# Ordinary Differential Equations 

for Engineers and Scientists

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## 2 First Order Equations

## Learning Outcome:

2. Solve first order differential equations and initial value problems; set up and solve first order differential equations modeling physical problems.

## Performance Criteria:

(a) Solve first order ODEs and IVPs by separation of variables.
(b) Demonstrate the algebra involved in solving a relation in $x$ and $y$ for $y$; in particular, change $\ln |y|=f(x)$ to $y=g(x)$, showing all steps clearly.
(c) Sketch solution curves to an ODE for different initial values. Given a set of solution curves for a first order ODE, identify the one having a given initial value.
(d) Sketch a small portion of the direction field for a first order ODE.
(e) Given the direction field and an initial value for a first order IVP, sketch the solution curve.
(f) Use an integrating factor to solve a first order linear ODE or IVP.
(g) Determine whether an ODE is autonomous.
(h) Create a one-dimensional phase portrait for an autonomous ODE.
(i) Determine critical points/equilibrium solutions of an autonomous ODE, and identify each as stable, unstable or semi-stable.
(j) Sketch solution curves of an autonomous ODE for various initial values.
(k) Solve an applied problem modeled by a first order ODE using separation of variables or an integrating factor.
(I) Give an ODE or IVP that models a given physical situation involving growth or decay, mixing, Newton's Law of Cooling or an RL circuit.
(m) Sketch the graph of the solution to a mixing or Newton's Law of Cooling problem, indicating the initial value and the steady-state asymptote.
(n) Identify the transient and steady-state parts of the solution to a first order ODE.

In the first chapter we found out what what ordinary differential equations (ODEs), initial value problems (IVPs) and boundary value problems (BVPs) are, and what it means for a function to be a solution to an ODE, IVP or BVP. We then saw how to determine whether a function is a solution to an ODE, IVP or BVP, and we looked at a few "real world" situations where ODEs, BVPs and IVPs arise from physical principles.

Our goal for the rest of the course is to solve ODEs, IVPs and BVPs and to see how the ODEs, IVPs, BVPs and their solutions apply to real situations. We can "solve" ODEs (and PDEs) in three ways:

- Analytically, which means "paper and pencil" methods that give exact solutions in the form of algebraic equations.
- Qualitatively, which means determining the general behavior of solutions without actually finding function values. Results of qualitative methods are often expressed graphically.
- Numerical methods which result in values of solutions only at discrete points in time or space. Results of numerical methods are often expressed graphically or as tables of values.

In this chapter you will learn how to find solutions qualitatively and analytically for first order ODEs and IVPs. (Numercial methods are discussed in Appendix C.) You will see two analytical methods, separation of variables and the integrating factor method.

- Separation of variables is the simpler of the two methods, but it only works for separable ODEs, which you learned about in Section 1.6. It is a useful method to look at because when it works it is fairly simple to execute, and it provides a good opportunity to review integration, which we will need for the other method as well.
- Solving with integrating factors is a method that can be used to solve any linear first order ODE, whether it is separable or not, as long as certain integrals can be found. The method of solution is more complicated than separation of variables, but not necessarily any more difficult to execute once you learn it.

After learning these two methods we will again look at applications, but only for first order ODEs and IVPs at this time.

### 2.1 Solving By Separation of Variables

## Performance Criteria:

2. (a) Solve first order ODEs and IVPs by separation of variables.
(b) Demonstrate the algebra involved in solving a relation in $x$ and $y$ for $y$; in particular, change $\ln |y|=f(x)$ to $y=g(x)$, showing all steps clearly.

So far you have learned how to determine whether a function is a solution to a differential equation, initial value problem or boundary value problem. But the question remains, "How do we find solutions to differential equations?" We will spend much of the course learning some analytical methods for finding solutions. If the ODE is separable, we can apply the simplest method for solving differential equations, called separation of variables. The bad news is that separation of variables only "works" for separable (so necessarily also first order) equations; the good news is that those sorts of equations actually occur in some "real life" situations. Let's look at an example of how we solve a separable equation.
$\diamond$ Example 2.1(a): Solve the differential equation $y^{\prime}-\frac{6 \sin 3 x}{y}=0$.
Another Example
Solution: Note that we can write the ODE as $\frac{d y}{d x}=6 \sin 3 x \cdot \frac{1}{y}=g(x) h(y)$, where $g(x)=$ $6 \sin 3 x$ and $h(y)=\frac{1}{y}$. (It doesn't really matter where the 6 is, it can be included in either $g$ or $h$.) Therefore the ODE is separable; let's separate the variables and solve:

$$
\begin{array}{rlrl}
y^{\prime}-\frac{6 \sin 3 x}{y} & =0 & & \text { The original equation. } \\
\frac{d y}{d x} & =\frac{6 \sin 3 x}{y} & & \begin{array}{l}
\text { Change to } \frac{d y}{d x} \text { notation and get the term with the } \\
\text { derivative alone on one side. }
\end{array} \\
d y & =\frac{6 \sin 3 x}{y} d x & & \text { Multiply both sides by } d x . \\
y d y & =6 \sin 3 x d x & \begin{array}{l}
\text { Do some algebra to get all the " } x \text { stuff" on one side } \\
\text { and the " } y \text { stuff" on the other. At this point the } \\
\text { variables have been separated. }
\end{array} \\
\int y d y & =\int 6 \sin 3 x d x & \begin{array}{l}
\text { Integrate both sides. }
\end{array} \\
\frac{1}{2} y^{2}+C_{1} & =-2 \cos 3 x+C_{2} & & \begin{array}{l}
\text { Compute the integrals. }
\end{array} \\
\frac{1}{2} y^{2} & =-2 \cos 3 x+C & \begin{array}{l}
\text { Subtract } C_{1} \text { from both sides and let } C=C_{2}-C_{1} . \\
\text { DO NOT solve for } y \text { unless asked to. } .
\end{array}
\end{array}
$$

The resulting solution for the above example is not a function, but is instead a relation. In some cases we will wish to solve for $y$ as a function of $x$ (or whatever other variables we might be using), but you should only do so when asked to.

In the next example you will see a simple, but a very useful, type of differential equations.
$\diamond$ Example 2.1(b): Solve the differential equation $\frac{d y}{d t}+0.5 y=0$ by separation of variables, and solve the result for $y$.

Solution: First let's solve the ODE by separation of variables:

$$
\begin{aligned}
\frac{d y}{d t}+0.5 y & =0 \\
\frac{d y}{d t} & =-0.5 y \\
d y & =-0.5 y d t \\
\frac{d y}{y} & =-0.5 d t \\
\int \frac{d y}{y} & =\int-0.5 d t \\
\ln |y|+C_{1} & =-0.5 t+C_{2} \\
\ln |y| & =-0.5 t+C_{3}
\end{aligned}
$$

where $C_{3}=C_{2}-C_{1}$. We now solve for $|y|$, using the facts that the inverse of the natural logarithm is the exponential function with base $e$ and if $|x|=u$, then $x= \pm u$ (the definition of absolute value):

$$
\begin{aligned}
\ln |y| & =-0.5 t+C_{3} & & \\
e^{\ln |y|} & =e^{-0.5 t+C_{3}} & & \text { take } e \text { to the power of each side } \\
|y| & =e^{-0.5 t} e^{C_{3}} & & \text { inverse of natural log and } x^{a} x^{b}=x^{a+b} \\
|y| & =C_{4} e^{-0.5 t} & & e^{C_{3}} \text { is just another constant, which we call } C_{4} \\
y & = \pm C_{4} e^{-0.5 t} & & \text { the definition of absolute value } \\
y & =C e^{-0.5 t} & & \text { "absorb" the } \pm \text { into } C_{4}, \text { calling the result } C
\end{aligned}
$$

The last step above might seem a bit "fishy," but it is valid. In most cases we have initial values, which then determine the constant $C$, including its sign:
$\diamond$ Example 2.1(c): Solve the initial value problem $\frac{d y}{d t}+0.5 y=0, \quad y(0)=7.3$.
Solution: We already solved the differential equation in the previous example, so we just need to find the value of the constant by substituting the initial values into the solution $y=C e^{-0.5 t}$ :

$$
\begin{aligned}
& 7.3=C e^{-0.5(0)} \\
& 7.3=C
\end{aligned}
$$

The solution to the IVP is $y=7.3 e^{-0.5 t}$.

Don't assume that the the constant is always the initial value!
$\diamond$ Example 2.1(d): Solve the initial value problem $y^{\prime}-\frac{6 \sin 3 x}{y}=0, \quad y(0)=4$.
Solution: We already solved the differential equation in Example 2.1(a), so we just need to find the value of the constant. Substituting $x=0$ and $y=4$ into the solution $\frac{1}{2} y^{2}=-2 \cos 3 x+C$ :

$$
\begin{aligned}
\frac{1}{2}(4)^{2} & =-2 \cos 3(0)+C \\
8 & =-2+C \\
C & =10
\end{aligned}
$$

The solution to the IVP is $\frac{1}{2} y^{2}=-2 \cos 3 x+10$.

The next, and last, example in this section illustrates something we will see again soon.
$\diamond$ Example 2.1(e): Solve the ODE $\left(x^{2}+4 x-5\right) y^{\prime}=x+17$.
Solution: The derivative $y^{\prime}$ is $\frac{d y}{d x}$. When we separate the variables we get

$$
d y=\frac{x+17}{x^{2}+4 x-5} d x
$$

If we do the partial fraction decomposition of the fraction on the right side (see Example A.4(a)) we can proceed as follows:

$$
\begin{aligned}
d y & =\left(\frac{3}{x-1}-\frac{2}{x+5}\right) d x \\
\int d y & =\int\left(\frac{3}{x-1}-\frac{2}{x+5}\right) d x \\
\int d y & =\int \frac{3}{x-1} d x-\int \frac{2}{x+5} d x \\
y+C_{1} & =3 \int \frac{d x}{x-1}-2 \int \frac{d x}{x+5} \\
y+C_{1} & =3 \ln |x-1|+C_{2}-2 \ln |x+5|+C_{3}
\end{aligned}
$$

