Ordinary Differential Equations

for Engineers and Scientists

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Contents

3 Second Order Linear ODEs

Learning Outcome:

3. Solve second order linear, constant coefficient ODEs and IVPs. Understand the nature of solutions to such ODEs and IVPs.

   Performance Criteria:

   (a) Solve an Euler equation.

   (b) Solve a second order, linear, constant coefficient, homogeneous ODE.

   (c) Set up and solve second order initial value problems modeling spring-mass systems and RLC circuits.

   (d) Sketch or identify the graph of the solution to an IVP for an undamped mass on a spring with no forcing function.

   (e) Write a function $y = A\sin\omega t + B\cos\omega t$ in the alternate form $y = C\sin(\omega t + \phi)$. From this, determine the amplitude, period, frequency, angular frequency and phase shift.

   (f) Determine from the coefficients of a second order, constant coefficient homogeneous ODE whether the system it models is (i) underdamped, (ii) critically damped, (iii) overdamped, or (iv) undamped.

   (g) Without finding the solution to the differential equation, sketch the graph of a solution of an overdamped or underdamped homogeneous second order, linear, constant coefficient ODE for given initial conditions. .

   (h) Find a particular solution to a second order linear, constant coefficient ODE using the method of undetermined coefficients.

   (i) Evaluate a differential operator for a given function.

   (j) Solve a second order linear, constant coefficient IVP.

   (k) Identify the transient and steady-state parts of the solution to a damped system with forced vibration.

In this course we are focusing on differential equations that can be solved by analytical ("pencil-and-paper") techniques. Many differential equations cannot be solved this way, and numerical methods must be employed to obtain solutions. (See Appendix B for an introduction to solving ODEs by numerical techniques.) Our chances of being able to solve an ODE analytically are much greater if it is linear.

A second order linear ODE has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \tag{1}$$

and when $f(x) = 0$ it is a *homogeneous* linear differential equation. Here a_2, a_1 and a_0 are functions of the independent variable x. In this chapter we will focus almost entirely on second order linear differential equations in which all the coefficients are constants and the independent variable is time t, rather than x. So our equations will generally have the form

$$ay'' + by' + cy = f(t), \tag{2}$$

where $a \neq 0$, b and c are constants and f is a function that we will refer to as the **forcing function**. (2) is called a *second order, linear, constant coefficient ODE*. Recall that (1) and (2) are **homogeneous** when $f = 0$ for all x or t.

The one other type of (linear) equation we will see in this chapter is a variety called an **Euler equation**. For such equations $a_2(x) = ax^2$, $a_1(x) = bx$ and $a_0(x) = c$, where b and c are constants, and $f(x) = 0$. Thus an Euler equation is one with the form

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0 \tag{3}$$

Equations of this form arise when solving certain partial differential equations. In the first section of the chapter we will solve equations of the forms

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0 \qquad \text{and} \qquad ay'' + by' + cy = 0, \tag{4}$$

both of which are clearly homogeneous. These equations will always have two solutions y_1 and y_2, and the general solution will be a linear combination

$$y = C_1 y_1 + C_2 y_2 \tag{5}$$

of the two solutions. (C_1 and C_2 are of course constants.) It should at this point be no surprise that the general solution to equations of either of the forms (4) contain two arbitrary constants!

Our main focus as the chapter goes on will be solving initial value problems of the form

$$ay'' + by' + cy = f(t), \qquad y(0) = y_0,\ y'(0) = y_0', \tag{6}$$

with a, b and c being constants, $a \neq 0$. In addition, the initial values y_0 and y_0' are constants as well. The method we will use to solve the IVP will consist of four steps:

(1) We will first solve the homogeneous equation obtained by replacing $f(t)$ with zero. This will give us a solution of the form (5), called the **homogenous solution**. We will denote it by y_h.

(2) Next we'll find something called a **particular solution**, denoted by y_p, for the ODE in (6). We will do this by a method called the **method of undetermined coefficients**.

(3) The **general solution** to the equation $ay'' + by' + cy = f(t)$ is

$$y = y_h + y_p = C_1 y_1(t) + C_2 y_2(t) + y_p(t), \tag{7}$$

the sum of the homogenous solution and the particular solution.

(4) The initial conditions $y(0) = y_0$ and $y'(0) = y_0'$ are used to determine the values of the constants C_1 and C_2. This gives us the final solution to the initial value problem (6).

