\dfrac{d^2 x}{dt^2} - 5\dfrac{dx}{dt} + 6x = 10 \sin t$

   (g) $L\dfrac{d^2 u}{dx^2} + g \sin u = 0$ (h) $EI\dfrac{d^4 y}{dx^4} = w$ (i) $\dfrac{\partial^2 u}{\partial t^2} - c^2\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial u}{\partial r}\right) = 0$

2. (a) For part (b) of the previous exercise you should not have been able to identify the independent variable. Given that the solution is $y = 3e^x - 5e^{-x}$, what is the independent variable?

   (b) The differential equation $y'' - 6y' + 9y = 0$ has solution $y = C_1 e^{3t} + C_2 te^{3t}$. What are the independent and dependent variables?

3. Is $y = \sin 2t$ a solution to $\dfrac{dy}{dt} + 2y = 0$?

4. Is $y = 3e^{x} - 5e^{-x}$ a solution to $y'' - y = 0$?

5. (a) Verify that $y = -5e^{-3x}$ is a solution to $\dfrac{dy}{dx} + 3y = 0$.

   (b) Verify that $y = Ce^{-3x}$, where C is any constant, is a solution to the same differential equation.

6. (a) Verify that $y = 2e^{-2t}$ is a solution to the differential equation $y'' + 9y = 26e^{-2t}$ and show that $y = 3e^{-2t}$ is *NOT* a solution to the same differential equation.

   (b) Verify that $y = C_1 \sin(3t) + C_2 \cos(3t) + 2e^{-2t}$ is a solution to the differential equation $y'' + 9y = 26e^{-2t}$.

7. Determine values of constants A and B for which $x = A \sin t + B \cos t$ is a solution to the differential equation $\dfrac{d^2x}{dt^2} - 7\dfrac{dx}{dt} + 10x = 8 \sin t$.

8. Consider the differential equation $y'' - 6y' + 9y = 0$.

   (a) Verify that $y = ce^{3t}$ is a solution to the differential equation.

   (b) Verify that $y = cte^{3t}$ is a solution to the differential equation.

9. Consider the differential equation $\dfrac{dy}{dx} - y = 4e^{3x}$.

   (a) Is there any value of c for which $y = ce^{x}$ a solution to the equation? If so, what is the value?

   (b) Is there any value of c for which $y = ce^{3x}$ a solution to the equation? If so, what is the value?

   (c) Recall that a solution to a differential equation that cannot have an arbitrary constant in it is called a **particular solution** to the equation. Give a particular solution to the differential equation $\dfrac{dy}{dx} - y = 4e^{3x}$.

   (d) Is there any value of c for which $y = ce^{x}$ a solution to the equation $\dfrac{dy}{dx} - y = 0$? If so, what is the value?

10. Consider the ODE $y'' + 3y' + 2y = 0$. For this exercise the independent variable is t, not x!

    (a) Because exponential functions are their own derivatives, it is conceivable that $y = e^{rt}$ is a solution for some *constant* value of r. Substitute it into the ODE.

    (b) You should be able to factor e^{rt} out of the left side of your result from (a). Now e^{u} is not zero for *any* value of u. What does this imply about our situation? (The answer to this is an equation!)

    (c) Your answer to (b) should be an equation. Solve it to determine what value(s) r must have in order for $y = e^{rt}$ to be a solution. Write the solution(s) *to the ODE.*

(d) You now should have at least one solution to the ODE. Write down all the solutions you have found and check each.

11. Repeat Exercise 10 for the ODE $2y'' + 3y' + y = 0$.

12. (a) Determine whether $x = e^{-3t}\sin 2t$ is a solution to the ODE $x'' + 4x' + 13x = 0$. *Note that you will need the product rule when taking derivatives.*

    (b) Determine whether $x = e^{-2t}\cos 3t$ is a solution to $x'' + 4x' + 13x = 0$.

13. An equation of the form $ax^2y'' + bxy' + cy = 0$ is called an Euler equation. (Euler is pronounced "oiler.")

    (a) Determine whether any of $y = x$, $y = x^2$, $y = x^3$ is a solution to the Euler equation $x^2y'' - 3xy' + 4y = 0$.

    (b) Another Euler equation is $4x^2y'' + 4xy' - y = 0$. Show that $y = C_1x^{\frac{1}{2}} + C_2x^{-\frac{1}{2}}$ is a solution to this equation.

    (c) Assume that a solution to the Euler equation $x^2y'' + 4xy' + 2y = 0$ has the form $y = x^r$, for some constant r. Substitute into the equation and do a bit of algebra to determine any values of r for which $y = x^r$ is in fact a solution.

1.6 Classification of Differential Equations

Performance Criteria:

1. (i) Classify differential equations as ordinary or partial; classify ordinary differential equations as linear or non-linear. Give the order of a differential equation.

   (j) Identify the functions $a_0(x)$, $a_1(x)$, ..., $a_n(x)$ and $f(x)$ for a linear ordinary differential equation. Classify linear ordinary differential equations as homogenous or non-homogeneous.

   (k) Write a first order ordinary differential equation in the form $\dfrac{dy}{dx} = F(x,y)$ and identify the function F. Classify first-order ordinary differential equations as separable or autonomous.

There are many different classifications and types of differential equations; we will focus on just a few classifications here. Let's consider the following examples, most of which we saw in the previous section.

Equation 1: $\dfrac{dy}{dx} + 3y = 0$ **Equation 2:** $x^2y'' + xy' + x^2y = 0$

Equation 3: $y'' + 9y = 26e^{-2t}$ **Equation 4:** $15.3\dfrac{d^4y}{dx^4} = 1.4$

Equation 5: $\dfrac{dy}{dx} = \dfrac{x}{y}$ **Equation 6:** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial u}{\partial t}$

Here are the classifications we'll be interested in: Video Discussion

- **Ordinary differential equations** (ODEs) versus **partial differential equations** (PDEs). We have already discussed this; Equations 1 - 5 are ODEs and Equation 6 is a PDE. It is worth mentioning here that a solution to an ODE is a function of just one variable, whereas a solution to a PDE is a function of more than one variable. The solution to Equation 6 is a function u of the three variables x, y and t.

- Differential equations are classified by **order**, which is the highest derivative occurring in the equation. Equations 1 and 5 are first order, Equations 2, 3 and 6 are second order (PDEs are classified by order the same way that ODEs are), and Equation 4 is fourth order.