From here we can combine the constants and apply properties of logarithms to obtain

$$
\begin{aligned}
& y=\ln |x-1|^{3}-\ln |x+5|^{2}+C \\
& y=\ln \frac{|x-1|^{3}}{|x+5|^{2}}+C
\end{aligned}
$$

which can also be written as

$$
y=\ln \left|\frac{(x-1)^{3}}{(x+5)^{2}}\right|+C
$$

1. Use separation of variables to solve each of the following ODEs. Don't solve for $y$.
(a) $\frac{d y}{d x}=-x \sec y$
(d) $y^{\prime}=\frac{y}{2 x+3}$
(b) $d x+x^{3} y d y=0$
(e) $x^{2} d y=e^{y} d x$
(c) $x^{2}+y^{4} \frac{d y}{d x}=0$
(f) $y^{\prime}=\frac{5 x+3}{y}$
2. Solve each of the following initial value problems. DO NOT solve for $y$, and give constants in exact form.
(a) $y^{\prime}=x y, \quad y(1)=3$
(c) $\frac{d y}{d x} \frac{e^{y}}{x}=3, \quad y(0)=2$
(b) $x \frac{d x}{d t}+5 t=3, \quad x(2)=4$
(d) $y^{\prime}=y^{4} \cos t, \quad y(0)=2$
3. Some of the following initial value problems can be solved by separation of variables. Solve the ones that CAN be solved by that method. DO solve for $y$ and give constants in exact form again.
(a) $\frac{d y}{d x}-3 y=0, \quad y(0)=4$
(b) $x \frac{d y}{d x}-y=x, \quad y(1)=2$
(c) $y^{\prime}-4 x y=0, \quad y(0)=2$
(d) $y^{\prime}-2 x=x y, \quad y(2)=5$
(e) $\frac{d y}{d x}-y=e^{3 x}, \quad y(0)=4$
(f) $\frac{d y}{d x}=\frac{y-1}{x+3}, \quad y(1)=3$
4. (a) Solve the initial value problem $y^{\prime}-2 x e^{-y}=e^{-y}, \quad y(0)=0$.
(b) Solve the initial value problem $y^{\prime}-2 x e^{-y}=e^{-y}, \quad y(1)=3$. Give the exact form for the unknown constant.
5. (a) Solve the differential equation $y^{\prime}+2 t y=0$. You should get $\ln y=-t^{2}+C$.
(b) We now want to get $y$ as a function of $t$. " $e$ " both sides of the equation and use the fact that $e^{\ln u}=u$. Use also the facts that $x^{a+b}=x^{a} x^{b}$ and $e$ raised to a constant power is yet another constant. You should now have a family of solutions to the differential equation.
(c) Use the initial condition $y(0)=7$ to determine the value of the arbitrary constant. You now have a solution to the initial value problem.
6. Later we will solve certain second order linear ODEs using a method called reduction of order. At one point in the process we will need to solve first order ODEs that will be expressed with independent variable $x$ and dependent variable $v$. An example of such an equation is the rather harmless looking equation

$$
x v^{\prime}+v=0
$$

Solving this requires a bit of delicate handling - you will be led through the process in this exercise.
(a) Separate the variables, noting that $x$ cannot be zero.
(b) When integrating the right side, note that there is a negative sign that can be taken out of the integral.
(c) The result of integrating the right side is $-\ln |x|$. Apply the property of logarithms stating that $\log \left(u^{c}\right)=c \log u$, and combine constants as usual.
(d) " $e$ " both sides and apply the fact that $|u|^{r}=\left|u^{r}\right|$ when $u^{r}$ is defined.
(e) Apply the fact that if $|u|=C|v|$, then $u= \pm C v$, absorb the $\pm$ into the constant, get rid of the negative exponent and you are done!
7. Solve each of the following ODEs. You will use separation of variables and partial fraction decomposition for each.
(a) $\left(x^{2}+3 x\right) \frac{d y}{d x}=2 x+3$
(b) $\left(x^{2}+3 x\right) \frac{d y}{d x}=3$
(c) $\left(x^{2}-3 x-10\right) \frac{d y}{d x}=-14$ (see Exercise 2(b) from Appendix A. 4 to check your partial fraction decomposition)
8. In this exercise you will solve the differential equation $\frac{d y}{d x}=-\frac{1}{3} y^{2}+y$ with various initial values. This will lead into Sections 2.2 and 2.4 , and will illustrate the sort of calculations that we must perform to solve certain applied problems related to something called the logistic equation. Some of these calculations are not really needed but make the expressions involved a bit simpler.
(a) Solve this system by separation of variables and partial fraction decomposition. (Be sure to begin by multiplying both sides by -3 to clear the fraction.) This situation is a bit different from the other ones you've encountered, in that you will be doing the partial fraction decomposition with the dependent variable this time.
(b) Check that your answer to (a) agrees with the solution given in the back of the book. Now " $e$ " both sides, putting the right side in the form demonstrated in Example 2.1(b). As in that example, the absolute value can be removed.
(c) Now here comes a bit of algebra: Get rid of the fraction on the left by multiplying both sides by its denominator. Multiply both sides by $e^{-x}$ and then solve for $y$. This is the solution to the ODE.
(d) Determine the values of the constant, and then the solution to the corresponding initial value, for each of the following initial conditions:

$$
y(0)=-\frac{1}{2}, \quad y(0)=0, \quad y(0)=1, \quad y(0)=4
$$

DO NOT give your answers as complex fractions: Multiply the numerator and denominator both by the same value in order to eliminate smaller fractions within them.
(e) Use your calculator or a graphing utility like www.desmos.com to graph your solutions. (For $y(0)=-\frac{1}{2}$ we want only the part of the solution that goes through that initial value.) Sketch a single grid with all four solutions on it. We call these solution curves for the ODE, each corresponding to a different initial value. In Section 2.4 we will see how to obtain these curves without even solving the ODE!
(f) Remembering that $e^{-x} \rightarrow 0$ as $x \rightarrow \infty$, give the limit of each of your solutions from part (d). This should agree with what you see in the graph from (e).
(g) Attempt to determine the value of the constant for the initial condition $y(0)=3$. What happens/what do you get?

### 2.2 Solution Curves and Direction Fields

## Performance Criteria:

2. (c) Sketch solution curves to an ODE for different initial values. Given a set of solution curves for a first order ODE, identify the one having a given initial value.
(d) Sketch a small portion of the direction field for a first order ODE.
(e) Given the direction field and an initial value for a first order IVP, sketch the solution curve.

Suppose that a tank contains 80 gallons of water with 10 pounds of salt dissolved in it. Fluid with a 0.3 pounds per gallon salt concentration is being pumped into the tank at a rate of 7 gallons per minute. The fluid is continually mixed and, at the same time, the fluid is being drained from the tank at a rate of 7 gallons per minute. (This is similar to the situation from Example 1.1(c).)

A quick computation reveals that the initial concentration of the solution in the tank is 0.125 pounds per gallon, less than the concentration of the fluid that is replacing it. Therefore the concentration of the fluid in the tank will increase, but it can never exceed the concentration of the incoming fluid. If all of the fluid in the tank had the concentration of the incoming fluid, there would be $(0.3)(80)=24$ pounds of salt. If we were then to graph the amount of salt in the tank as a function of time we would get the solid curve graphed below and to the right. The limit of the amount of salt in the tank is 24 pounds, indicated by the dotted line.

Now suppose that the tank had $A_{0}=60$ pounds of salt initially, giving an initial concentration of 0.5 pounds per gallon, higher than the concentration of the incoming fluid. In this case the amount of salt in the tank will decrease, with a limit of 24 pounds again. The dashed line on the graph to the right shows the amount of salt as a function of time, for the initial amount of 60 pounds. The dotted curve is the solution curve for the initial amount of 24 pounds, and it is also the asymptote for the other solutions.


As we saw before, the situation with the tank can be modeled with a differential equation, and the general solution to that differential equation is a family of functions. The graphs of the functions in the family are called solution curves for the ODE. Each curve is associated with a particular initial value. The graph above shows the graphs of the solutions for the initial values 10,24 and 60 pounds of salt in the tank. Notice that none of the solution curves cross each other; this is not always the case, but will be for most of the ODEs that we'll look at. For an initial salt amount of 15 pounds, the solution curve will lie midway between the curves for initial amounts of 10 and 24 pounds, without crossing either.

In the exercises you will use your calculator or a graphing utility to plot solution curves for various ODEs.

## Direction Fields

To obtain a graph like the one above we need to either find actual solutions to the differential equation for various initial values, or we have to have a good intuitive idea of what is happening. What if we don't have either of those two things? Well, for first order equations it is usually fairly easy to determine what solution curves look like from just the differential equation itself, as we will now see.

Consider the first order linear differential equation $\frac{d y}{d x}+y=x$. We can solve for $\frac{d y}{d x}$ to get $\frac{d y}{d x}=x-y$. Now remember that by "solving" the differential equation we mean finding a function $y=y(x)$ that makes the equation true; there are infinitely many such functions, with the graph of each representing a particular solution curve. Recall also that when considering the graph of a function, the derivative of the function at some point is the slope of the tangent line to the graph of the function at that point. So the equation $\frac{d y}{d x}=x-y$ gives us a formula for finding the slope of the tangent line to the unknown function $y(x)$ at any point $(x, y)$.

To be more specific, consider the point $(3,1)$. The equation

$$
\frac{d y}{d x}=x-y
$$

tells us that at that point the slope of the tangent line to the solution curve will be

$$
\left.\frac{d y}{d x}\right|_{(3,1)}=3-1=2
$$

So the tangent line of the solution curve passing through the point $(3,1)$ has slope 2 at that point. The dotted line to the right has slope two and passes through the point $(3,1)$. We will just keep the small part of it that actually goes through the point. In the following example we continue on to find slopes at other points with integer coordinates.

$\diamond$ Example 2.2(a): Find slopes for the remaining grid points, for $\frac{d y}{d x}=x-y$.

## Another Example

Solution: It is often easiest to determine slopes not by going point to point, but to find all points where the slope is the same. For this equation, the slope will be zero at every point where $x=y$, so at $(0.0,(1,1),(2,2)$, and so on. Similarly, the slope will be one at all the points where $x$ is one unit larger than $y$; for the above grid those are the points $(1,0),(2,1),(3,2)$ and $(4,3)$. Similarly, the slope will be two at the points where $x$ is two greater than $y$ : $(0,-2),(1,-1),(2,0),(3,1)$ and $(4,2)$. The slope lines for slopes zero, one and two are plotted on the grid shown to the right. The remaining slopes can be seen on the left graph at the top of the next page.


The graph of the result of what we have been doing is something called a direction field or slope field; the completed direction field can be seen to the left at the top of the next page. Direction fields are a way of studying the behavior of solutions to first order differential equations without actually solving the equations analytically. The slope lines that we have drawn in on the direction field are not all that are possible - such a slope line exists for every point in the plane where the derivative exists. Given a direction field and an initial value, we can sketch a solution curve by drawing a curve that starts at the initial value point and that is tangent to the "imagined" slope lines at all points that a curve
goes through. To the right below you can see the solution curves corresponding to the initial values $y(0)=2, y(0)=-1$ and $y(0)=-3$. Each curve is began by sketching a curve that is tangent to the slope line at the initial value, then continues to be tangent to other slope lines it passes through or near as the curve is constructed.

Video Example



## Section 2.2 Exercises

## To Solutions

1. The general solution to the ODE $\frac{d y}{d x}+y=x$ from Example 2.2(a) is $y=x-1+C e^{-x}$. Find the values of the constant $C$ and graph the solution curves for each of the following initial values. Sketch each of the curves on the same grid as each other, for $-1 \leq x \leq 4$. Use a graphing tool if you wish.
(a) $y(0)=2$
(b) $y(0)=0$
(c) $y(0)=-1$
(d) $y(0)=-3$
2. On your graph from Exercise 1, sketch what you think the graphs for initial conditions $y(0)=-2$, $y(0)=1$ and $y(0)=3$ would look like. Then graph them with a graphing tool to check yourself. (You will need to find the values of $C$ for each to do this.)
3. The graph of some solution curves for a differential equation are shown to the right. Give the Roman numeral that corresponds to each given initial condition.
(a) $y(0)=1$
(b) $y\left(-\frac{1}{2}\right)=1$
(c) $y(1)=\frac{1}{2}$
(c) $y(0)=0$

4. (a) On the grid for Exercise 3, sketch in what you think the solution curve for the initial value $y(0)=\frac{3}{4}$ would look like.
(b) The general solution for the ODE for which some solution curves are shown in Exercise 3 is $y=\frac{1}{2}(\sin x+\cos x)+C e^{x}$. Determine the value of $C$ for the initial value $y(0)=\frac{3}{4}$ and plot the solution curve using technology. Compare with the curve you sketched for part (a).
5. (a) The graph below shows some solutions to $\frac{d y}{d x}=x y^{2}$. Label each that you can with its initial value $y(0)=$ $\qquad$ .
(b) The solution to the ODE is $y=\frac{-2}{x^{2}+C}$. Find a point on one of the curves for which you couldn't find an initial value and substitute it into the solution to determine the value of $C$.
(c) Use technology to graph the solution, for the value of $C$ that you found in (b). Explain what is going on. What are the asymptotes for the parts of the graph that go out the top and bottom edges of the grid?