Initial value problems of the form

$$ay'' + by' + cy = f(t), \qquad y(0) = y_0,\ y'(0) = y_0', \tag{6}$$

model certain electrical circuits and simple mechanical vibration. We will proceed through a series of variations on this initial value problem, developing an understanding of how the particular model describes the system and how the solution $y(t)$ behaves:

- In Section 1.2 we saw how a system consisting of a mass on a spring is modeled by the second order ODE

$$m\frac{d^2y}{dt^2} = -ky \qquad \text{or} \qquad m\frac{d^2y}{dt^2} + ky = 0,$$

  which is $ay'' + by' + cy = f(t)$ with $b = 0$ and $f(t) = 0$. When $b = 0$ we refer to the system as **undamped**, and when $f(t) = 0$ the system is **free**. We begin our study of applications with free, undamped systems in Section 3.2.

- In Section 3.3 we will add damping, but still no forcing function. That is, we'll have $b \neq 0$ and $f(t) = 0$. In this case we'll see that there are three possible scenarios - the system is just one of (a) **underdamped**, **overdamped**, or **critically damped**. In Section 3.3 we will also introduce an electrical circuit for which the mathematical model is exactly the same as a mass on a spring.

- Next, in Section 3.4, we will consider systems that are both **damped** (so $b \neq 0$ and **forced** (so $f(t) \neq 0$). We will see in that section how to find **particular solutions** to forced systems.

- Differential operators are introduced in Section 3.5. These give us a convenient way to understand the nature of solutions to forced systems.

- In Section 3.6, we'll put together everything from the previous sections to solve initial value problems with second order, linear, constant coefficient differential equations. We'll consider such IVPs in applied settings, and we'll examine the nature of solutions to such IVPs.

As a specific example of an ODE of the form $ay'' + by' + cy = f(t)$, let's consider

$$y'' + 5y' + 6y = 5\sin 3t.$$

We want to know not only how to solve such ODEs and their associated IVPs, but also to know what kind of behavior to expect from solutions, based on the original ODE or IVP. Here is a summary of some of the ideas you will run across, which were previously brought up in Chapter 2:

- The left hand side of the ODE represents some sort of "system" like a mass on a spring or a simple electric circuit. The function on the right represents some kind of input to the system, which we call a **forcing function**, since it forces movement (for a mass on a spring) or current (for an electrical circuit) in the system.

- The solution function y is the **response** of the system, and will often consist of exponential or trigonometric functions, or products of the two. The solution may have several terms, some of which may "die out" (go to zero) as time goes on, and others that may not. The parts that die out are called **transient** parts of the solution, and parts that are constant or periodic are called **steady-state** parts. These are also referred to as the transient and steady-state response of the system.

3.1 Homogeneous Second-Order Equations

Performance Criterion:

3. (a) Solve an Euler equation.
   (b) Solve a second order, linear, constant coefficient, homogeneous ODE.

In this section we will solve second order homogeneous equations of the forms

$$ax^2\frac{d^2y}{dx^2}+bx\frac{dy}{dx}+cy=0 \qquad \text{and} \qquad ay''+by'+cy=0, \tag{1}$$

where $a\neq 0$, b and c are constants. In both cases, solutions are obtained by guessing what the general form of a solution might be and then substitution that guess into the ODE and "making it work." The first equation in (1) is called an **Euler equation** and the second is a second order, linear, constant coefficient, homogeneous ODE. The bulk of this chapter is about constant coefficient equations. In this section we'll first see how to solve Euler equations, then look at homogeneous constant coefficient equations, whose solutions take a variety of forms.

Consider the Euler equation

$$x^2y''+2xy'-6y=0.$$

Note that if y was some power of x then y' would be one power lower and y'' two powers lower. If we then substituted our y, y' and y'' into the ODE we would have three terms of the same power due to the multiplication of y' by x and y'' by x^2 in the ODE. Thus there might be some hope that the three terms add up to zero.

◇ **Example 3.1(a):** Solve the ODE $x^2y''+2xy'-6y=0$ by guessing a solution of the form $y=x^p$ and determining the value of p.

Solution: First we compute $y'=px^{p-1}$ and $y''=p(p-1)x^{p-2}$. Substituting these into the ODE we get

$$\begin{aligned}
x^2p(p-1)x^{p-2}+2xpx^{p-1}-6x^p &= 0\\
p(p-1)x^p+2px^p-6x^p &= 0\\
x^p\big[p(p-1)+2p-6\big] &= 0\\
x^p(p^2+p-6) &= 0
\end{aligned}$$

Now the quantity x^p is only zero when $x=0$, leading to the trivial solution $y=0$. Because we would like nonzero solutions, it must be the case that $p^2+p-6=0$. Solving this gives us $p=-3,2$, so both $y=x^{-3}$ and $y=x^2$ are solutions. The general solution is then $y=C_1x^{-3}+C_2x^2$.

We now contemplate the second order linear, *constant coefficient*, homogeneous ODE

$$y''+5y'+6y=0.$$

In this case we are looking for some function $y=y(t)$ (we will be interested in applications in which time is the independent variable) for which we can multiply the function and its first and second derivatives by constants, add the results and get zero. This indicates that y must essentially be equal to its first and second derivatives, which only occurs for exponential functions. Therefore we guess that the solution has the form $y = e^{rt}$, where r is some constant.