- An ODE that can be written in the form
$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \tag{1}$$
where $a_0(x), ..., a_n(x)$ are functions of x (possibly constants), is called a **linear** ordinary differential equation. Equations 1 through 4 are linear ODEs: Another Video Discussion

  - Equation 1 is first order linear, with $a_1(x) = 1$, $a_0(x) = 3$ and $f(x) = 0$.
  - Equation 2 is second order linear, with $a_2(x) = x^2$, $a_1(x) = x$, $a_0(x) = x^2$ and $f(x) = 0$. This particular equation is known as a Bessel equation of order zero (where "order" does not refer to the order of the ODE - how confusing!). It is obtained when working with a PDE called the **wave equation**, used for things like modeling the vibration of a drumhead.

- Equation 3 is second order linear with $a_2(t) = 1$, $a_1(t) = 0$, $a_0(t) = 9$ and $f(t) = 26e^{-2t}$. Note the variable is t, rather than x, because the independent variable in this case is t.
- Equation 4 is fourth order linear with $a_4(x) = 15.3$, $a_3(x) = a_2(x) = a_1(x) = a_0(x) = 0$ and $f(x) = 1.4$.

- A linear equation, so an ODE of the form (1) above, is called **homogeneous** if $f(x) = 0$. Equations 1 and 2 are homogeneous, Equations 3 and 4 are **non-homogeneous**. *One must be a bit careful, because there is another meaning of homogenous associated with ODEs!* The difference between the two uses must be determined by the context in which they are used. The definition just given is the only one we'll be using.

- If we multiply both sides of Equation 5 by y we get $y\dfrac{dy}{dx} = x$, which is not of the form (1). Any effort to get the coefficient of $\dfrac{dy}{dx}$ to be a function of x will fail, so Equation 5 is **non-linear**. Note that if the original equation had instead been $\dfrac{dy}{dx} = \dfrac{y}{x}$, we could multiply both sides by x and subtract y to get $x\dfrac{dy}{dx} - y = 0$. This equation *IS* linear, with $a_1(x) = x$, $a_0(x) = -1$ and $f(x) = 0$.

Suppose that we have a *first* order ODE with independent variable x and dependent variable y. Such an equation can always be written in the form $\dfrac{dy}{dx} = F(x, y)$, where F is simply a function of the two variables x and y. Consider for example the Equation B below; it can be written as $\dfrac{dy}{dx} = x + 2xy$, so $F(x, y) = x + 2xy$ for that equation.

A. $\dfrac{dy}{dx} - \dfrac{x}{y} = 0$ B. $y' - 2xy = x$ C. $y' + 2y = y^2$ D. $5\dfrac{dy}{dx} - 3y = \sin x$

◇ **Example 1.6(a):** Determine the functions $F(x, y)$ for Equations A, C and D above.

Solution: Each of the equations can be solved for $\dfrac{dy}{dx}$ to get

A. $\dfrac{dy}{dx} = \dfrac{x}{y}$ C. $\dfrac{dy}{dx} = y^2 - 2y$ D. $\dfrac{dy}{dx} = \dfrac{3}{5}y + \dfrac{1}{5}\sin x$

We can see now that the functions F for the three equations are

A. $F(x, y) = \dfrac{x}{y}$ C. $F(x, y) = y^2 - 2y$ D. $F(x, y) = \dfrac{3}{5}y + \dfrac{1}{5}\sin x$

We can now define two other categories of first order differential equations.

- When F is the product of a function of x and a function of y, written compactly as $F(x, y) = g(x)h(y)$, the ODE is called **separable**.
- When F is really just a function of y (so $F(x, y) = f(y)$) the ODE is called **autonomous**. Note that by letting $g(x) = 1$, any autonomous equation is also separable (but not vice-versa!).

◇ **Example 1.6(b):** Determine whether any of the the equations

A. $\dfrac{dy}{dx} - \dfrac{x}{y} = 0$ B. $y' - 2xy = x$ C. $y' + 2y = y^2$ D. $5\dfrac{dy}{dx} - 3y = \sin x$

are separable or autonomous.

Solution: The only one of the equations that can be written in the form $\dfrac{dy}{dx} = f(y)$ is C, so it is autonomous (and therefore separable as well). For Equation A, $F(x, y) = \dfrac{x}{y} = x \cdot \dfrac{1}{y}$, so it is separable, with $g(x) = x$ and $h(y) = \dfrac{1}{y}$. For Equation B, $F(x, y) = x + 2xy = x(1 + 2y)$, so it is also separable, with $g(x) = x$ again and $h(y) = 1 + 2y$. In the case of Equation D $F(x, y) = \frac{3}{5}y + \frac{1}{5}\sin x$, which is clearly not just a function of y, so it is not autonomous. We also see that it is not possible to write $F(x, y)$ in the form $g(x)h(y)$ either, so it is also not separable.

We will see the significance of separable and autonomous equations later. For now we should note a bit of algebra that can be performed with a separable equation. To begin with, we need to think of the derivative $\dfrac{dy}{dx}$ as being the quotient of the two **differentials** dy and dx. Treating each like we would a variable, when we are working with a separable equation we can get all the x "stuff" on one side of the equation and the y "stuff" on the other side:

$$\begin{aligned}
\frac{dy}{dx} - \frac{x}{y} &= 0 \\
\frac{dy}{dx} &= \frac{x}{y} \\
dy &= \frac{x}{y}\,dx \\
y\,dy &= x\,dx
\end{aligned}$$

Separable equations are often easy to find solutions for, as we'll do in Chapter 2, and computations like the above will be part of the process.

Section 1.6 Exercises To Solutions

1. List the letters of all the following that are ordinary differential equations. Assume that any letters not used in derivatives represent constants except $w(x)$ is a function of x.

(a) $\dfrac{dy}{dx} - 2y = 0$ (b) $y'' - y = 0$ (c) $\dfrac{\partial^2 u}{\partial t^2} = 3\left(\dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \dfrac{\partial^2 u}{\partial x_3^2}\right)$

(d) $y'' + 9y = 26e^{-2t}$ (e) $\dfrac{\partial u}{\partial t} = 0.5\dfrac{\partial^2 u}{\partial x^2}$ (f) $\dfrac{d^2x}{dt^2} - 5\dfrac{dx}{dt} + 6x = 10\sin t$

(g) $L\dfrac{d^2u}{dx^2} + g\sin u = 0$ (h) $EI\dfrac{d^4y}{dx^4} = w(x)w($ (i) $u_{tt} - c^2(u_{rr} + \frac{2}{r}u_r) = 0$

2. Give the order of each of the following ordinary differential equations. Assume that any letters not used in derivatives represent constants.