6. Suppose that a group of $N_{0}$ individuals is put in an environment that can only support $K$ individuals, and suppose that the growth rate of the population without any restrictions would be $r$ percent (in decimal form!) per year. Then the population $N$ at any time $t$ years is given by

$$
N=\frac{K}{1-\left(1-K / N_{0}\right) e^{-r t}}
$$

The value $N_{0}$ is called the initial population, $K$ is called the carrying capacity. Suppose that for some population the carrying capacity is 100 and the growth rate is $20 \%$. Graph the functions $N$ for the initial populations below all on the same grid, for zero to forty years, using technology. Your graph will need to go up to at least 150 individuals. Sketch the graph.
(a) $N_{0}=20$
(b) $N_{0}=150$
(c) $N_{0}=100$
(d) $N_{0}=0$
7. Think about the graph you got in the previous exercise, and make sure that you understand (from a population growth point of view) why each curve looks the way it does.
8. For each ODE given, plot the direction field at integer coordinates over the values given for each variable.
(a) $\frac{d x}{d t}=\frac{1}{2} x t, \quad-1 \leq t \leq 2, \quad-2 \leq x \leq 2$
(b) $\frac{d y}{d x}=x^{2}-2 x, \quad 0 \leq x \leq 4, \quad-2 \leq y \leq 4$
(c) $\frac{d y}{d x}=y^{2}-2 y, \quad 0 \leq x \leq 4, \quad-2 \leq y \leq 4$
(d) $\frac{d y}{d t}=y+t . \quad-2 \leq t \leq 2,-2 \leq y \leq 2$
9. On the direction field below and to the left, sketch the solution curves going through the given points.
(a) $(-4,5)$
(b) $(-2,-2)$
(c) $(6,-4)$


10. On the direction field above and to the right, sketch the solution curves going through the given points.
(a) $(-2,0)$
(b) $(-1.5,-1)$
(c) $(3,0)$

### 2.3 Solving With Integrating Factors

## Performance Criterion:

2. (f) Use an integrating factor to solve a first order linear ODE or IVP.

Let's begin with an example that demonstrates the limitation of separation of variables.
$\diamond$ Example 2.3(a): Solve $\frac{d y}{d x}-3 y=e^{5 x}$.
Solution: Note that if we try to separate the variables we get

$$
\begin{aligned}
\frac{d y}{d x}-3 y & =e^{5 x} \\
\frac{d y}{d x} & =3 y+e^{5 x} \\
d y & =\left(3 y+e^{5 x}\right) d x
\end{aligned}
$$

Here we see that there is no way to get the $3 y$ term back over to the left side with $d y$. (This is because $3 y+e^{5 x}$ cannot be written in the form $g(x) h(y)$.) Therefore this equation cannot be solved by separation of variables.

The following derivative computation provides the key for solving equations like the one above.
$\diamond$ Example 2.3(b): Suppose that $y=y(x)$ is some function of $x$. Find the derivative of $y e^{-3 x}$ (with respect to $x$ ).

Solution: Because both $y$ and $e^{-3 x}$ are functions of $x$, we must use the product rule:

$$
\frac{d}{d x}\left(y e^{-3 x}\right)=y \frac{d}{d x}\left(e^{-3 x}\right)+e^{-3 x} \frac{d}{d x}(y)=-3 y e^{-3 x}+e^{-3 x} \frac{d y}{d x}=e^{-3 x}\left(\frac{d y}{d x}-3 y\right)
$$

Notice that multiplying the left side of the ODE of Example 2.3(a) by $e^{-3 x}$ gives the result of Example 2.3(b). This indicates an idea for solving the ODE:

Video Example

$$
\begin{aligned}
\frac{d y}{d x}-3 y & =e^{5 x} & & \text { Original equation } \\
e^{-3 x}\left(\frac{d y}{d x}-3 y\right) & =e^{-3 x} e^{5 x} & & \text { Multiply both sides by } e^{-3 x} \\
e^{-3 x} \frac{d y}{d x}-3 e^{-3 x} y & =e^{2 x} & & \text { Distribute } e^{-3 x} \text { and apply } x^{a} x^{b}=x^{a+b} \\
\frac{d\left(y e^{-3 x}\right)}{d x} & =e^{2 x} & & \text { From Example 2.3(b) } \\
d\left(y e^{-3 x}\right) & =e^{2 x} d x & & \text { Multiply both sides by } d x
\end{aligned}
$$

$$
\begin{aligned}
\int d\left(y e^{-3 x}\right) & =\int e^{2 x} d x & & \text { Integrate both sides } \\
y e^{-3 x}+C_{1} & =\frac{1}{2} e^{2 x}+C_{2} & & \text { Carry out the integrations } \\
y e^{-3 x} & =\frac{1}{2} e^{2 x}+C & & \text { Combine constants } \\
y e^{-3 x} e^{3 x} & =\frac{1}{2} e^{2 x} e^{3 x}+C e^{3 x} & & \text { Multiply both sides by } e^{3 x} \\
y & =\frac{1}{2} e^{5 x}+C e^{3 x} & & \text { Apply properties of exponents }
\end{aligned}
$$

Thus the solution to $\frac{d y}{d x}-3 y=e^{5 x}$ is $y=\frac{1}{2} e^{5 x}+C e^{3 x}$. The reason for multiplying both sides by $e^{3 x}$ was to get $y$ alone on the left side.

The method just shown for finding the solution to $\frac{d y}{d x}-3 y=e^{5 x}$ probably seems a bit mysterious, to say the least! This is called the integrating factor method, which we now summarize. Note that it only applies to linear first order ODEs, which can always be put into the form $\frac{d y}{d x}+p(x) y=q(x)$.

## Solving a 1st Order Linear ODE Using An Integrating Factor

To solve a first order ODE of the form $\frac{d y}{d x}+p(x) y=q(x)$,

1) Compute $u=\int p(x) d x$. The integrating factor is $e^{u}$ (not just $u$ ).
2) Multiply both sides of the equation by the integrating factor $e^{u}$. The left side of the differential equation then becomes $\frac{d\left(y e^{u}\right)}{d x}$.
3) Multiply both sides of the equation by $d x$ and integrate both sides. The left side will become $y e^{u}$.
4) Solve for $y$ by multiplying both sides by $e^{-u}$.

Note that after integrating both sides of the equation there will be a constant added to the right side. This constant will be multiplied by $e^{-u}$ in the solution. For the equation $\frac{d y}{d x}-3 y=e^{5 x}$, $p(x)=-3$ so $u=\int p(x) d x=-3 \int d x=-3 x$ and $e^{u}=e^{-3 x}$.

Any first order linear ODE can be solved using the integrating factor method, as long as $p(x)$ and $e^{u} q(x)$ can be integrated; sometimes you can use either this method or separation of variables and they both will work. Now let's take a look at executing the above steps with another example.

Example 2.3(c): Solve $\frac{d y}{d x}+\frac{y}{x}=x^{2}$ for $x>0$ by the integrating factor method.
Solution: First we note that $p(x)=\frac{1}{x}$ and $q(x)=x^{2}$. Because $x>0,|x|=x$ and $u=\int \frac{1}{x} d x=\ln x$. Therefore $e^{u}=e^{\ln x}=x$. We now carry out steps (2) through (4) above, as shown at the top of the next page.

$$
\begin{aligned}
\frac{d y}{d x}+\frac{y}{x} & =x^{2} & & \text { original equation } \\
x \frac{d y}{d x}+y & =x^{3} & & \text { multipy both sides by } e^{u}, \text { which in this case is } x \\
\frac{d(x y)}{d x} & =x^{3} & & \text { use the product rule "in reverse" to "collapse" the left side } \\
\int d(x y) & =\int x^{3} d x & & \text { multiply both sides by } d x \text { and integrate } \\
x y & =\frac{1}{4} x^{4}+C & & \text { include a single constant of integration on the right side } \\
y & =\frac{1}{4} x^{3}+\frac{C}{x} & & \text { multiply both sides by } e^{-u}=\frac{1}{e^{u}}=\frac{1}{x}
\end{aligned}
$$

Note that in the next to last step we simply put the constant on the right that results from combining the constants from both sides. From here on we will simply put a constant on one side (usually the right side) when we integrate both sides of an equation.

\section*{| Section 2.3 Exercises | To Solutions |
| :--- | :--- |}

1. Solve the IVP $\frac{d y}{d x}-3 y=e^{5 x}, \quad y(0)=-1$.
2. (a) Use an integrating factor to solve $y^{\prime}+2 y=0.4 e^{-2 t}$. (Note that $y$ is now a function of t.) Solve for $y$.
(b) Solve the IVP $y^{\prime}+2 y=0.4 e^{-2 t}, \quad y(0)=3$.
3. (a) Use an integrating factor to solve $\frac{d y}{d x}-\frac{1}{2} y=0$.
(b) Solve the same ODE by separation of variables. Solve for $y$ and compare with your answer to (a) (and take any action that might be suggested by this comparison!).
(c) Solve the IVP $\frac{d y}{d x}-\frac{1}{2} y=0, \quad y(0)=\frac{3}{2}$.
4. (a) Solve the ODE $y^{\prime}-5 y=3 \cos 2 t$. Use your formula sheet to avoid some very messy integration.
(b) Solve the IVP $y^{\prime}-5 y=3 \cos 2 t, \quad y(0)=-4$.
5. (a) Solve the ODE $\frac{d y}{d t}+3 y=t^{2}+5 t-1$.
(a) Solve the IVP $\frac{d y}{d t}+3 y=t^{2}+5 t-1, \quad y(0)=2$.
6. The IVPs from Exercises 3(b) and 3(e) of Section 2.1 couldn't be solved by separation of variables, but they can be done with integrating factors. You will do them here.
(a) Solve the IVP $\frac{d y}{d x}-y=e^{3 x}, \quad y(0)=4$.
(b) Solve the IVP $x \frac{d y}{d x}-y=x, \quad y(1)=2$. Begin by multiplying through by $\frac{1}{x}$.
7. The IVP $y^{\prime}-2 x=x y, y(2)=5$ from Exercise 3(d) of Section 3.1 can be solved by both separation and using an integrating factor. Solve it using an integrating factor. Be sure to get it in the right form before multiplying by the integrating factor!
8. In this exercise you will see another method for solving the ODE $y^{\prime}-5 y=3 \cos 2 t$ from Exercise 4. This method will be used later when we solve second order ODEs.
(a) The equation $y^{\prime}-5 y=0$ is the homogeneous equation associated with $y^{\prime}-5 y=3 \cos 2 t$. Substitute $y=C e^{r t}$ into the homogenous equation to determine what value $r$ must have in order for $y=C e^{r t}$ to be a solution. For that value of $r, y=C e^{r t}$ is called the homogeneous solution to $y^{\prime}-5 y=3 \cos 2 t$.
(b) Find the values of $A$ and $B$ for which $y=A \sin 2 t+B \cos 2 t$ is a solution to $y^{\prime}-5 y=$ $3 \cos 2 t$. Do this as follows:

- Find $y^{\prime}$ and substitute it and $y$ into the differential equations to get an equation involving sines and cosines of $2 t$.
- Combine the like terms on the left side of the equation to get only one sine term and one cosine term.
- You will need to note that on the right side of your equation $3 \cos 2 t$ is the same as $3 \cos 2 t+0 \sin 2 t$. Equate the coefficient of $\cos 2 t$ on the left side with the coefficient of $\cos 2 t$ on the right side to get an equation involving the unknowns $A$ and $B$. Then repeat for sine to get another equation with $A$ and $B$.
- Solve two equations for the two unknowns $A$ and $B$. The resulting $y=A \sin 2 t+$ $B \cos 2 t$ is called the particular solution to $y^{\prime}-5 y=3 \cos 2 t$.
(c) Write the sum of the homogeneous and particular solutions. This is known as the general solution, and should match what you found in Exercise 4(a).