◇ **Example 3.1(b):** Solve $y'' + 5y' + 6y = 0$, assuming that the independent variable is t.

Solution: We substitute the guess $y = e^{rt}$ into the ODE, resulting in

$$r^2e^{rt} + 5re^{rt} + 6e^{rt} = (r^2 + 5r + 6)e^{rt} = 0$$

Now the quantity e^{rt} is never zero, so $r^2 + 5r + 6 = 0$. Solving this gives us $r = -3, -2$, so both $y = e^{-3t}$ and $y = e^{-2t}$ are solutions. It is not hard to show that if y_1 and y_2 are both solutions to a homogeneous ODE and C_1 and C_2 are constants, then $y = C_1y_1 + C_2y_2$ is also a solution. So in this case, the general solution is $y = C_1e^{-3t} + C_2e^{-2t}$.

When using the above method to solve

$$ay'' + by' + cy = 0 \tag{2}$$

we will arrive at $(ar^2 + br + c)e^{rt} = 0$, leading to $ar^2 + br + c = 0$. We will refer to the equation $ar^2 + br + c = 0$ as the **auxiliary equation** (also called the **characteristic equation**) associated with the ODE $ay'' + by' + cy = 0$, and we will call solutions to this equation **roots** of the equation. The following summarizes what we saw in the above example, which is only one of several possible forms that a solution to (2) can take.

> When the auxiliary equation for a second order, linear, constant coefficient homogeneous equation has two real roots r_1 and r_2, the solution to the ODE is
>
> $$y = C_1e^{r_1t} + C_2e^{r_2t}.$$

Video Example - watch from 1:20 to 3:10

The most efficient method for finding the roots of the auxiliary equation $r^2 + 5r + 6 = 0$ from Example 3.1(b) is to factor the left side, but we could have used the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2}$$

instead. We will at times want or need to use the quadratic formula, and at other times factoring or another method may be more efficient for finding the roots of the auxiliary equation. Regardless of what we might do in practice, an examination of (2) is useful for determining the various possibilities when obtaining the roots to the auxiliary equation:

- When $b^2 - 4ac > 0$ there will be two real roots, as in Example 3.1(b).
- When $b^2 - 4ac = 0$ there will only be one real root, which is a rather special case.
- When $b^2 - 4ac < 0$ and $b \neq 0$ the roots will be complex conjugates. That is, they will be complex numbers of the form $r = k + \lambda i$ and $r = k - \lambda i$, where $i = \sqrt{-1}$.

- When $b^2 - 4ac < 0$ and $b = 0$, the roots will be two purely imaginary numbers $r = \lambda i$ and $r = -\lambda i$. (This is just the previous case, with $k = 0$.)

Video Discussion

Each of the above cases results in different forms of the solution to the homogeneous equation $ay'' + by' + cy = 0$, and the methods for solving the auxiliary equations are generally different as well, although the quadratic formula *could* be used in all cases. We've already taken care of the first case with Example 3.1(b).

We now consider the second case above. Suppose that we were to try the method of Example 3.1(b) for the ODE $y'' + 6y' + 9y = 0$. We would only get one value for r, -3. By the same reasoning as used before we then have the solution $y = C_1e^{-3t} + C_2e^{-3t}$; because these are like terms, this can be written $y = Ce^{-3t}$, where $C = C_1 + C_2$. But (for reasons we'll go into in more depth later) the general solution to a second order ODE must be the sum of two "different" functions, each multiplied by arbitrary constants, so something is wrong! The following example shows that there is in fact another solution besides e^{-3t}.

◇ **Example 3.1(c):** Verify that $y = te^{-3t}$ is also a solution to the ODE $y'' + 6y' + 9y = 0$.

Solution: Using the product and chain rules to compute the derivatives,

$$y' = t(e^{-3t})' + t'e^{-3t} = -3te^{-3t} + e^{-3t}$$

$$y'' = -3(te^{-3t})' - 3e^{-3t} = -3(-3te^{-3t} + e^{-3t}) - 3e^{-3t} = 9te^{-3t} - 6e^{-3t}$$

Substituting into the ODE,

$$\begin{aligned} y'' + 6y' + 9y &= \big(9te^{-3t} - 6e^{-3t}\big) + 6\big(-3te^{-3t} + e^{-3t}\big) + 9\big(te^{-3t}\big) \\ &= 9te^{-3t} - 6e^{-3t} - 18te^{-3t} + 6e^{-3t} + 9te^{-3t} \\ &= 0 \end{aligned}$$

The general solution to the ODE $y'' + 6y' + 9y = 0$ is then $y = C_1e^{-3t} + C_2te^{-3t}$. Any time that our auxiliary equation has a repeated root r (both solutions are the same), one of the solutions is e^{rt}, and the other solution is te^{rt}. We'll see in Section 4.2 how this solution is obtained.