(a) $\dfrac{dy}{dt} - 2y = 0$ (b) $y'' - y = 0$ (c) $\dfrac{1}{y}\dfrac{dy}{dx} + y = 1$

(d) $y'' + 9y = 26e^{-2t}$ (e) $\dfrac{1}{x}\dfrac{dy}{dx} + y = 1$ (f) $\dfrac{d^2x}{dt^2} - 5\dfrac{dx}{dt} + 6x = 10\sin t$

(g) $L\dfrac{d^2u}{dx^2} + g\sin u = 0$ (h) $EI\dfrac{d^4y}{dx^4} = w$ (i) $\dfrac{dy}{dx} + xy = 1$

3. For each of the first order equations from Exercise 2, give the function F if the the equation was to be written in the form $\dfrac{dy}{dx} = F(x,y)$. (Use the appropriate variables for the equation.)

4. For each of the ODEs from Exercise 2 that are linear, give the values of the functions f, a_0, a_1, a_2, ... (Include the independent variable, like $a_1(x) = x^2$, for example.) If the independent variable cannot be determined, use x.

5. For each of the first order equations from Exercise 2 that are separable, give the functions g and h, using the appropriate independent variable.

6. Which of the first order equations from Exercise 2 are autonomous?

1.7 Initial Value Problems and Boundary Value Problems

Performance Criterion:

1. (I) Determine whether a function satisfies an initial value problem (IVP) or boundary value problem (BVP); determine values of constants for which a function satisfies an IVP or BVP.

Initial Value Problems

Consider again the ODE $\frac{dy}{dx} + 3y = 0$, for which any function of the form $y = Ce^{-3x}$ is a solution. Suppose we impose the additional condition that $y = 4$ when $x = 0$. This is called an **initial condition** and we often write such a condition in the form $y(0) = 4$, where the number four is called an **initial value**. (As discussed in Section 1.4, we will often blur the distinction between initial conditions and initial values.) Substituting these values into $y = Ce^{-3x}$ gives $4 = Ce^{-3(0)}$, leading to $C = 4$.

When we combine a differential equation with one or more initial values, we have what is called an **initial value problem** (IVP). The solution to an initial value problem is a function or equation that satisfies *both* the differential equation and the initial value(s). Thus $y = 4e^{-3x}$ is a solution to the IVP

$$\frac{dy}{dx} + 3y = 0\,, \qquad y(0) = 4$$

The term "initial" implies "starting," so the independent variable for initial value problems is often time. To be a solution to an initial value problem means the following:

Solution to an Initial Value Problem

A **solution to an initial value problem** is a function that is a solution to the differential equation and that satisfies all of the initial conditions.

◇ **Example 1.7(a):** Verify that $y = \frac{7}{2}e^{-5t} + \frac{5}{2}\sin t - \frac{1}{2}\cos t$ is the solution to the initial value problem

$$\frac{dy}{dt} + 5y = 13\sin t, \qquad y(0) = 3$$

Solution: $\frac{dy}{dt} = -\frac{35}{2}e^{-5t} + \frac{5}{2}\cos t + \frac{1}{2}\sin t$, so

$$\begin{aligned}
\text{LHS} &= -\tfrac{35}{2}e^{-5t} + \tfrac{5}{2}\cos t + \tfrac{1}{2}\sin t + 5\left(\tfrac{7}{2}e^{-5t} + \tfrac{5}{2}\sin t - \tfrac{1}{2}\cos t\right) \\
&= -\tfrac{35}{2}e^{-5t} + \tfrac{5}{2}\cos t + \tfrac{1}{2}\sin t + \tfrac{35}{2}e^{-5t} + \tfrac{25}{2}\sin t - \tfrac{5}{2}\cos t \\
&= \tfrac{26}{2}\sin t \\
&= 13\sin t \\
&= \text{RHS}
\end{aligned}$$

This shows that the function satisfies the differential equation. We must now show that the function satisfies the initial condition. When $t = 0$,

$$y = \tfrac{7}{2}e^{-5(0)} + \tfrac{5}{2}\sin 0 - \tfrac{1}{2}\cos 0 = \tfrac{7}{2} + 0 - \tfrac{1}{2} = \tfrac{6}{2} = 3,$$

so the function satisfies the initial condition also. Therefore it is a solution to the IVP.

We should observe in the above example that the process of checking the initial condition is easier than checking the differential equation; this is often the case.

◇ **Example 1.7(b):** Determine whether $y = 5e^{-3t} - 2e^{3t}$ is a solution to the initial value problem

$$y'' - 9y = 0\,, \qquad y(0) = 3\,, \qquad y'(0) = -8$$

Solution: This time let's check the initial conditions first. We see that $y(0) = 5 - 2 = 3$, so the first initial condition is met. We next compute $y'(t) = -15e^{-3t} - 6e^{3t}$, so $y'(0) = -15 - 6 \neq -8$, So the second initial condition is not met. Therefore the function is NOT a solution to the IVP.

Let's reiterate that in order to be a solution to an IVP, the function must satisfy *BOTH* the ODE and the initial conditions. Since the function in this last example failed to satisfy one of the initial conditions, it doesn't matter whether it satisfies the ODE or not (it does in this case), it still fails to satisfy the IVP.

We will now see how initial values or boundary values are used to determine the values of arbitrary constants for solutions to ODEs for which we also know initial or boundary conditions.

◇ **Example 1.7(c):** It can be shown that Another Example

$$x = C_1e^{-t} + C_2e^{-3t} + 2\sin t \tag{1}$$

is the general solution to the differential equation $x'' + 4x' + 3x = 4\sin t + 8\cos t$. Find the values of C_1 and C_2 for which the function also satisfies the initial conditions

$$x(0) = -2, \qquad x'(0) = 5.$$

Solution: First we can substitute $t = 0$ and $x = -2$ into (1) to get

$$-2 = C_1 + C_2. \tag{2}$$

Next we compute the derivative of (1) to get $x' = -C_1e^{-t} - 3C_2e^{-3t} + 2\cos t$. Substituting $t = 0$ and $x' = 5$ into this gives us

$$5 = -C_1 - 3C_2 + 2. \tag{3}$$

We can simply add (2) and (3) at this point to get

$$3 = -2C_2 + 2,$$

which can then be solved to obtain $C_2 = -\frac{1}{2}$. We then substitute this value into (2) and solve to obtain $C_1 = -\frac{3}{2}$.