9. Use the method of Exercise 8 to solve the ODE $\frac{d y}{d x}-3 y=e^{5 x}$ from Exercise 1 with this difference: For part (b), find the value of $A$ for which $y=A e^{5 x}$ is a solution to the ODE.
10. Use the method of Exercise 8 to solve the ODE $\frac{d y}{d t}+3 y=t^{2}+5 t-1$ from Exercise 5, but for part (b), find the values of $A, B$ and $C$ for which $y=A t^{2}+B t+C$ is a solution to the ODE.

### 2.4 Phase Portraits and Stability

## Performance Criteria:

2. (g) Determine whether an ODE is autonomous.
(h) Create a one-dimensional phase portrait for an autonomous ODE.
(i) Determine critical points/equilibrium solutions of an autonomous ODE, and identify each as stable, unstable or semi-stable.
(j) Sketch solution curves of an autonomous ODE for various initial values.

Recall that a first order ODE is called autonomous if it can be written in the form $\frac{d y}{d x}=f(y)$. That is, when we get the derivative alone on the left side of the equation, the right hand side is a function of only the dependent variable. $\frac{d y}{d t}=y^{2}-2 y$ is an example of an autonomous ODE. Note that $\frac{d y}{d t}=0$ whenever $y=0$, so if we had an initial condition of $y(0)=0$, then the value of $y$ would never change because the rate of change with respect to time is zero. Therefore the solution to the IVP

$$
\frac{d y}{d t}=y^{2}-2 y, \quad y(0)=0
$$

is the constant function $y=0$. We have solved the IVP without doing any calculations! Now suppose that, for the same ODE, $y(0)=2$. We see that when $y=2$ we again have $\frac{d y}{d t}=0$, so the value of $y$ will again not change. This means that $y=2$ is the solution to the IVP

$$
\frac{d y}{d t}=y^{2}-2 y, \quad y(0)=2
$$

The graph to the right shows the two solution curves that we have obtained so far for the ODE $\frac{d y}{d t}=y^{2}-2 y$, for the initial values $y(0)=0$ and $y(0)=2$. (The dots represent the initial values themselves - note the position of zero on the horizontal axis.) We will call constant solutions like those two equilibrium solutions; the word equilibrium essentially meaning unchanging as time goes on. The question that should occur to you is "What happens for other initial values of $y$ ?" With a little thought we should be able to figure that out. There are three key observations we can make that will help answer the question:


Video Discussion/Example

- The direction field depends only on $y$ alone, so for any given value of $y$ the slope remains constant.
- The right hand side of the ODE can be factored to $y(y-2)$. From that we see that if $y<0$ both $y$ and $y-2$ will be negative, so $\frac{d y}{d t}=y(y-2)$ will be positive. Therefore any solution with an initial value less than zero will be increasing. When $0<y<2, y$ is positive and $y-2$ is negative, so $\frac{d y}{d t}$ is negative and any solution with an initial value between zero and two will be decreasing. Finally, when $y>2$ we have $\frac{d y}{d t}>0$ because both $y$ and $y-2$ are positive when $y>2$. Any solution with an initial value greater than two will be increasing.
- Whether positive or negative, the value of $\frac{d y}{d t}$ approaches zero as $y$ gets nearer to either zero or two. Therefore the direction field lines become "flatter" (closer to horizontal) for values of $y$ close to zero and two.

From these observations we can deduce that the direction field for $\frac{d y}{d t}=y^{2}-2 y$ has the appearance shown to the left below. The direction field with solutions corresponding to four different initial conditions is shown in the center.


We will summarize the information in the three bullets above with something called a phase portrait, shown above and to the right. (Technically it is a one-dimensional phase portrait. Those of you taking the second term of this course may see two-dimensional phase portraits.) The vertical line indicates $y$ values, with the two critical points zero and two indicated. The critical points divide the line into three intervals, and the arrow in each interval indicates whether $y$ is increasing or decreasing, in the particular interval, as time increases.

Look again at the direction field in the middle above, with the four solutions curves drawn in. Note that solutions with an initial $y$ value less than two (including less than zero) all tend toward the constant solution $y=0$, as the phase portrait tells us they will. When this occurs we say that the solution $y=0$ is a stable equilibrium solution. You can sort of think that if we have an initial condition of zero the solution will be zero, and if we "bump off" from zero a bit with our initial condition, the new solution obtained will tend back toward zero as time goes on. This is indicated by putting a solid dot at $y=0$ on the phase portrait.

On the other hand you can see that if $y$ starts with the value two it will remain two, but if it starts at any value close to, but not equal to, two the solution will diverge away from two. For this reason we call the solution $y=2$ an unstable equilibrium solution and we indicate it on the phase portrait with an open circle at $y=2$. Other language you might hear is that $y=0$ is a stable critical point on the phase diagram, and $y=2$ is an unstable critical point. As just stated, on our phase portraits we will indicate stable critical points with solid dots, and unstable critical points with open circles. (We will also use open circles for semi-stable equilibria, which you will see later.)

Again, the entire analysis you have just seen is based on the fact that the ODE is autonomous. Two points of interest here are

- Autonomous first order ODEs are not just a curiosity - they occur naturally in many applications.
- Autonomous ODEs can be difficult or impossible to solve. However, an analysis like we just did can make it very easy to determine how solutions to such an equation behave, in a qualitative sense.

It is important to be able to recognize autonomous differential equations.
$\diamond$ Example 2.4(a): Determine which of the following first order ODEs are autonomous:

$$
\frac{d y}{d x}=y^{2}-x \quad y^{\prime}=2 y-1 \quad \frac{d y}{d t}=t^{2}-5 t+1 \quad x \frac{d x}{d t}=x+1
$$

Solution: Clearly the first equation is not autonomous and the second is. The third and fourth equations might be a bit confusing, as the variables are no longer $x$ and $y$. In the third equation the dependent variable is $y$ and the independent variable is $t$. Because the right hand side is a function of only the independent variable $t$, the equation is not autonomous. In the fourth equation the dependent variable is $x$, the independent variable is $t$, and the equation can be rewritten as $\frac{d x}{d t}=\frac{x+1}{x}$. The right hand side is then a function of $x$ alone, so the equation is autonomous.

If we can determine the phase portrait for an autonomous ODE, we then have a pretty good idea what all solutions to the ODE look like, without having to go to the trouble of creating a direction field. The next example shows how this is done.

Example 2.4(b): Sketch the phase portrait for $\frac{d y}{d t}=-y^{3}+6 y^{2}-9 y$ and use it to sketch solution curves for the initial conditions $y(0)=4, y(0)=2$ and $y(0)=-1$. Identify each equilibrium solution as stable or unstable.

Solution: We factor the right hand side of the ODE, starting by factoring $-y$ out:

$$
\frac{d y}{d t}=-y^{3}+6 y^{2}-9 y=-y\left(y^{2}-6 y+9\right)=-y(y-3)^{2}
$$

From this we can see that the equilibrium solutions are $y=0$ and $y=3$. Testing values of $y$ in each of the three intervals $(-\infty, 0),(0,3)$ and $(3, \infty)$ gives us the following:

$$
\text { - When } y<0, \frac{d y}{d t}>0 \quad \text { - When } 0<y<3, \frac{d y}{d t}<0 \quad \text { - When } y>3, \frac{d y}{d t}<0
$$

This gives us the phase portrait shown to the left below, which indicates that there are equilibrium solutions of $y=0$ and $y=3$. If $y(0)<0$ the solution is increasing, but will approach the equilibrium solution $y=0$; if $0<y(0)<3$ the solution decreases toward $y=0$. If $y>3$ the solution also decreases, but toward $y=3$. These behaviors are shown to the right below, for the three solutions with the given initial values. The solution $y=0$ is a stable equilibrium solution, and the solution $y=3$ is neither stable nor unstable, but is what we call a semi-stable equilibrium solution.



## Section 2.4 Exercises

## To Solutions

1. Determine which of the following first order ODEs are autonomous.
(a) $\frac{d y}{d t}-2 y=0$
(b) $\frac{d y}{d x}+x y=1$
(c) $\frac{1}{y} \frac{d y}{d x}+y=3$
(d) $y^{\prime}=y^{2}-7 y+10$
(e) $\frac{1}{x} \frac{d y}{d x}+y=1$
(f) $\frac{d x}{d t}+6 x^{2}=x^{3}+9 x$
2. For each of the ODEs in Exercise 1 that are autonomous,

- draw a phase portrait
- on a single separate graph, sketch in all equilibrium solution curves, and one solution curve with an initial value in each interval of the real line created by the values of the equilibrium solutions
- give all equilibrium solutions and, for each, tell what kind of equilibrium it is

3. In (i) below, the graph of some solution curves to an ODE are given. (ii) and (iii) are phase portraits for two other ODEs.
(i)

(ii)

(iii)

(a) For the solution curves in (i) above, identify each equilibrium solution and tell whether it is a stable equilibrium, unstable equilibrium, or semi-stable equilibrium.
(b) Repeat (a) for the phase portrait (ii).
(c) Repeat (a) for the phase portrait (iii).
4. (a) Draw the phase portrait for the ODE with solution curves shown in (i) of the previous exercise.
(b) Draw some solution curves for the ODE whose phase portrait is shown in (ii) of the previous exercise. Be sure to include curves for the equilibrium solutions and at least one solution with initial value in each of the intervals created by the equilibrium points.
(c) Draw some solution curves for the ODE whose phase portrait is shown in (iii) of the previous exercise. Be sure to include curves for the equilibrium solutions and at least one solution with initial value in each of the intervals created by the equilibrium points.
5. (a) Give an ODE of the form $\frac{d y}{d x}=f(y)$, with $f(y)$ in factored form, that could have the solution curve graph shown in Exercise 3(a).
(b) Repeat (a) for the phase portrait shown in Exercise 3(b).
(c) Repeat (a) for the phase portrait shown in Exercise 3(c).
6. In Exercise 7 of Section 2.1 you solved the ODE $\frac{d y}{d x}=-\frac{1}{3} y^{2}+y$. When working with it in this exercise, you will find it useful to factor $-\frac{1}{3}$ out of the right hand side first.
(a) When we tried to solve the ODE with the initial value $y(0)=3$ we were not able to obtain a solution. What do we now know happens for that initial condition?
(b) Give the equilibrium solutions, and what kind each is.
(c) Sketch the phase portrait and some solution curves for the ODE.
7. In the next section the ODE

$$
\frac{d A}{d t}=2.1-0.0875 A
$$

will arise in a mixing problem. Give all equilibrium points and identify each as stable, semi-stable or unstable. The sketch a phase portrait and a graph of solution curves for the initial values $A=10, A=40$ and $A=80$.
8. Consider the ODE $\frac{d T}{d t}=-k(T-50)$, where $k>0$. Sketch a graph of the equilibrium solution and several other solution curves with initial values different from that of the equilibrium solution.
9. Sketch several solution curves for the ODE $\frac{1}{4} \frac{d i}{d t}+15 i=12$, including any equilibrium solutions.