> When the auxiliary equation for a second order, linear, constant coefficient homogeneous equation has only one real root r, the solution to the ODE is
>
> $$y = C_1e^{rt} + C_2te^{rt}.$$

Video Example

We now consider the case where $b^2 - 4ac < 0$ and $b = 0$, resulting in two purely imaginary roots $r = \pm\lambda i$. We'll consider the homogeneous equation $y'' + 9y = 0$ that we looked at previously, which can be rewritten as $y'' = -9y$. Through our familiarity with derivatives and the chain rule, we guessed (correctly) that both $y = \sin 3t$ and $y = \cos 3t$ are solutions, and it is not hard to show that

$y = C_1 \sin 3t + C_2 \cos 3t$ is a solution. Now how would it work to try the method of Example 3.1(b) for this equation? Letting $y = e^{rt}$ we have

$$y'' + 9y = r^2 e^{rt} + 9e^{rt} = (r^2 + 9)e^{rt} = 0$$

so we need to solve $r^2+9 = 0 \Rightarrow r^2 = -9$. If we allow complex solutions, the solution to this equation is $r = \pm 3i$, so the solution to the ODE is $y = Ae^{3it} + Be^{-3it}$, where A and B are arbitray constants. (We could have used C_1 and C_2 for the constants as we have been doing, but we are "saving" them, as you'll see.) To continue we will need the following:

> **Euler's Formula:** $e^{i\theta} = \cos\theta + i \sin\theta$

To see where Euler's Formula comes from, see Appendix B.5. We will also use the two basic trig identities:

$$\cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta$$

Combining the two solutions e^{3it} and e^{-3it}, we get

$$\begin{aligned} y &= Ae^{3it} + Be^{-3it} \\ &= A\big(\cos 3t + i \sin 3t\big) + B\big[\cos(-3t) + i\sin(-3t)\big] \\ &= A\cos 3t + Ai \sin 3t + B\cos 3t - Bi\sin 3t \\ &= (A+B)\cos 3t + (A-B)i \sin 3t \\ &= C_1 \cos 3t + C_2 \sin 3t. \end{aligned}$$

Here A and B are constants, and $C_1 = A + B$, $C_2 = (A - B)i$. Now it seems that C_2 should be a complex number, which is a bit disconcerting. However, it is possible that A and B are complex as well, in such a way that perhaps C_2 turns out to be real! You will find that this method "works," regardless. The following summarizes what we have just seen.

> When the auxiliary equation for a second order, linear, constant coefficient homogeneous equation has two purely imaginary roots $r = \pm\lambda i$, the solution to the ODE is
>
> $$y = C_1 \sin \lambda t + C_2 \cos \lambda t.$$

Let's use a specific example to examine the final situation, where the auxiliary equation has two complex roots $r = k \pm \lambda i$.

◇ **Example 3.1(d):** Solve $y'' + 10y' + 28y = 0$.

Solution: Guess $y = e^{rt}$, so $y' = re^{rt}$ and $y'' = r^2 e^{rt}$. Then

$$y'' + 10y' + 28y = r^2e^{rt} + 10re^{rt} + 28e^{rt} = (r^2 + 10r + 28)e^{rt} = 0$$

Again, e^{rt} cannot equal zero, so we solve $r^2 + 10r + 28 = 0$; this is done in this case using the quadratic formula:

$$r = \frac{-10 \pm \sqrt{10^2 - 4(28)}}{2} = -5 \pm 2i\sqrt{2}$$

Therefore

$$\begin{aligned}
y &= Ae^{(-5+2i\sqrt{2})t} + Be^{(-5-2i\sqrt{2})t} \\
&= Ae^{-5t}e^{2i\sqrt{2}\,t} + Be^{-5t}e^{-2i\sqrt{2}\,t} \\
&= e^{-5t}\Big[\big(A\cos(2\sqrt{2}\,t) + Ai\sin(2\sqrt{2}\,t)\big) + \big(B\cos(-2\sqrt{2}\,t) + Bi\sin(-2\sqrt{2}\,t)\big)\Big] \\
&= e^{-5t}\Big[\big(A\cos(2\sqrt{2}\,t) + Ai\sin(2\sqrt{2}\,t)\big) + \big(B\cos(2\sqrt{2}\,t) - Bi\sin(2\sqrt{2}\,t)\big)\Big] \\
&= e^{-5t}\big[(A+B)\cos(2\sqrt{2}\,t) + (A-B)i\sin(2\sqrt{2}\,t)\big] \\
&= e^{-5t}[C_1\cos(2\sqrt{2}\,t) + C_2\sin(2\sqrt{2}\,t)]
\end{aligned}$$

The solution to $y'' + 10y' + 28y = 0$ is $y = e^{-5t}\big[C_1\cos(2\sqrt{2}\,t) + C_2\sin(2\sqrt{2}\,t)\big]$.