Note that the above shows that

$$x = -\tfrac{3}{2}e^{-t} - \tfrac{1}{2}e^{-3t} + 2\sin t$$

is a solution to the initial value problem

$$x'' + 4x' + 3x = 4\sin t + 8\cos t, \qquad x(0) = -2,\ x'(0) = 5.$$

Boundary Value Problems

When the independent variable we are working with is distance along a line, rather than time, we have **boundary conditions** rather than initial conditions. An example of such a situation occurs when we model the deflection of a horizontal beam. Often when we have boundary conditions they are given at two different values of the independent variable. The numerical values of the dependent variable that describe the boundary conditions are called **boundary values**. A solution to a boundary value problem is a function that satisfies both the differential equation and the boundary conditions.

◇ **Example 1.7(d):** Determine whether $y = C\cos\frac{2}{3}x$ is a a solution to the boundary value problem

$$y'' + \tfrac{4}{9}y = 0, \qquad y'(0) = 0,\ y'(3\pi) = 0.$$

Solution: First we see that

$$y = C\cos\tfrac{2}{3}x \qquad \Longrightarrow \qquad y' = -\tfrac{2}{3}C\sin\tfrac{2}{3}x \qquad \Longrightarrow \qquad y'' = -\tfrac{4}{9}C\cos\tfrac{2}{3}x,$$

from which we get

$$y'' + \tfrac{4}{9}y = -\tfrac{4}{9}C\cos\tfrac{2}{3}x + \tfrac{4}{9}C\cos\tfrac{2}{3}x = 0,$$

so $y = C\cos\frac{2}{3}x$ is a solution to the differential equation. Noting that we found y' above, we have

$$y'(0) = -\tfrac{2}{3}C\sin\tfrac{2}{3}(0) = 0 \qquad \text{and} \qquad y'(3\pi) = \tfrac{2}{3}C\sin\tfrac{2}{3}(3\pi) = \tfrac{2}{3}C\sin 2\pi = 0.$$

These show that $y = C\cos\frac{2}{3}x$ also satisfies the boundary condtions, so it is indeed a solution to the boundary value problem.

◇ **Example 1.7(e):** Consider the boundary value problem

$$y'' + \tfrac{1}{4}y = 0, \qquad y(0) = 3,\ y(\pi) = -4. \tag{2}$$

Show that

$$y = C_1\sin\tfrac{1}{2}x + C_2\cos\tfrac{1}{2}x \tag{3}$$

is a solution to the ODE $y'' + \frac{1}{4}y = 0$. Then find values of C_1 and C_2 for which the function (3) satisfies the boundary conditions $y(0) = 3,\ y(\pi) = -4$.

Solution: It is easy to calculate

$$y' = \tfrac{1}{2}C_1\cos\tfrac{1}{2}x - \tfrac{1}{2}C_2\sin\tfrac{1}{2}x \qquad \text{and} \qquad y'' = -\tfrac{1}{4}C_1\sin\tfrac{1}{2}x - \tfrac{1}{4}C_2\cos\tfrac{1}{2}x,$$

leading to

$$y'' + \tfrac{1}{4}y = (-\tfrac{1}{4}C_1\sin\tfrac{1}{2}x - \tfrac{1}{4}C_2\cos\frac{1}{2}x) + \tfrac{1}{4}(C_1\sin\tfrac{1}{2}x + C_2\cos\tfrac{1}{2}x) = 0,$$

so $y = C_1\sin\frac{1}{2}x + C_2\cos\frac{1}{2}x$ is a solution to $y'' + \frac{1}{4}y = 0$.

For the boundary condition $y(0) = 3$ we substitute $x = 0$ and $y = 3$ into (3) to get

$$3 = C_1\sin\tfrac{1}{2}(0) + C_2\cos\tfrac{1}{2}(0).$$

This gives us $C_2 = 3$. Substituting $x = \pi$ and $y = -4$ into (3) gives us $C_1 = -4$.

We now know that $y = -4\sin\frac{1}{2}x + 3\cos\frac{1}{2}x$ is a solution to the boundary value problem (2).

Section 1.7 Exercises

To Solutions

1. Verify that $y = -\dfrac{1}{x} + 3$ is a solution to the initial value problem

$$x^2 \frac{dy}{dx} = 1, \qquad\qquad y(1) = 2$$

2. Determine whether $y = 2\sin(3t) + e^{-2t}$ is a solution to the initial value problem

$$y'' + 9y = 13e^{-2t}, \qquad\qquad y(0) = 1, \qquad\qquad y'(0) = 4$$

3. For each of the following, determine whether the given function is a solution to the initial value problem that is given after it. *If it is not, tell why not.*

   (a) $y = \frac{1}{4}e^{-x} + \frac{1}{2}e^{2x} + \frac{3}{4}e^{3x}$ IVP: $y'' - y' - 2y = e^{3x}, \quad y(0) = \frac{3}{2}, \quad y'(0) = 1$

   (b) $y = 3e^{-x} + \frac{1}{2}\sin x$ IVP: $y' + y = \sin x, \quad y(0) = 3$

   (c) $y = \frac{5}{2}e^{x^2} - \frac{1}{2}$ IVP: $\dfrac{dy}{dx} - 2xy = x, \quad y(0) = 2$

   (d) $x = 2\sin 2t + 3\cos 2t$ IVP: $\dfrac{d^2x}{dt^2} + 4x = 0, \quad x(0) = 3, \quad x'(0) = 4$

4. (a) $y = Ce^{-2t} + 3\cos 2t$ is the general solution to the ODE $y' + 2y = 6\cos 2t - 6\sin 2t$. Determine the solution to the initial value problem

$$y' + 2y = 6\cos 2t - 6\sin 2t, \qquad y(0) = 5.$$

   (b) $x = C_1e^{-t} + C_2e^{-4t} + 3t + 1$ is the general solution to the ODE $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + 5x = 12t + 19$. Find the solution to the IVP

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 5x = 12t + 19, \qquad x(0) = -2,\ x'(0) = 1.$$

   (c) $y = A\sin\sqrt{5}\ t + B\cos\sqrt{5}\ t$ is the general solution to the ODE $y'' + 5y = 0$. Find the solution to the initial value problem

$$y'' + 5y = 0\ , \qquad y(0) = -3,\ y'(0) = \tfrac{2}{3}.$$

   (d) $y = C_1\sin 2t + C_2\cos 2t + e^{-3t}$ is the general solution to the ODE $y'' + 4y = 13e^{-3t}$. Find the solution to the initial value problem

$$y'' + 5y = 13e^{-3t}\ , \qquad y(0) = 7,\ y'(0) = -4.$$

5. We showed in Example 1.7(e) that $y = C_1 \sin \frac{1}{2}x + C_2 \cos \frac{1}{2}x$ is a solution to the differential equation $y'' + \frac{1}{4}y = 0$.