### 2.5 Applications of First Order ODEs

## Performance Criteria:

2. (k) Solve an applied problem modeled by a first order ODE using separation of variables or an integrating factor.
(I) Give an ODE or IVP that models a given physical situation involving growth or decay, mixing, Newton's Law of Cooling or an RL circuit.
(m) Sketch the graph of the solution to a mixing or Newton's Law of Cooling problem, indicating the initial value and the steady-state asymptote.
(n) Identify the transient and steady-state parts of the solution to a first order ODE.

## Radioactive Decay and Population Growth

In general, we can assume that the rate at which a quantity of radioactive material decays is proportional to the amount present. For example, $20 \%$ of the material might decay in any 600 year period. If there were 1000 pounds initially, 200 pounds would decay over 600 years, but if there were only 100 pounds initially, only 20 pounds would decay over 600 years. If we let $A$ represent the amount of material at any time $t$, then the rate at which the material decays is given by the derivative $\frac{d A}{d t}$. The above discussion tells us that there is some constant of proportionality $r$ for which

$$
\frac{d A}{d t}=r A
$$

We will find that $r$ is negative because the amount $A$ (which is positive) is decreasing.
Similarly, suppose that $N$ represents the number of individuals (which could be people or any other animals) in a region, and assume the that the population is growing. If there were no constraints like famine, disease and such, the population should grow continuously. The derivative $\frac{d N}{d t}$ would represent the rate of change of population with respect to time. When the population is small we would expect a small change in population over a fixed time period, but when the population is large we'd expect a greater increase in population over the same time period, because there is a larger population having offspring. We'd again expect the rate of change to be proportional to the population itself, resulting in the differential equation $\frac{d N}{d t}=r N$, but in this case the constant $r$ would be positive because here there is growth rather than decay.

Clearly the differential equations for both radioactive decay and population growth are the same, and both can easily be solved by separation of variables, or even just by guessing, as long as we remember that the solution must contain an arbitrary constant! The arbitrary constant is in addition to the constant $r$; the additional constant is introduced by the fact that we must essentially integrate once to determine the function $N=N(t)$.
$\diamond$ Example 2.5(a): Five hundred rainbow trout are introduced into a previously barren (no fish in it) lake. Three years later, biologists estimate that there are 1730 rainbow trout in the lake. Assuming that the population satisfies the ODE $\frac{d N}{d t}=r N$, determine the function $N=N(t)$ that gives the number $N$ of rainbow trout in the lake at time $t$.

Solution: The ODE $\frac{d N}{d t}=r N$ says that we are looking for a function $N(t)$ whose derivative is $r$ times the function itself, and $N=e^{r t}$ clearly is such a function. This function contains no constant of integration, but we recall that $N=C e^{r t}$ is also a solution, for any value of $C$. The general solution is then $N=C e^{r t}$. The fact that there are 500 fish in the lake at time zero gives us $500=C e^{0}$, so $C=500$ and $N=500 e^{r t}$.

To find $r$ we substitute $N=1730$ when $t=3$ into our solution to get $1730=500 e^{3 r}$, and solve to find $r=0.414$. The equation for the number of rainbow trout at any time $t$ is then $N=500 e^{0.414 t}$.

Here's how one would solve the differential equation from the last example by separation of variables, with some of the steps combined:

$$
\begin{array}{rll}
\frac{d N}{d t} & =r N & \\
\frac{d N}{N} & =r d t \quad & \text { multiply by } d t \text { and divide by } N \\
\ln N & =r t+C & \begin{array}{l}
\text { integrate both sides - absolute value is not } \\
\text { needed because } N \text { must be greater than zero } \\
\text { exponentiate both sides and apply } x^{a} x^{b}=x^{a+b}
\end{array} \\
N & =C e^{r t} & \begin{array}{l}
\text { exponem }
\end{array}
\end{array}
$$

## Mixing Problems

We've discussed this sort of situation previously, so let's go straight to an example:
$\diamond$
Example 2.5(b): A tank contains 80 gallons of water with 10 pounds of salt dissolved in it. Fluid with a 0.3 pounds per gallon salt concentration is being pumped into the tank at a rate of 7 gallons per minute. The fluid is continually mixed and, at the same time, the fluid is being drained from the tank at a rate of 7 gallons per minute. Letting $A$ represent the amount of salt in the tank, in pounds, sketch a graph of $A$ as a function of time $t$. Label any values you can on the $A$ axis.

Solution: In the next two examples we will set up and solve the initial value problem for this situation analytically, arriving at an equation that will give us the amount of salt at any time $t$. Before doing that it would be good to have some idea of what the behavior of the solution would be. We did this previously, in Example 1.1(j), but let's repeat the reasoning here.

The initial amount of salt in the tank is 10 pounds. We know that as time goes on the concentration of salt in the tank will approach that of the incoming solution, 0.3 pounds per gallon. This means that the amount of salt in the tank will approach $0.3 \mathrm{lbs} / \mathrm{gal} \times 80 \mathrm{gal}=24$ pounds, resulting in the graph shown to the right, where $A$ represents the amount of salt, in pounds, and $t$ represents the time, in minutes.


Now let's set up an IVP and solve it:

Example 2.5(c): Letting $A$ represent the amount of salt in the tank, in pounds, give an initial value problem describing this situation.
Solution: Taking the concentration of salt in the incoming fluid times the rate at which the fluid is coming in, we get that salt is entering the tank at a rate of

$$
\left(0.3 \frac{\mathrm{lbs}}{\mathrm{gal}}\right)\left(7 \frac{\mathrm{gal}}{\mathrm{~min}}\right)=2.1 \frac{\mathrm{lbs}}{\mathrm{~min}}
$$

Now let $A=A(t)$ be the amount (in pounds) of salt in the tank at any time $t$ minutes. Then the concentration of salt in the tank is $\frac{A}{80}$, so by the same sort of calculation that we just did the rate at which salt is leaving the tank is $\frac{7 A}{80}=0.0875 A \frac{\mathrm{lbs}}{\min }$. The net rate of change of salt in the tank is the amount coming in minus the amount going out, or $(2.1-0.0875 A) \frac{\mathrm{lbs}}{\mathrm{min}}$. But this quantity, being a rate of change, is also a derivative. Namely it is $\frac{d A}{d t}$, giving us the differential equation

$$
\frac{d A}{d t}=2.1-0.0875 A
$$

We also have the initial value $A(0)=10$ pounds, so we have the initial value problem

$$
\begin{equation*}
\frac{d A}{d t}=2.1-0.0875 A, \quad A(0)=10 \tag{1}
\end{equation*}
$$

Note that the ODE is autonomous, and the graph obtained in the previous exercise can be obtained from the ODE by the methods of the previous section.
$\diamond$ Example 2.5(d): The ODE in (1) above is autonomous. Determine the equilibrium solution and whether it is stable, unstable, or semi-stable. Sketch a phase portrait for the situation.

Solution: The equilibrium solution occurs when

$$
\frac{d A}{d t}=2.1-0.0875 A=0
$$

Solving $2.1-0.0875 A=0$ for $A$ gives us an equilibrium solution of $A=24$ pounds. When $A<24$ we find that $\frac{d A}{d t}>0$, and when $A>24, \frac{d A}{d t}<0$. Therefore $A=24$ is a stable equilibrium. The phase portrait is shown to the right.

Note that the phase portrait agrees with the solution curve obtained in Example 2.5(b).
We will see differential equations like the one in (1) above in several contexts, and it can always be solved using separation of variables or an integrating factor (and you should be able to do it either way). When we separate variables we get

$$
\frac{d A}{2.1-0.0875 A}=d t .
$$

The left side can be integrated by $u$-substitution, but such integrals come up often enough in practice that we should use the following formula instead, obtained by $u$-substitution:

$$
\begin{equation*}
\int \frac{1}{a x+b} d x=\frac{1}{a} \ln |a x+b|+C \tag{2}
\end{equation*}
$$

We can now use this result to solve the IVP (1).
$\diamond$ Example 2.5(e): Solve the IVP $\frac{d A}{d t}=2.1-0.0875 A, A(0)=10$.
Solution: Multiplying both sides of the ODE by $d t$ and dividing by the quantity 2.1 $0.0875 A$ gives us

$$
\frac{d A}{2.1-0.0875 A}=d t
$$

To use equation (2) from the previous page we note that our left side is like the left side of (2), but with $x=A, a=-0.0875$ and $b=2.1$. The formula then tells us that the left side integrates to $-\frac{1}{0.0875} \ln (2.1-0.0875 A)+C$. (We don't need absolute value because the quantity $2.1-0.0875 A$ is the rate at which the amount of salt is changing, and this is positive due to the concentration of the incoming solution being higher than the initial concentration in the tank.) Thus when we integrate both sides and combine the constants we have

$$
\begin{aligned}
-\frac{1}{0.0875} \ln (2.1-0.0875 A) & =t+C \\
\ln (2.1-0.0875 A) & =-0.0875 t+C \\
2.1-0.0875 A & =e^{-0.0875 t+C} \\
2.1-0.0857 A & =C e^{-0.0875 t} \\
-0.0875 A & =-2.1+C e^{-0.0875 t} \\
A & =24+C e^{-0.0875 t}
\end{aligned}
$$

Applying the initial value $A(0)=10$ we get $10=24+C$, so $C=-14$ and the solution to the IVP is $A=24-14 e^{-0.0875 t}$.

Note that when $t=0$ the solution $A=24-14 e^{-0.0875 t}$ gives us $A=10$, as it should. Also, as $t \rightarrow \infty, A$ goes to 24 as expected.

## Newton's Law of Cooling

## Newton's Law of Cooling

Suppose that a solid object with initial temperature $T_{0}$ is placed in a medium with a constant temperature $T_{m}$, and let $T=T(t)$ be the temperature of the object at any time $t$ after it is placed in the medium. The rate of change of the temperature $T$ with respect to time is proportional to the difference between the temperatures of the medium and that of the object. That is,

$$
\begin{equation*}
\frac{d T}{d t}=-k\left(T-T_{m}\right) \tag{3}
\end{equation*}
$$

for some constant $k>0$. Together with $T(0)=T_{0}$, this gives us an initial value problem for the temperature of the object.