In general, we have the following.

When the auxiliary equation for a second order, linear, constant coefficient homogeneous equation has two complex roots $r = k \pm \lambda i$, the solution to the ODE is

$$y = e^{kt}(C_1\sin\lambda t + C_2\cos\lambda t).$$

Example 3.1(d) and the discussion before it show how sine and cosine functions arise when r_1 and r_2 are complex numbers, but you need not show all those steps when solving. Unless asked to show more, we will just set up and solve the equation $ar^2 + br + c = 0$, then give the solution that arises from the form of the roots. In the Chapter 3 Summary you will find a flowchart detailing the process of finding the solution to a second order, linear, constant coefficient, homogeneous differential equation. *You will need to "memorize" the forms of the solution for the various forms of* r_1 *and* r_2, but if you do enough exercises, you should know them by the time you are done.

Section 3.1 Exercises To Solutions

1. Solve each Euler equation.

   (a) $x^2y'' - 4xy' + 4y = 0$ (b) $x^2y'' + 4xy' + 2y = 0$ (c) $3x^2y'' - xy' + y = 0$

2. Solve each ODE, assuming the independent variable is t. Use exact values except for (i) - use decimals rounded to the hundredth's place there.

   (a) $y'' - 2y' - 3y = 0$ (b) $y'' + 2y' + 10y = 0$ (c) $y'' + 10y' + 25y = 0$

   (d) $y'' + 6y' + 11y = 0$ (e) $y'' + 3y' + 2y = 0$ (f) $y'' + 2y = 0$

   (g) $y'' + 2y' + y = 0$ (h) $y'' + 16y = 0$ (i) $y'' + 3.1y' + 4.5y = 0$

3. (a) Give the auxiliary equations for the ODEs $y'' + 25y = 0$ and $y'' + 25y' = 0$.

   (b) Solve the ODE $y'' + 25y' = 0$ using the method of this section.

4. Give the general form of the solution (that is, give one of the forms found in the boxes in this section) to $ay'' + by' + cy = 0$ under each of the following conditions:

   (a) $b = 0$ (b) $b^2 - 4ac > 0$ (c) $b^2 - 4ac < 0,\ b \neq 0$ (d) $b^2 - 4ac = 0$

5. (a) Solve the Euler equation $x^2y'' - 3xy' + 4y = 0$. You should obtain only one solution.

   (b) We know that there should be two solutions. Show that $y = x^2 \ln x$, $x > 0$, is also a solution.

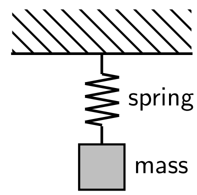
3.2 Free, Undamped Vibration

Performance Criteria:

3. (c) Set up and solve second order initial value problems modeling spring-mass systems and RLC circuits.
   (d) Sketch or identify the graph of the solution to an IVP for an undamped mass on a spring with no forcing function.
   (e) Write a function $y = A\sin\omega t + B\cos\omega t$ in the alternate form $y = C\sin(\omega t + \phi)$. From this, determine the amplitude, period, frequency, angular frequency and phase shift.

Let's return to the following example:

◇ **Example 1.1(a):** Suppose that a mass is hanging on a spring that is attached to a ceiling, as shown to the right. If we lift the mass, or pull it down, and let it go, it will begin to oscillate up and down. Its height (relative to some fixed reference, like its height before we lifted it or pulled it down) varies as time goes on from when we start it in motion. We say that *height is a function of time*.



In Section 1.2 we derived the differential equation

$$m\frac{d^2y}{dt^2} + ky = 0 \tag{1}$$

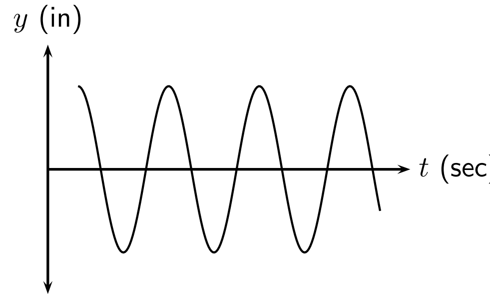
that governs the motion of the mass. Here y is the height (from equilibrium) of the mass at any time t after it is set in motion by some means, *with positive being upward*. m is the mass of the mass (the first of these being a quantity, and the second being an object) and k is the spring constant. We will assume that there is no external force acting on the mass once it is set in motion - in such cases we call the vibration **free**. We will also consider the system to be **undamped**, which means that there is nothing hindering the motion of the mass once it begins moving up and down.