   (a) Determine values of C_1 and C_2 for which $y = C_1 \sin \frac{1}{2}x + C_2 \cos \frac{1}{2}x$ is a solution to the boundary value problem

   $$y'' + \tfrac{1}{4}y = 0, \qquad y'(0) = 1,\ y'(\pi) = 2.$$

   Note that both boundary conditions are on the first derivative!

   (b) Determine values of C_1 and C_2 for which $y = C_1 \sin \frac{1}{2}x + C_2 \cos \frac{1}{2}x$ is a solution to the boundary value problem

   $$y'' + \tfrac{1}{4}y = 0, \qquad y(0) = 5,\ y'(2\pi) = -3.$$

6. In each of the following, a boundary value problem and function are givne. In each case, determine whether the function is a solution to the boundary value problem (see Example 1.7(d)). If it is not a solution, tell why not.

   (a) **BVP:** $y'' + \frac{\pi^2}{25}y = 0, \ \ y(0) = 0,\ y'(5) = 0$ **Function:** $y = C \sin \frac{\pi}{5}x$

   (b) **BVP:** $y'' + \frac{\pi^2}{25}y = 0, \ \ y(0) = 0,\ y(5) = 0$ **Function:** $y = C \sin \frac{\pi}{5}x$

   (c) **BVP:** $y'' + \frac{25}{4}y = 0, \ \ y(0) = 0,\ y'(\pi) = 0$ **Function:** $y = C \sin \frac{5}{2}x$

   (d) **BVP:** $y'' + \frac{\pi}{14}y = 0, \ \ y'(0) = 0,\ y(7) = 0$ **Function:** $y = C \cos \frac{\pi}{14}x$

   (e) **BVP:** $y'' + \frac{\pi^2}{25}y = 0, \ \ y'(0) = 0,\ y'(10) = 0$ **Function:** $y = C \cos \frac{\pi}{5}x$

   (f) **BVP:** $y'' + \frac{9}{25}y = 0, \ \ y(0) = 0,\ y'(5\pi) = 0$ **Function:** $y = C \cos \frac{3}{5}x$

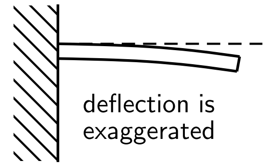
1.8 Chapter 1 Summary

- For our purposes, a function is a dependent variable that depends on one or more independent variables. (For most of this course we are concerned only with functions of one independent variable.)
- The values of the independent variable for which values of the dependent variable are obtained are called the domain of the function.
- The graph of a function gives us a quick way to determine the general behavior (sometimes called the qualitative behavior) of the function.
- The derivative of a function gives the rate of change of the dependent variable with respect to the independent variable.
- Exponential functions are essentially their own derivatives (of any order). Sine and cosine are essentially their own second derivatives. "Essentially" means that the function and the derivative differ only by a factor of (multiplication by) a constant.
- Differential equations are equations containing derivatives of a function. When the function is a function of one variable, the differential equation is an ordinary differential equation; when the function is a function of more than one variable, the differential equation is a partial differential equation.
- Parameters are values that change from situation to situation, but that do not change once the situation is set. Variables are values that change once the situation is set. Another way of looking at this is that parameters are characteristics of the physical system, variables are quantities that vary within the physical system.
- When examining a situation in which time is the independent variables, we generally have initial conditions, which describe the state of the dependent variable at time zero (or some other point in time).
- When considering a situation in which position in space (or along a rod, on a surface) is(are) the independent variable(s), we have values of the dependent variable on the boundary of our domain. These are called boundary values.
- Even though they are slightly different, we use initial values and initial conditions synonymously, and the same for boundary values and boundary conditions. Technically, initial conditions are physical states of the system at time zero and initial values are numerical values describing those states. A similar distinction holds for boundary conditions and boundary values.
- A solution to a differential equation is a function (or relation) for which the function(relation) and its relevant derivatives can be substituted into the equation to obtain a true statement.
- When a solution contains arbitrary constants we call it a family of solutions. A family that includes all possible solutions to a differential equation is called a general solution; a solution that contains no arbitrary constants is called a particular solution.
- General solutions to first order differential equations contain one arbitrary constant, and general solutions to second order differential equations contain two arbitrary constants.
- To verify that a function is a solution to a differential equation, we substitute the function and its derivative into the left side and see if the result is the right side, *OR* we substitute the function and its derivatives into both sides and see if the results are equal.

- We classify ordinary (and partial) differential equations:
  - The order of a differential equation is the order of the highest derivative in the equation.
  - In addition to classifying ordinary differential equations by order, we also classify them as linear or non-linear.
  - Linear ODEs are classified as homogeneous or non-homogeneous.
  - First order ordinary differential equations can also be classified as separable (or not), and autonomous (or not).
- Values of arbitrary constants are determined by initial (or boundary) conditions. For a first order equation, one initial condition is needed to determine the one constant. For a second order equation, two initial (or boundary) conditions are needed to determine the two constants.
- A differential equation together with either initial values or boundary values is called an initial value problem or a boundary value problem.
- Recognizing the classification(s) of an ordinary differential equation is important for knowing how to solve the equation.

1.9 Chapter 1 Exercises

1. For the ten foot beam in Example 1.1(b), shown to the right, the boundary value problem modeling the situation is given below. In this exercise you will solve the boundary value problem.



$$\frac{d^4y}{dx^4} = 2, \qquad y(0) = 0,\ y'(0) = 0,\ y''(10) = 0,\ y'''(10) = 0$$

   (a) The original equation can be written $y^{(4)} = 2$. Given that $y^{(4)}$ is the derivative of $y^{(3)} = y'''$, what must $y^{(3)}$ look like? *Your answer should include a constant.*

   (b) Use the boundary condition $y'''(10) = 0$ to determine the constant from your answer to (a). Write the function y''' with that value substituted into your answer to (a).