The medium that the object is placed in might be something like air, water, etc., and $T_{m}$ stands for "temperature of the medium," sometimes called ambient temperature. We will always consider situations for which this temperature is constant. Note that if the temperature of the medium is greater
than the temperature of the object, the rate of change of temperature must be positive, which is why $k$ must be positive. A little thought will tell you that $k$ must be positive if the temperature of the medium is less than the temperature of the object as well.
$\diamond$ Example 2.5(f): Suppose that a solid object with initial temperature $88^{\circ} \mathrm{F}$ is placed in a medium with ambient temperature $50^{\circ} \mathrm{F}$, and after one hour the object has a temperature of $65^{\circ} \mathrm{F}$. Determine the equation for the temperature $T$ as a function of the time $t$.

Solution: Because the object's initial temperature of $88^{\circ} \mathrm{F}$ is higher than the ambient temperature of $50^{\circ} \mathrm{F}$, it will cool after being placed in the medium. Newton's Law of Cooling tells us that it cools more rapidly at first, when the temperature difference between the object and the medium is large. As the object cools to temperatures closer to the ambient temperature, the rate of cooling decreases. This is shown by the graph to the right.


Now the initial value problem for this situation is

$$
\frac{d T}{d t}=-k(T-50), \quad T(0)=88
$$

The differential equation becomes $\frac{d T}{T-50}=-k d t$, and the method of Example 2.4(e) gives us $T=50+C e^{-k t}$ before applying the initial condition. Using the initial condition, we obtain the solution $T=50+38 e^{-k t}$. To determine $k$ we apply $T(1)=65$ to get $k=0.930$, so the final solution is $T=50+38 e^{-0.930 t}$.

## Electric Circuits

We will work with some basic electrical circuits in this class. The first sort of circuit we'll look at consists of a voltage source, a resistor and an inductor. The voltage source can be constant (so-called "direct current," or "DC"), or it can be variable, usually in an oscillating manner ("alternating current," or " AC "). The voltage source causes electrons to move in the circuit, and the flow of electrons is called current. (Somewhat confusingly, the current flows in the direction opposite the flow of the electrons.) The voltage source provides an electromotive force which we can think of as sort of "pushing" current through the circuit, analogous to a pump pushing fluid through a network of pipes. The units of the electromotive force are volts. We will use the symbol $i$ for current, and it is measured in units called amperes. ("Amps," for short.)

The resistor has a characteristic called resistance, which is measured in units called ohms. The inductor's characteristic is called inductance, which is measured in henries. Although resistance and inductance could be variable, they will always be constants in our considerations. We will use $E=E(t)$ for the voltage, $R$ for the resistance and $L$ for the inductance. To the right is a schematic diagram of such a circuit. We will usually think of our circuits as having a switch as well, which is "open" (off) until time zero, when it is "closed" (turned on). From that point on the current is (usually) changing, and is a function of time:


Figure 2.5(a) $i=i(t)$.

## RL Circuit

Consider an electric circuit as described above, with an applied voltage $E(t)$ volts (possibly a function of time) and constant resistance of $R$ ohms and constant inductance of $L$ henries. The current $i=i(t)$, in amperes, satisfies the first order linear differential equation

$$
\begin{equation*}
L \frac{d i}{d t}+R i=E(t) \tag{4}
\end{equation*}
$$

Because the ODE (4) is first order linear, it can be solved using an integrating factor. If the voltage $E(t)$ is constant, the equation can be solved by separation of variables as well. Let's examine the case where the voltage source $E(t)$ is a constant function, to observe why mathematics is so powerful in science and engineering. Suppose that $L=\frac{1}{4}$ henry, $R=15$ ohms, and $E=12$ volts; in this case the ODE is $\frac{1}{4} \frac{d i}{d t}+15 i=12$. If we separate the variables we obtain $\frac{d i}{12-15 i}=4 d t$ or $\frac{d i}{48-60 i}=d t$. Let's look at the first of these along with the separated equations arising in Examples 2.5(e) and 2.5(f): Video Example

$$
\frac{d i}{12-15 i}=4 d t \quad \frac{d A}{2.1-0.0875 A}=d t \quad \frac{d T}{T-50}=-k d t
$$

Note that these are all of the form $\frac{d x}{a x+b}=c d t$, where $a, b$ and $c$ are constants. This illustrates the fact that

## physical situations that seem to have nothing in common lead to the same differential equation.

We will see this principle in action again when we study second order ODEs.

## Response of a System

Suppose that we have a circuit like that show in Figure 2.5(a), but without a voltage source $E$ (but with the circuit closed). If there is no current in the circuit initially, then there will not be any current at any future time. However, if there is some current in the circuit initially (which can be made to happen by including a voltage source, then removing it and completing the circuit in its absence), then there will be current in the future as well. Similarly, if we set a mass on a spring (like shown in Example 1.1(a)) in motion it will continue to oscillate for some time.

We will refer to the circuit without a voltage and the mass on a spring as systems. In the case of the circuit the variable of interest is the current in the circuit at any time, and for the mass we are interested in the vertical position at any time. With some initial "stimulus" (non-zero initial conditions) in each case the current or vertical position will vary with time. The current or vertical position will be referred to as the response of the system to the initial conditions.

Now suppose that we have either an $R L$ circuit with a voltage source or a spring-mass system with some outside force pushing or pulling on the mass. The outside force or voltage (which is also sometimes referred to as an electromotive force) we will refer to as a forcing function for the system. In the presence of a voltage source, the circuit will have current at all future times. Similarly, a mass on a spring with a forcing function will continue to oscillate

We will revisit the spring-mass system, along with slightly more complex electrical circuits, in Chapters 3 and 4. For the time being, let's focus on the circuit shown in Figure 2.5(a) and the governing differential equation

$$
\begin{equation*}
L \frac{d i}{d t}+R i=E(t) \tag{5}
\end{equation*}
$$

Here the left side $L \frac{d i}{d t}+R i$ of the equation represents the system, and the right side $E(t)$ represents the forcing function. Our goal is usually to solve an associated initial value problem for the current $i=i(t)$. That current, the solution to the IVP, is the response of the system to the forcing function $E(t)$ and initial current. Another way of thinking about this is that the forcing function and initial condition(s) are "inputs" to the system, and the response is the "output" of the system for that input.

Let us consider for example an $R L$ circuit where again $L=\frac{1}{4}$ henry and $R=15$ ohms, and for which $E(t)=\sin 3 t$. In this case we would not be able to separate the variables, so we'd solve the equation using an integrating factor. In doing so, we would obtain the solution

$$
i=\frac{4}{63} \sin 3 t+C e^{-60 t}
$$

where $C$ is a constant that would be determined by an initial condition. Note that the solution has two parts:

- The part $\frac{4}{63} \sin 3 t$, which is periodic and "goes on forever in the same way." This part of the solution is called the steady-state solution or steady-state response of the system.
- The part $C e^{-60 t}$, approaches zero as time goes on, so it "dies out." Such a solution or part of a solution is called the transient solution or transient response of the system.

To clarify a little, we will define a steady-state solution to be any solution that is either constant (and, to avoid triviality, not zero) or periodic.

In practice, when a system like a machine with moving parts or an electrical circuit is "turned on," it often exhibits a certain behavior as it starts up, which is the transient response of the system. Then it will "settle in" to a steady state behavior or response. (Note that the ideas of transient and steady state solutions only make sense when the independent variable is time.) In general only the steady-state response is important in terms of what we want the system to do from a practical viewpoint, but the transient response might be of interest because it might cause some sort of stress on the system that could cause a problem.

For the scenario described above but with $E$ having the constant value of 12 volts, the solution to the differential equation is

$$
i=\frac{12}{15}+C e^{-60 t}
$$

where $C$ is again determined by the initial current in the circuit. (Ordinarily we would be expected to reduce $\frac{12}{15}$, but we'll leave it as is to see the voltage and resistance.) We can see that here the steady-state solution is $i=\frac{12}{15}$, where 12 is the voltage and 15 is the resistance, saying that "in the long run" the circuit will exhibit Ohm's Law $V=I R \quad\left(\right.$ solved for $\left.I=\frac{V}{R}\right)$. This is because the current approaches a constant value, and the inductor only affects the circuit when there is change in the current, causing flux in the coil of the inductor.
$\diamond$ Example 2.5(g): In Example 2.5(f) we found that when a solid object with initial temperature $88^{\circ} \mathrm{F}$ is placed in a medium with ambient temperature $50^{\circ} \mathrm{F}$, and after one hour the object has a temperature of $65^{\circ} \mathrm{F}$, the temperature $T$ at any time $t$ is given by $T=50+38 e^{-k t}$. Give the transient and steady-state parts of the solution.

Solution: The transient part of the solution is $38 e^{-k t}$ and the steady-state part is 50 .

1. When a person takes a medication, the amount in their body decreases exponentially, in the same way that a radioactive element decays. Suppose that a person takes 80 grams of some medication, and that we somehow know (???) that 12 hours later they still have 23 grams in their system.
(a) Give the initial value problem for this situation, using $A$ for amount in grams and $t$ for time in hours.
(b) The ODE is autonomous - what is the equilibrium solution? Is it stable?
(c) Solve the IVP. Your answer should still contain an unknown constant $k$.
(d) Determine $k$ and give the amount function $A$.
2. An underground storage tank contains 1000 gallons of water with 87 pounds of contaminant in it. At some time we will call time zero, clean water is pumped into the tank at a rate of 300 gallons per hour and the thoroughly mixed solution is pumped out at the same rate.
(a) Set up an IVP for this situation, using $A$ for the amount of contaminant, in pounds, and $t$ for time, in minutes.
(b) Solve the IVP.
(c) Determine when the amount of contaminant has decreased to five pounds.
(d) Give the transient and steady-state solutions.
3. (a) A solid object is placed in a medium with ambient temperature $70^{\circ} \mathrm{F}$. Solve the differential equation (2) for this situation. The constant $k$ will be unknown for now, and there will be another constant that arises as well.
(b) Suppose that the initial temperature of the object is $32^{\circ} \mathrm{F}$. Solve for the constant that arose in solving the ODE.
(c) After one hour the object has a temperature of $58^{\circ} \mathrm{F}$. Use this information to determine the constant $k$. Give units with your answer.
(d) What is the steady-state solution? What is the transient solution?
4. (a) Suppose that the voltage in resistor-inductor series circuit is supplied by a 12 volt battery, so $E(t)=12$. The inductance of the circuit is $\frac{1}{2}$ henry, and the resistance is 10 ohms. At time zero the current in the circuit is zero. Find the current function $i(t)$ by solving the initial value problem just described.
(b) Now suppose that the voltage is variable, with equation $E(t)=10 \sin 2 t$, and the initial current is zero. Solve the IVP.
(c) Give the transient and steady-state parts of your solution to part (b). (Make it clear of course which is which!)
5. In general, the solution to the differential equation for Newton's Law of Cooling is

$$
\begin{equation*}
T(t)=T_{m}+\left(T_{0}-T_{m}\right) e^{-k t}, \tag{5}
\end{equation*}
$$