First let's remind ourselves of how we can solve such ODEs:

◇ **Example 3.2(a):** Solve Equation 1 for a mass of 10.3 and a spring constant of 28.7 (both with appropriate units).

Solution: The auxiliary equation corresponding to the ODE is $10.3r^2 + 28.7 = 0$. Subtracting 28.7 from both sides and dividing by 10.3 gives $r^2 = -2.786....$ If we then take the square root of both sides and round to the nearest hundredth, we have $r = \pm 1.67i$. Therefore the solution to the ODE is $y = C_1\sin 1.67t + C_2\cos 1.67t$.

Intuitively it is reasonable that, once set in motion, a mass on a spring with no damping will oscillate forever with the same amplitude. The graph of such motion would look something like what is shown to the right; this is referred to as **harmonic motion**. It may not be clear how the solution to the above differential equation, which contains a sum of sine and cosine functions, gives a graph like the one shown. We'll soon see a computation that makes it clear how this happens.



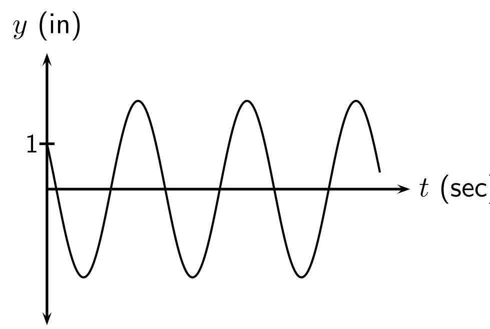
A mass hanging on a spring can be set in motion by doing one of three things:

- moving the mass away from its equilibrium position and letting it go
- giving the mass some initial velocity upward or downward from its initial position
- moving the mass away from its initial position *and* giving it an initial velocity

If we move the mass upward or downward, that will give $y(0)$ equal to some value other than zero, and if we impart an initial velocity the value of $y'(0)$ will be nonzero.

◇ **Example 3.2(b):** Suppose that the mass from Example 3.2(a) is set in motion by lifting it up one inch and giving it an initial velocity of 2 inches per second downward, both at time zero. Give the initial conditions in function form and sketch a graph of what you expect the motion to look like. Then determine the solution to the initial value problem and graph it to check your graph.

Solution: The condition of raising the mass up by one inch is given by $y(0) = 1$, and the initial velocity of two inches per second downward is given by $y'(0) = -2$, with the negative indicating that the velocity is in the downward direction. Because the mass starts above its equilibrium point, we expect the y-intercept of the graph to be at positive one. The condition of being given an initial velocity downward means that the slope of the tangent line to the graph at $t = 0$ will be negative. We therefore expect a graph something like that shown to the right.



Applying the first initial condition with the solution from Example 3.2(a) gives $C_2 = 1$. Taking the derivative of the solution obtained in Example 3.2(a) gives

$$y' = 1.67C_1 \cos 1.67t - 1.67C_2 \sin 1.67t.$$

Substituting the initial velocity initial condition and $C_2 = 1$ into this gives us $C_1 = -1.20$ (the zero indicates accuracy to the hundredth's place), so the solution to the IVP is

$$y = -1.20 \sin 1.67t + \cos 1.67t.$$

The graph of this function agrees with that shown above.

It is sometimes useful to change an expression of the form $-1.20 \sin 1.67t + \cos 1.67t$ into a single sine function with a phase shift, and here is what we use to do it:

$C\sin(\omega t + \phi)$ Form

$$A\sin\omega t + B\cos\omega t = C\sin(\omega t + \phi), \quad \text{where} \quad C = \sqrt{A^2 + B^2}$$

$$\text{and} \quad \phi = \tan^{-1}\frac{B}{A} \quad \text{if} \quad A > 0, \qquad \phi = \pi + \tan^{-1}\frac{B}{A} \quad \text{if} \quad A < 0$$

$$\text{If} \quad A = 0, \quad \text{then} \quad B\cos\omega t = B\sin\left(\omega t + \tfrac{\pi}{2}\right)$$

Note that radian mode must be used when using a calculator to compute ϕ!

◇ **Example 3.2(c):** Change the solution $y = -1.20\sin 1.67t + \cos 1.67t$ into the form $y = C\sin(\omega t + \phi)$, where C, ω and ϕ are all decimals rounded to the hundredth's place.

$$C = \sqrt{(-1.20)^2 + 1^2} = 1.56, \qquad \phi = \pi + \tan^{-1}\frac{B}{A} = 2.44$$

Thus the solution can be written as $y = 1.56\sin(1.67t + 2.44)$.

We see that the amplitude of the motion for the situation from Examples 3.2(a), (b) and (c) is 1.56, greater than the initial position of one unit. That is due to the initial velocity - you will see in the exercises what happens if there is no initial velocity.