   (c) Now that you know y''', you can use the fact that it is the derivative of y'' to find what y'' looks like. It contains a constant - use the boundary condition $y''(10) = 0$ to find the value of that constant. Then give y''.

   (d) Repeat to find y', and then y.

2. In this exercise you will solve what is perhaps the easiest boundary value problem there is. Remember the situation from Exercise 5 of Section 1.4: A 70 centimeter metal rod is perfectly insulated along the length of the rod, so that no heat can enter or leave along its length, but heat *CAN* enter or leave at its ends. We impose a one-dimensional coordinate system that puts zero at the left end of the rod and 70 cm at the right end, and we let $u(x)$ represent the temperature at any point x along the length of the rod, using our coordinate system. The temperature at the left end of the rod is held at a constant temperature of 32° Fahrenheit and the right end is held at 115° F.

   We now leave the rod in this state for a very long (infinite) period of time. The temperature at each point in the rod will eventually reach a constant value, called its **equilibrium** temperature. Let $u(x)$, for $0 \leq x \leq 70$, represent the function giving the equilibrium temperature at every point in the rod. Physical principles of heat flow dictate that the function u must satisfy the following BVP:

$$\frac{d^2u}{dx^2} = 0, \qquad u(0) = 32,\ u(70) = 115.$$

   The equation $\dfrac{d^2u}{dx^2} = 0$ is called the one-dimensional **Laplace's equation** or the **steady-state heat equation**, and $u(0) = 32$ and $u(70) = 115$ are associated boundary values. Let's solve the boundary value problem!

   (a) Draw a graph of what you think the equilibrium temperatures will look like. As we will always do, put the independent variable x on the horizontal axis and the dependent variable u (the temperature) on the vertical axis. Label each axis with its variable *and the units for that variable.*

   (b) The ODE can be written as $u'' = 0$. Remembering that u'' is the derivative of u', what sort of function must u' be if it has a derivative of zero? Write an equation for u'.

(c) What must u look like in order to have the derivative found in (b)? Write an equation for u - it should contain two arbitrary constants.

(d) Substitute the first boundary condition into your answer to (b) to get an equation containing both arbitrary constants but neither of the variables x or u. Repeat for the second boundary condition.

(e) You now have a system of two equations containing the two unknown constants. Solve the system to find their values.

(f) Give the function u that is the solution to the BVP. Would the graph of this function look like what you drew for part (a)?

NOTE: When we have a similar problem, but for temperatures in a two-dimensional plate of metal rather than a rod, the differential equation will be the two-dimensional **Laplace's equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This is a partial differential equation - we can tell by the facts that the derivatives are partial derivatives and that there are two independent variables, x and y. You should be able to guess what the three-dimensional Laplace's equation would look like.

If we were to, instead of waiting for the temperatures to reach equilibrium in our rod, watch the temperature as time progressed from time zero, then time itself would be an independent variable as well. The temperature $u(x,t)$ at any point and time in the rod is then a function of two variables, and it obeys the one-dimensional **heat equation**

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}.$$

Here the constant κ is a parameter that depends on various properties of the material the rod is made of. If our object was three dimensional, the corresponding heat equation would be

$$\frac{\partial u}{\partial t} = \kappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right).$$

When we work with Laplace's equation or the heat equation in more than one dimension, the boundary value situation becomes more complicated, as there are infinitely many boundary points. When working with the heat equation (in any number of dimensions) there will also be infinitely many initial values. In all of these cases the boundary values and initial values are given by functions rather than constants (unless the functions happen to be constant functions!).

There are three important classes of partial differential equations, called **elliptic**, **parabolic** and **hyperbolic**. Laplace's equation and the heat equation are the standard examples of elliptic and parabolic equations, respectively. (Mathematicians love to call such examples "canonical" examples.) The canonical example of a hyperbolic equation is the **wave equation**. In three dimensions the wave equation is usually written as

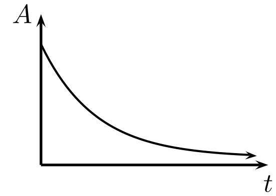
$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

where c is a constant.

D Solutions to Exercises

D.1 Chapter 1 Solutions

Section 1.1 Solutions Back to 1.1 Exercises



1. (a) The independent variable is time, and the dependent variable is the amount of radioactive material.

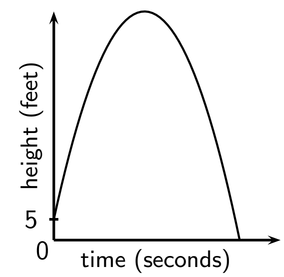
   (b) See graph to the right.

2. (a) The independent variables are time t and the distance x out from the edge of the table, and the dependent variable is the deflection of the ruler at any point and time.

   (b) $0 \le t$, $0 \le x \le 6$, where x is measured in inches. (Substitute 0.5 for 6 if measuring in feet instead of inches.)

3. $0 \le r \le 5$ inches, $0 \le \theta \le 2\pi$, $0 \le t$ Note that, unless we are doing right triangle trigonometry, angles will be measured in radians.

4. If measuring in feet, $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.



Exercise 5(a)

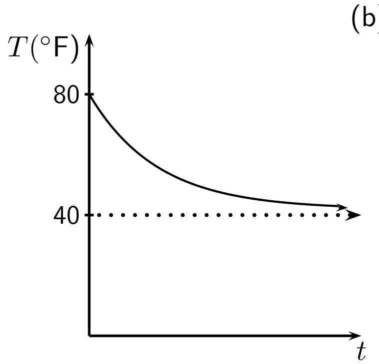
5. (a) See graph to the right.

   (b) The shape is a parabola, opening downward.

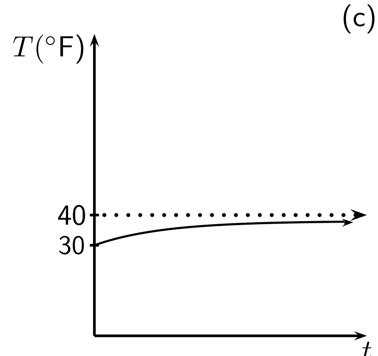
   (c) We would model the height with a quadratic function.