where $T_{0}$ is the initial temperature.
(a) What happens if $T_{0}=T_{m}$ ?
(b) What is the steady-state solution to (5)?
(c) What is the transient solution to (5)?
6. The ODE $\frac{d A}{d t}=2.1-0.0875 A$ from Example 2.5(c) is autonomous.
(a) Determine all equilibrium solutions, and tell whether each is stable, unstable or semi-stable.
(b) Sketch some solution curves for $A(0)>0$, including some with $A(0)$ greater than the largest equilibrium solution.
7. When an owner arrives home in their car, it is at $29^{\circ} \mathrm{F}$ from being outside all day. The owner parks it in a heated garage, which is at a temperature of $73^{\circ} \mathrm{F}$.
(a) We wish to determine the temperature $T$ of the car as a function of time $t$, assuming it follows Newton's Law of Cooling. Write a differential equation to be solved to find that function, the solution we are looking for. Also, give any initial condition(s).
(b) Will there be a steady-state solution? If so, what will it be?
(c) Solve the initial value problem. Your answer will still contain a constant, but you should be able to determine the value of another.
(d) Suppose that two hours later the owner is ready to go back out in the car, and by that time it has warmed up to $47^{\circ} \mathrm{F}$. Determine the function modeling the temperature of the car in the two hours that it was in the garage.
8. An RL circuit contains an inductor with an inductance of $\frac{3}{4}$ henry and a 15 ohm resistor. It is driven by a variable voltage $E(t)=6 \cos 2 t$, and the initial current in the circuit is 2 amperes.
(a) Give the initial value problem to be solved, and solve it. Determine exact values for all constants.
(b) Give the steady-state and transient solutions.
9. A 150 gallon tank contains a 3 pounds (lbs) per gallon (gal) salt solution. At time zero, solution will begin being pumped out of the tank at a rate of 7 gallons per minute and a 1 pounds per gallon solution will begin being pumped into the tank at the same rate. Assume that there is constant mixing in the tank, so that it has the same concentration all over in the tank at any given time. Let $A$ represent the amount of salt in the tank (in pounds) and let $t$ represent time (in minutes).
(a) Sketch the graph of the amount $A$ of salt in the tank as a function of time, from just the given information.
(b) Give the initial value problem to be solved for $A$, and solve it.
(c) Graph your solution using some technology, and compare with your answer to (a).
(d) Give the steady-state and transient solutions.
10. Suppose that we have an RL circuit with no voltage, as shown to the right. The resistor has a resistance of 8 ohms, and the inductor has an inductance of $\frac{1}{3}$ henry, and there is an initial current in the circuit of 5 amperes.
(a) Solve the initial value problem.
(b) Give the transient and steady-state parts of the solution.


### 2.6 Chapter 2 Summary

- We can "solve" ODEs (and PDEs) in three ways:
- Analytically, which means "paper and pencil" methods that give exact algebraic solutions.
- Qualitatively, which means determining the general behavior of solutions without actually finding function values. Results of qualitative methods are often expressed graphically.
- Numerically, which result in values of solutions only at discrete points in time or space. Results of numerical methods are often expressed graphically or as tables of values.
- The two most commonly applicable methods for solving first order ODEs analytically are separation of variables and the integrating factor method.
- Separation of variables only works for equations that can be written $\frac{d y}{d x}=g(x) h(y)$, and for which the antiderivatives $\int g(x) d x$ and $\int \frac{d y}{h(y)}$ can be determined.
- The integrating factor method only applies to linear first order ODEs. Such ODEs can be put in the form $\frac{d y}{d x}+p(x) y=q(x)$, and to carry out the integrating factor method the antiderivatives $u=\int p(x) d x$ and $\int e^{u} q(x) d x$ must exist.
- Some applications of first order ODEs are population growth and radioactive decay, mixing problems, Newton's Law of Cooling problems, and $R L$ electric circuits.
- Very different physical situations often result in the same differential equation.
- Suppose that the independent variable for an ODE is time, so the solution is a function of time. Any part of the solution that goes to zero as time goes to infinity is called the transient part of the solution, and any part of the solution that is a nonzero constant or periodic is called the steady state part of the solution.
- It is not necessary that all parts of solutions exhibit transient or steady state behavior, but it is often the case that they do.


### 2.7 Chapter 2 Exercises

1. In Example 2.1(b) the ODE $\frac{d y}{d t}+0.5 y=0$ is solved by separation of variables, and it can also easily be solved using an integrating factor.
(a) Instead of either of these, assume that it has a solution of the form $y=C e^{r t}$ and determine the value of $r$ by substituting this solution into the equation. After finding the value of $r$, give the solution to the ODE.
(b) Solve the IVP $\frac{d y}{d t}+0.5 y=0, \quad y(0)=4.7$
2. Consider the situation of Example 2.5(b) with the following change: Suppose that the 0.3 pounds per gallon fluid is coming in at a rate of 5 gallons per minute, rather than 7 gallons per minute. (The mixed fluid is still being drained from the tank at 7 gallons per minute.) The goal here is to determine the amount $A$ of salt in the tank at any time $t$.
(a) Give an expression for the amount of fluid in the tank at any time $t$.
(b) Give an expression for the concentration of salt in the tank at any time $t$.
(c) Give the initial value problem to be solved in order to determine the amount of salt in the tank as a function of time.
(d) The differential equation is linear. What are the functions $p(t)$ and $q(t)$ ?
(e) Solve the differential equation, using the integrating factor method. Graph the solution and make sure it behaves as expected.

## Reduction of Order

The term reduction of order usually refers to a method for finding a second solution to a second order ODE from one solution that is already known. We will use the term more generally, for any process in which one or more ODEs is turned into one or more other ODEs of smaller order. This can be done in a variety of ways, the simplest of which is illustrated in the next few exercises.
3. In this exercise we'll use reduction of order to solve $u^{\prime \prime}+2 u^{\prime}=0$, where the independent variable is $x$. This equation would likely not show up in any application, but it provides us with an easy introduction to how reduction of order works.
(a) We begin by letting $v=u^{\prime}$ where $v$, like $u$, is a function of $x$. What then is $u^{\prime \prime}$ ? Substitute the appropriate expressions in $v$ in for $u^{\prime \prime}$ and $u^{\prime}$, then solve the resulting ODE for $v$. For simplicity, assume that $v \geq 0$. (Make sure you see why I am allowing this assumption!)
(b) Now that you have found $v$, substitute $u^{\prime}$ for $v$ and solve the new ODE. Note that the original ODE is second order, so your solution should have two arbitrary constants.
4. A classic problem in the study of PDEs is the equilibrium distribution of heat in a circular disk. In the course of solving that problem one obtains the ODE

$$
\begin{equation*}
r^{2} \frac{d^{2} R}{d r^{2}}+r \frac{d R}{d r}-n^{2} R=0 \tag{1}
\end{equation*}
$$

Note that $r$ and $R$ are two different variables! $R$ is the dependent variable, and is a function of the independent variable $r$. Later we will see how to solve this equation for $n \neq 0$, but here we will solve for $n=0$.
(a) Write the equation with $n=0$.
(b) Let $S=\frac{d R}{d r}$. What then is $\frac{d^{2} R}{d r^{2}}$, in terms of $S$ ?
(c) Substitute what you were given and what you determined in (b) into your equation from (a) to obtain a first order ODE with dependent variable $S$.
(d) Solve your equation from (c), solving for $S$ eventually.
(e) Replace $S$ in your answer to (d) with $\frac{d R}{d r}$ and solve the resulting first order ODE for $R$. Note that the original ODE (1) is second order, so your solution should have two arbitrary constants.

Here you used reduction of order to start with a second order ODE but make a substitution that gives us a first order equation.

## Logistic Growth

When we assume that a population will increase exponentially for all time, the differential equation for the number $N$ of individuals at time $t$ is

$$
\begin{equation*}
\frac{d N}{d t}=r N \tag{2}
\end{equation*}
$$

where $r$ is a constant that represents the growth rate. (See Example 2.5(a).) This model is somewhat unrealistic, however - usually we expect some upper limit to the population due to the fact that as it gets large it will begin to be constrained by some factor like the food available or disease. Thus the growth rate should approach zero at some point, or even become negative if the population becomes too large. On the other hand, the growth rate should be relatively constant for very small numbers $N$ of individuals. These conditions can be incorporated into our model by including a factor $1-\frac{N}{K}$ for some other constant $K>0$ :

$$
\begin{equation*}
\frac{d N}{d t}=r\left(1-\frac{N}{K}\right) N . \tag{3}
\end{equation*}
$$

Equation (3) is one of several forms of what is called the logistic equation.
5. (a) Equation (3) is autonomous. Determine all equilibrium solutions (in terms of $K$ ), and classify each as stable or unstable. Sketch a phase diagram and some solution curves.
(b) Discuss the significance of the constant $K$.
(c) What effect should changing $r$ have on solution curves? Be as specific as possible. (Hint: Think in terms of the direction field for the equation.)
6. (a) Solve equation (3) for $K=3000$ and $N(0)=500$. You don't need to know the value of $r$ to do this, but $r$ will appear in your solution.
(b) Suppose that $N(4)=2000$, Use this to determine the value of $r$.
(c) Using your solution to (a) with the value of $r$ determined in (b), determine when the population will reach $N=2500$.

## RC Circuits

In Section 2.5 there is a discussion of RL circuits, ones containing a voltage source, resistor and inductor. Another simple circuit of interest is one containing a voltage source, resistor and capacitor, called an RC circuit. Capacitors are devices that store something called charge, which we'll denote as $q$. The units of charge are coulombs. The ability of a capacitor to store charge is quantified by a characteristic called capacitance, denoted by $C$. The units of capacitance are farads.

To the right is a schematic diagram of an RC circuit, which we will also assume has a switch that allows current to begin flowing at some time. The differential equation that models the charge $q$ (in coulombs) "on" (stored by) the capacitor at any time $t$ (in seconds) is

$$
R \frac{d q}{d t}+\frac{1}{C} q=E(t)
$$

where $R$ is resistance in ohms and $C$ is capacitance in farads, and
 $E(t)$ is the voltage, which may or may not be constant.

Finally, we note that the derivative $\frac{d q}{d t}$, the rate at which the charge on the capacitor is changing with respect to time, is the current in the circuit.
7. A 12 volt battery is attached to a circuit containing a $0.5 \mu F$ (microfarad, $10^{-6}$ farad) and an $8 k \Omega$ (kilo-ohm, $10^{3}$ ohm) resistor. At time zero, when the circuit is closed by a switch, the capacitor has a charge of $10^{-9}$ coulombs.
(a) Give the charge $q$ on the capacitor and the current $i$ in the circuit as functions of time $t$ in seconds after closing the switch. Note that constants and variables have to be in terms of volts, ohms, farads in order to get results in terms of coulombs and amperes.
(b) Plot the charge on the capacitor and the current in the circuit as two separate graphs. Indicate clearly any asymptotes.
8. Consider the same situation as the previous exercise, but with the 12 volt battery replaced by a variable voltage source $E(t)=10 \cos 240 \pi t$. Repeat parts (a) and (b) of the previous exercise, but do not expect asymptotic behavior of either the charge or the current.