There are some important parameters associated with a function $y = C\sin(\omega t + \phi)$:

Amplitude, frequency, period and phase shift

For a function $y = C\sin(\omega t + \phi)$,

- C is the **amplitude**
- ω is the **angular frequency**
- $T = \dfrac{2\pi}{\omega}$ is the **period**
- $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$ is the **frequency**
- $-\dfrac{\phi}{\omega}$ is the **phase shift**
- $\omega = 2\pi f$

◇ **Example 3.2(d):** For the function $y = 1.56\sin(1.67t + 2.44)$, give the amplitude, period, frequency and phase shift.

Solution: The amplitude is 1.56, the period is $T = \dfrac{2\pi}{1.67} = 3.76$, the frequency is $f = \dfrac{1}{T} = 0.64$ and the phase shift is $-\dfrac{2.44}{1.67} = -1.46$.

With a bit of careful thought, the following should be clear: *Only the amplitude and and phase shift are influenced by the initial conditions.* The period, frequency and angular frequency in this case are all determined by m and k, the parameters of the system.

Section 3.2 Exercises

To Solutions

1. Suppose that the mass is set in motion by moving it upward by 2.5 cm and releasing it with no initial velocity.

   (a) Sketch what you think the graph of y versus t will look like, taking care with the fact that positive y is upward. Make the amplitude of the motion clear on your graph.

   (b) Express the initial conditions mathematically by giving values for $y(0)$ and $y'(0)$.

2. Repeat Exercise 1 for a mass that is set in motion by hitting it upward (as it hangs at equilibrium), giving it an initial speed of 3 inches per second.

3. Repeat Exercise 1 for a mass that is set in motion by giving it an initial speed of 8 cm per second downward from a point 2 cm above equilibrium. *The amplitude is not two* - make it clear whether it is more or less than two.

4. A mass of $\frac{3}{4}$ is attached to a spring with a spring constant of 15. It is set into motion by raising it 2.5 cm and releasing it. (See Exercise 1.)

   (a) Set up an initial value problem consisting of a differential equation and two initial conditions.

   (b) Solve the initial value problem to find the position function y. Give all numbers as decimals, rounded to the hundredth's place.

   (c) Graph your solution using some technology and compare with your answer to Exercise 1. They should agree; if they don't, figure out what is wrong and fix it!

   (d) Put your solution in the form $C\sin(\omega t+\phi)$. Graph the resulting function and make sure the graph agrees with what you got for (c). If not, try to find and correct your error.

   (e) Give the amplitude, angular frequency, period, frequency and phase shift of the solution.

5. (a) Consider again a mass of $\frac{3}{4}$ attached to a spring with a spring constant of 15, as in Exercise 4. Solve the initial value problem obtained with the initial conditions of Exercise 3, where the initial speed was 8 cm downward from a point 2 cm above equilibrium. Again give all numbers as decimals, rounded to the hundredth's place.

   (b) Graph your solution using technology and compare with your answer to Exercise 3.

   (c) Put your solution in the form $y = A\sin(\omega t+\phi)$. Check it by graphing it and your solution to part (a) together; they should be the same!

   (d) Give the amplitude, angular frequency, period, frequency and phase shift of the solution.

6. A mass of $\frac{4}{10}$ on a spring with spring constant 4 is given an initial velocity of 9 cm/sec upward from an initial position of 4 cm below equilibrium.

   (a) Give the initial value problem.

   (b) Find the equation of motion of the mass, $y(t)$, in the form $y = A\sin(\omega t+\phi)$.

   (c) Give the amplitude, angular frequency, period, frequency and phase shift of the solution.

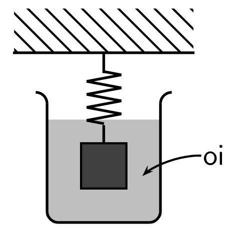
3.3 Free, Damped Vibration

Performance Criteria:

3. (c) Set up and solve second order initial value problems modeling spring-mass systems and RLC circuits.

   (f) Determine from the coefficients of a second order, constant coefficient homogeneous ODE whether the system it models is (i) underdamped, (ii) critically damped, (iii) overdamped, or (iv) undamped.

   (g) Without finding the solution to the differential equation, sketch the graph of a solution of an overdamped or underdamped homogeneous second order, linear, constant coefficient ODE for given initial conditions.



Consider again a mass on a spring, but suppose that we submerge the mass in an oil bath, as shown to the right (think about an oil-damped shock absorber). As the mass moves up and down there is now an additional force acting on it, the resistance of the oil. We will make the assumption that the force is directly proportional to the velocity but in the opposite direction; that is, for some *positive* constant β (the Greek letter *beta*), the force of resistance is given by $-\beta\,\dfrac{dy}{dt}$.