   (d) In this case the domain is finite, of the form $0 \le t \le a$ for the time a when the rock hits the ground, whereas in Exercise 1(b) the domain was infinite, from time zero "to infinity."

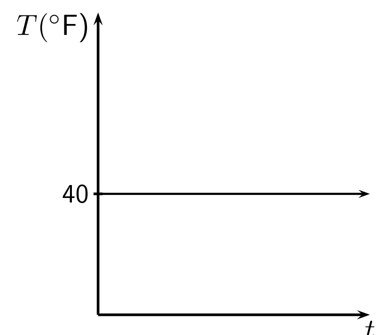
6. (a)



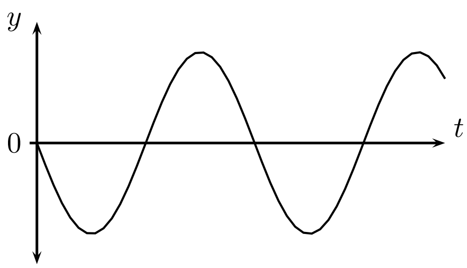
(b)



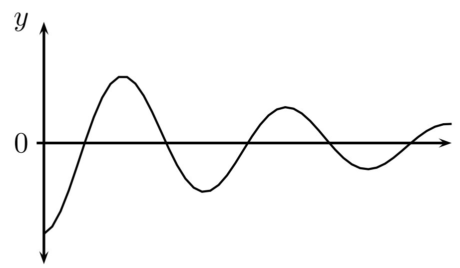
(c)



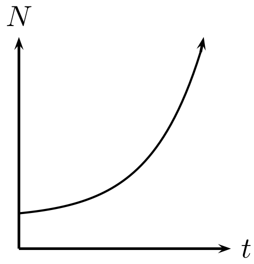
7. (a)

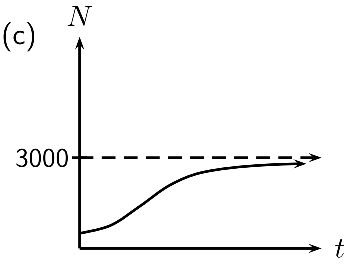


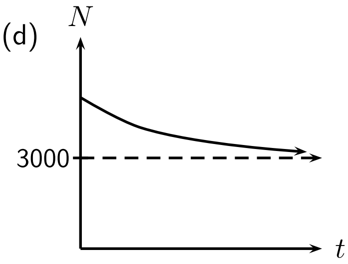
(b)



8. (a) Time is independent, number of individuals is dependent.

(b) 

(c) 

(d) 

9. (c) Changing A affects the amplitude, changing w affects the period. The phase is not affected by changing A or w.
   (d) Changing A affects both the amplitude and the phase, and changing w affects only the period. Changing B also affects both the amplitude and phase.

11. (b) a is the value of the horizontal asymptote, and it also affects the y-intercept. b affects the y-intercept and (sort of) affects the rate at which the function approaches the asymptote. r affects only the rate at which the function approaches the asymptote.

12. (a) $a = 30$, $b = -20$, r cannot be determined.

13. (b) $a = 70$, $b = 90$, r cannot be determined.

14. (a) $t = 0$ at the y-intercept, the y-intercept is $a + b$.
   (b) The limit of y as t goes to infinity is a , so the graph has a horizontal aymptote of $y = a$.

16. (a) III (b) I (c) II

Section 1.2 Solutions

Back to 1.2 Exercises

1. (a) $\dfrac{dy}{dx} = 6\cos 3x$ (b) $y' = -2e^{-0.5t}$ (c) $x' = 2t + 5$
   (d) $y' = -4.42\sin(1.3t - 0.9)$ (e) $\dfrac{dy}{dt} = -3te^{-3t} + e^{-3t}$
   (f) $x' = 12e^{-2t}\cos(3t + 5) - 8e^{-2t}\sin(3t + 5)$

2. (a) $\dfrac{d^2y}{dx^2} = -18\sin 3x$ (b) $y'' = e^{-0.5t}$ (c) $x' = 2$

3. At seven minutes, the temperature is increasing at 2.7°F per minute.

4. At 12.5 minutes, the amount of salt in the tank is decreasing at 1.3 pounds per minute.

5. (a) At 2 seconds the mass is moving downward at 5 inches per second.
   (b) At 2 seconds the mass is accelerating *upward* at 3 inches per second per second (in/sec^2).
   (c) At 2 seconds the mass is slowing down because the acceleration is in the direction opposite the velocity.

6. At 5.4 hours, the number of bacteria in the dish is increasing at a rate of 430 bacteria per hour.

7. (a) For $x > 0$ the derivative will be positive, because the deflection increases as x increases.
   (b) The absolute value of the derivative at x_2 will be greater than the absolute value of the derivative at x_1.

8. (a) $y = e^{-3x}$, $y = Ce^{-3x}$ for any constant C.
   (b) $y = \sin 3t$ or $y = \cos 3t$. Any function of the form $y = C_1 \sin 3t + C_2 \cos 3t$ will do it, for any constants C_1 and C_2.
   (c) $y = e^{3t}$, $y = Ce^{3t}$ for any constant C.
   (d) $y = \sin\sqrt{5}\,x$ or $y = \cos\sqrt{5}\,x$. Any function $y = C_1 \sin\sqrt{5}\,x + C_2 \cos\sqrt{5}\,x$ will do it, for any constants C_1 and C_2.

9. (a) $\dfrac{dy}{dx} = -3y$ (b) $\dfrac{d^2y}{dt^2} = -9y$ (c) $\dfrac{d^2x}{dt^2} = 9x$ (d) $\dfrac{d^2y}{dx^2} = -5y$

Section 1.3 Solutions

Back to 1.3 Exercises

1. (a) The parameters are the the initial temperature T_0, the temperature T_m of the medium, and the constant k.
   (b) The independent variable is time t.
   (c) The dependent variable is temperature T.

2. The parameters are the amount of the mass, the stiffness of the spring (the spring constant), the amount the mass is pulled downward before letting it go, and the viscosity of the oil in the oil bath. The shape of the mass is probably a parameter as well.

3. The independent variable is time t and the dependent variable is current i. The parameters are the inductance L, the resistance R, the voltage E and the initial current i_0.