## Falling Body With Air Resistance

9. Here is another situation which is basically reduction of order, but disguised a little. Recall (from Section 0.2) that the differential equation governing the motion of a falling object (or one that has been projected upward) is

$$
\begin{equation*}
\frac{d^{2} h}{d t^{2}}=-32 \tag{4}
\end{equation*}
$$

where $h$ is the height of the object at any time $t$. For the value -32 on the right hand side, $h$ is measured in feet and $t$ in seconds. Now $\frac{d^{2} h}{d t^{2}}$ is the acceleration due to gravity. Remembering that acceleration is the derivative of velocity, (4) can be rewritten as

$$
\begin{equation*}
\frac{d v}{d t}=-32 \tag{5}
\end{equation*}
$$

The negative sign here is based on a coordinate system where up is positive. For the sake of simplicity, let's consider a falling body (so it never goes upward), and let's take down to be positive. (5) then becomes

$$
\begin{equation*}
\frac{d v}{d t}=32 \tag{6}
\end{equation*}
$$

(4) and (5) are based on the assumption that there is no air resistance, but now lets remove that assumption. A reasonable alternative premise is that the air resistance is proportional to the velocity, but in the opposite (upward, so negative in our new cordinate system) direction. Letting the constant of proportionality be $k>0$, (6) then becomes

$$
\begin{equation*}
\frac{d v}{d t}=32-k v \tag{7}
\end{equation*}
$$

(a) Equation (7) is autonomous; what is the equilibrium solution? (Your answer will contain the constant $k$.) Is it stable, or unstable? Sketch the phase diagram and and a graph of some solution curves. The equilibrium solution is what people are referring to when they talk about terminal velocity.
(b) Solve (7) using one of the methods from this chapter. Take the limit of your solution as $t \rightarrow \infty$ and make sure it matches your equilibrium solution from (a).
(c) Give the equation for the velocity of an object that begins its motion by being dropped with no initial velocity, with the assumption that air resistance is proportional to velocity.

## D Solutions to Exercises

## D. 2 Chapter 2 Solutions

## Section 2.1 Solutions

Back to 2.1 Exercises
1.
(a) $\sin y=-\frac{1}{2} x^{2}+C$
(b) $\frac{1}{2} y^{2}=\frac{1}{2 x^{2}}+C$ or $y^{2}=\frac{1}{x^{2}}+C$
(c) $\frac{1}{5} y^{5}=-\frac{1}{3} x^{3}+C$ or
(d) $\ln |y|=\frac{1}{2} \ln |2 x+3|+C$ $3 y^{5}=-5 x^{3}+C$
(e) $-e^{-y}=-\frac{1}{x}+C$ or
$e^{-y}=\frac{1}{x}+C$
(f) $\frac{1}{2} y^{2}=\frac{5}{2} x^{2}+3 x+C$ or $y^{2}=5 x^{2}+6 x+C$
2. (a) $\ln |y|=\frac{1}{2} x^{2}+\ln 3-\frac{1}{2}$
(b) $\frac{1}{2} x^{2}=-\frac{5}{2} t^{2}+3 t+12$ or $x^{2}=-5 t^{2}+6 t+24$
(c) $e^{y}=\frac{3}{2} x^{2}+e^{2}$
(d) $-\frac{1}{3 y^{3}}=\sin t-\frac{1}{24}$
3. (a) $y=4 e^{3 x}$
(b) not separable
(c) $\ln |y|=2 x^{2}+\ln 2$
(d) $y=\frac{7}{e^{2}} e^{\frac{1}{2} x^{2}}-2$
(e) not separable
(f) $y=\frac{1}{2} x+\frac{5}{2}$
4. (a) $y=\ln \left(x^{2}+x+1\right)$
(b) $y=\ln \left(x^{2}+x+e^{3}-2\right)$
5. (b) $y=C e^{-t^{2}}$
(c) $y=7 e^{-t^{2}}$
6. The final solution is $v=\frac{C}{x}$.
7. (a) $y=\ln |x(x+3)|+C$
(b) $y=\ln \left|\frac{x}{x+3}\right|+C$
(c) $y=\ln \left(\frac{x+2}{x-5}\right)^{2}+C$
8.
(a) $\frac{|y|}{|y-3|}=x+C$ or $\left|\frac{y}{y-3}\right|=x+C$
(b) $\frac{y}{y-3}=C e^{x}$
(c) $y=\frac{3 C}{C-e^{-x}}$
(d) In order, the constants are $C=\frac{1}{7}, 0,-\frac{1}{2}, 4$ and the solutions are $y=\frac{3}{1-7 e^{-x}}, \quad y=0, \quad y=\frac{3}{1+2 e^{-x}}, \quad y=\frac{12}{4-e^{-x}}$
(f) $y \rightarrow 3$ as $x \rightarrow \infty$ in all cases
(g) The value of the constant cannot be determined - the equation to be solved has no solution.

1. Constants for each initial value are given below, graph is to the right.
(a) $C=3$
(b) $C=1$
(c) $C=0$
(d) $C=-2$

2. (a) II
(b) 1
(c) 111
(d) IV
3. (a) The top, U-shaped curve is for initial value $y(0)=1$. The next curve down has initial value $y(0)=-1$, and the one below that has initial value $y(0)=-2$.
(b) Using the point $(2,-1)$, the value obtained for $C$ is -2 . If instead one uses the point $(-2,-1)$, the same value of $C$ is obtained!
(c) When the solution is graphed for $C=-2$, the graph includes the $U$-shaped curve as well as the two curves in the lower left and lower right. They are all parts of the same graph, which has vertical asymptotes at $x=-\sqrt{2}$ and $x=\sqrt{2}$.
4. (a)

(c)

(b)

(d)

5. 


10.


## Section 2.3 Solutions

Back to 2.3 Exercises

1. $y=-\frac{3}{2} e^{3 x}+\frac{1}{2} e^{5 x}$
2. (a) $y=0.4 t e^{-2 t}+C e^{-2 t}$
(b) $y=0.4 t e^{-2 t}+3 e^{-2 t}$
3. (a) and (b) $y=C e^{\frac{1}{2} x}$
(c) $y=\frac{3}{2} e^{\frac{1}{2} x}$
4. (a) $y=\frac{6}{29} \sin 2 t-\frac{15}{29} \cos 2 t+C e^{5 t}$
(b) $y=\frac{6}{29} \sin 2 t-\frac{15}{29} \cos 2 t-\frac{101}{29} e^{5 t}$
5. (a) $y=C e^{-3 t}+\frac{1}{3} t^{2}+\frac{13}{9} t-\frac{22}{27}$
(b) $y=\frac{76}{27} e^{-3 t}+\frac{1}{3} t^{2}+\frac{13}{9} t-\frac{22}{27}$
6. 

(a) $y=\frac{1}{2} e^{3 x}+\frac{7}{2} e^{x}$
(b) $y=x \ln x+2 x$
7. $y=\frac{7}{e^{2}}{ }^{\frac{1}{2} x^{2}}-2$
8. (a) $r=5$
(b) $y=\frac{6}{29} \sin 2 t-\frac{15}{29} \cos 2 t$
(c) $y=\frac{6}{29} \sin 2 t-\frac{15}{29} \cos 2 t+C e^{5 t}$

## Section 2.4 Solutions

1. The ODEs in parts (a), (c), (d) and (f) are autonomous.
2. (a)

$y=0$ is an unstable equilibrium
(c)

$y=0$ is an unstable equilibrium $y=3$ is a stable equilibrium
(d)

$y=2$ is a stable equilibrium $y=5$ is an unstable equilibrium
(f)

$y=0$ is an unstable equilibrium $y=3$ is a semi-stable equilibrium
3. (a) $y=0$ is a stable equilibrium, $y=2$ is an unstable equilibrium, $y=6$ is a stable equilibrium
(b) $y=1$ is a semi-stable equilibrium, $y=4$ is a stable equilibrium
(c) $y=0$ is an unstable equilibrium, $y=2$ is a stable equilibrium, $y=5$ is an unstable equilibrium
4. 

(a)


(c)

5.
(a) $\frac{d y}{d t}=-y(y-2)(y-6)$
(b) $\frac{d y}{d t}=-(y-4)(y-1)^{2}$
(c) $\frac{d y}{d t}=y(y-2)(y-5)$
6. (c)

6. We can factor the right side of the ODE to get $\frac{d y}{d x}=-\frac{1}{3} y(y-3)$.
(a) $y=3$ is a critical value, so there is an equilibrium solution of $y=3$.
(b) $y=3$ is a stable equilibrium solution, $y=0$ is an unstable equilibrium solution.
(c) See above and to the right.

1. (a) $A(0)=80$
(b) $A=0$ is a stable equilibrium solution
(c) $k=-0.104, \quad A=80 e^{-0.104 t}$
2. (a) $\frac{d A}{d t}=-\frac{3}{10} A, \quad A(0)=87$
(b) $A=87 e^{-\frac{3}{10} t}$
(c) $t=9.52$ hours
(d) The transient solution is $87 e^{-\frac{3}{10} t}$ and the steady state solution is zero. (Saying there is no steady-state solution could be considered correct as well.)
3. 

(a) $T=70+C e^{-k t}$
(b) $C=-38$, so $T=70-38 e^{-k t}$
(c) $k=\ln \frac{19}{6} \approx 1.15$
(d) Steady-state: $70 \quad$ Transient: $-38 e^{-\ln \frac{19}{6} t} \approx-38 e^{-1.15 t}$
4. (a) $i=\frac{6}{5}-\frac{6}{5} e^{-20 t}$ or $i=\frac{6}{5}\left(1-e^{-20 t}\right)$
(b) $i=\frac{5}{101}(20 \sin 2 t-2 \cos 2 t)-\frac{10}{101} e^{-20 t}$ or $i=\frac{100}{101} \sin 2 t-\frac{10}{101} \cos 2 t-\frac{10}{101} e^{-20 t}$
(c) The transient part is $-\frac{10}{101} e^{-20 t}$ and the steady-state part is $\frac{5}{101}(20 \sin 2 t-2 \cos 2 t)$ or $\frac{101}{101} \sin 2 t-\frac{10}{101} \cos 2 t$.
5. (a) If $T_{0}=T_{m}$ the temperature will not change because the initial temperature of the object will be the same as the temperature of the medium; the solution will be completely steady-state.
(b) The steady-state solution is $T_{m}$.
(c) The transient solution is $\left(T_{0}-T_{m}\right) e^{-k t}$
6. (a) $A=24$ is a stable equilibrium solution
(b) See to the right.
7. (a) $\frac{d T}{d t}=-k(T-73), \quad T(0)=29$

(b) There will be a steady state solution of $T=73$.
(c) $T=73-44 e^{-k t}$
(d) $T=73-44 e^{-0.263 t}$
8. (a) Initial value problem: $\frac{3}{4} \frac{d i}{d t}+15 i=6 \cos 2 t, \quad i(0)=2$

Solution: $\quad i=\frac{2}{101}(20 \cos 2 t+2 \sin 2 t)+\frac{162}{101} e^{-20 t}$
(b) Steady-state: $\frac{2}{101}(20 \cos 2 t+2 \sin 2 t) \quad$ Transient: $\frac{162}{101} e^{-20 t}$
9. (b) Initial value problem: $\frac{d A}{d t}=7-\frac{7 A}{150}, \quad A(0)=450$

Solution: $A=150+300 e^{-\frac{7}{150} t}$
(a) Steady-state: $150 \quad$ Transient: $300 e^{-\frac{7}{150} t}$
10. (a) $i=5 e^{-54 t}$
(b) The transient part of the solution is $5 e^{-54 t}$ and there is no steady-state part.