4. (a) The independent variables are time and the distance along the string.
   (b) Some parameters would be physical properties of the material the string is made of, the thickness and cross-section of the string (which is probably circular, but could be different) and the tension in the string.

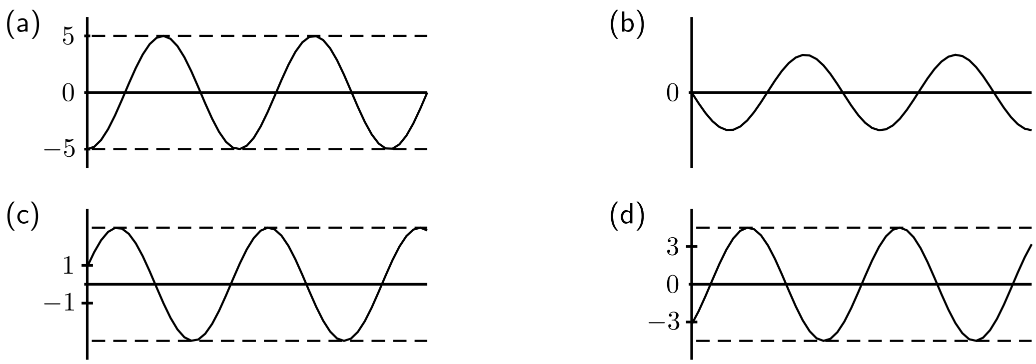
5. (a) The independent variable is time, and the dependent variable is the amount of salt in the tank.
   (b) The parameters are the rate at which fluid is entering and leaving the tank, and the concentration of the incoming fluid. We might think that the amount of salt in the tank to begin with is a parameter, but it is instead what we call an **initial value**. It is a value of the dependent variable when the independent variable is zero.

Section 1.4 Solutions

Back to 1.4 Exercises

1. (a) $y = 7$ when $x = 0$, $y' = -3$ when $x = 0$
   (b) $x = 1$ when $t = 0$, $x' = 5$ when $t = 0$
   (c) $y = 0$ when $x = 0$, $y'' = 0$ when $x = 0$, $y = 0$ when $x = 15$, $y' = 0$ when $x = 0$

2. (a) $y(0) = -5,\ y'(0) = 0$ (b) $y(0) = 0,\ y'(0) = -2$
   (c) $y(0) = 1,\ y'(0) = 2$ (d) $y(0) = -3,\ y'(0) = 1$

3. 

4. (a) $y(0) = y(20) = 0,\ y''(0) = 0,\ y'(20) = 0$ (b) $y(0) = y(12) = 0,\ y''(0) = y''(12) = 0$

5. $u(0) = 32, \quad u(70) = 115$

Section 1.5 Solutions Back to 1.5 Exercises

1. (a) Independent variable: x Dependent variable: y
   (b) Independent variable: can't determine Dependent variable: y
   (c) Independent variables: x_1, x_2, x_3, t Dependent variable: u
   (d) Independent variable: t Dependent variable: y
   (e) Independent variable: x, t Dependent variable: u
   (f) Independent variable: x Dependent variable: u
   (g) Independent variable: x Dependent variable: y
   (h) Independent variable: x Dependent variable: y
   (i) Independent variable: r, t Dependent variable: u

2. (a) Independent variable: x Dependent variable: y
   (b) Independent variable: t Dependent variable: y

3. no 4. yes 7. $A = \frac{36}{65}$, $B = \frac{28}{65}$.

9. (a) no (b) $y = ce^{3x}$ is only a solution if $c = 2$ (c) $y = 2e^{3x}$
   (d) $y = ce^{x}$ is a solution for any value of c

Section 1.6 Solutions Back to 1.6 Exercises

1. (a), (b), (d), (f), (g), (h)

2. (a), (c), (e) and (i) are first order. (b), (d), (f) and (g) are second order, and (h) is fourth order.

3. (a) $F(x, y) = 2y$ (c) $F(x, y) = y - y^2$ (e) $F(x, y) = x - xy$ (i) $F(x, y) = 1 - xy$

4. (a) $f(x) = 0,\ a_0(x) = -2,\ a_1(x) = 1$
   (b) $f(x) = 0,\ a_0(x) = -1,\ a_1(x) = 0,\ a_2(x) = 1$

(c) not linear

(d) $f(t) = 26e^{-2t},\ a_0(t) = 9,\ a_1(t) = 0, a_2(t) = 1$

(e) $f(x) = 1,\ a_0(x) = 1,\ a_1(x) = \frac{1}{x}$

(f) $f(t) = 10\sin t,\ a_0(t) = 6,\ a_1(t) = -5,\ a_2(t) = 1$

(g) not linear

(h) $f(x) = w,\ a_0(x) = a_1(x) = a_2(x) = a_3(x) = 0,\ a_4(x) = 1$

(i) $f(x) = 1,\ a_0(x) = x,\ a_1(x) = 1$

5. (a) $g(x) = 1,\ h(y) = 2y$ (c) $g(x) = 1,\ h(y) = y - y^2$

(e) $g(x) = x,\ h(y) = 1 - y$ (i) not separable

6. (a) and (c)

Section 1.7 Solutions Back to 1.7 Exercises

2. Yes, it is a solution.

3. (a) Not a solution, $y'(0) \neq 1$ and y is not a solution to the ODE.

(b) Not a solution, y is not a solution to the ODE.

(c) Solution. (d) Solution.

4. (a) $y = 2e^{-2t} + 3\cos 2t$ (b) $x = -\frac{14}{3}e^{-t} + \frac{5}{3}e^{-4t} + 3t + 1$

(c) $y = \frac{2}{3\sqrt{5}}\sin\sqrt{5}t - 3\cos\sqrt{5}t$ (d) $y = -\frac{1}{2}\sin 2t + 6\cos 2t + e^{-3t}$

5. (a) $C_1 = 2,\ C_2 = -4$ (b) $C_1 = 6,\ C_2 = 5$

6. (a) The function is not a solution to the BVP, it doesn't satisfy the boundary condition $y'(5) = 0$.
(b) The function is a solution to the BVP.
(c) The function is a solution to the BVP.
(d) The function is not a solution to the BVP, it doesn't satisfy the ODE.
(e) The function is a solution to the BVP.
(f) The function is not a solution to the BVP, it doesn't satisfy the boundary condition $y(0) = 0$.