Suppose that we also have some variable external force acting on the mass as well. (Think force exerted upward on a shock absorber by the road, as a car drives along.) If this force is some function $f(t)$, then the net force on the mass at any time t is

$$F_{\text{net}} = m\,\frac{d^2y}{dt^2} = f(t) - \beta\,\frac{dy}{dt} - ky.$$

If we rearrange this equation we get

$$m\,\frac{d^2y}{dt^2} + \beta\,\frac{dy}{dt} + ky = f(t), \tag{1}$$

the second order differential equation that models the motion of a spring-mass system with damping and an external forcing function f.

Now suppose that we have an electric circuit consisting of a resistor, an inductor and a capacitor in series with each other, along with (perhaps) a voltage source. We will refer to such a circuit as an **RLC circuit**, shown schematically to the right. The differential equation that models an RLC circuit is derived from Kirchoff's voltage law which tells us that the sum of the voltage drops across each of the three components is equal to the voltage imposed on the system by the voltage source. By Ohm's law the voltage drop across the resistor is $V = iR$, where i is the current and R is the resistance of the resistor.

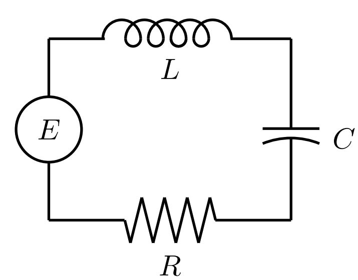


Figure 3.1

The voltage drops across the inductor and the capacitor are $L\dfrac{di}{dt}$ and $\dfrac{1}{C}q$, where q is the charge on the capacitor. From Kirchoff's voltage law we now have

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t), \tag{2}$$

where $E(t)$ is the voltage as a function of time. (It may of course be constant, such as in the case of DC voltage supplied by a battery.)

But the current i is the rate at which charge is passing by a point in the circuit: $i = \dfrac{dq}{dt}$ Thus $\dfrac{di}{dt} = \dfrac{d}{dt}\left(\dfrac{dq}{dt}\right) = \dfrac{d^2q}{dt^2}$ and the above equation becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)\ , \qquad (3),$$

where L is the inductance of the inductor in henries, R is the resistance of the resistor in ohms, and C is the capacitance of the capacitor in farads. *All are positive quantities; this will be important!* If we would rather work with current rather than charge we can differentiate both sides of (3) to get

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'(t)\ , \qquad (4),$$

Note that both (3) and (4) are completely analogous to equation (1), which models a spring-mass system. This illustrates the principle, first encountered in Section 2.5, that different physical situations are often modeled by the same differential equation.

In this section we will consider the situation in which there is no forcing function. That is, the right hand sides of (1), (3) and (4) are zero. Of course something needs to happen to get the mass moving or to get current to flow in the circuit, and each can be accomplished in a variety of ways. For example, the spring and the capacitor both have the ability to store energy by compressing or stretching the spring, or by storing charge in the capacitor (with the positivity or negativity of that charge being analogous to compressing or stretching the spring). As that energy is released it will cause the mass to move or current to flow in the circuit. Here is a summary of initial conditions we can have for the spring, which we previously gave in Section 3.2:

- the mass can be displaced and let go, with no initial velocity
- the mass can be given some velocity at its resting position
- the mass can be displaced *AND* given some initial velocity upward or downward

The analogous conditions for the electric circuit are as follows:

- there is no current in the circuit, but there is an initial charge on the capacitor
- there is no charge on the capacitor, but there is an initial current
- there is both an initial charge on the capacitor and an initial current

Our main objective in this section is to understand the behavior of the solution function $y(t)$ of equation (1), (3) or (4) when $f(t) = 0$ or $E(t) = 0$, and how that behavior varies depending on the *parameters* m, k and β or R, L and C. You will be looking at some exercises to do this. Realize that the values given for the parameters may not be realistic - they were chosen in such a way as to make the mathematics reasonable.

Section 3.3 Exercises To Solutions

For all exercises in this section you will be working with the equation

$$m\,\frac{d^2y}{dt^2} + \beta\,\frac{dy}{dt} + ky = f(t), \tag{1}$$

for various values of m, β and k, **but always with** $f(t) = 0$**.**

1. (a) Solve the initial value problem consisting of Equation (1) with $m = 5$, $\beta = 6$ and $k = 80$, and initial conditions $y(0) = 2$, $y'(0) = -6$. Give your answer in the form $y = Ce^{at}\sin(\omega t + \phi)$ and all numbers in decimal form, rounded to the nearest tenth. (Note that 5.0007 rounded to the nearest tenth is 5.0, *not* 5! What is the difference?)

   (b) Graph the solution to the IVP on your calculator. Adjust the viewing window to get about three cycles of the motion displayed fairly large. Sketch your graph.

   (c) Graph $y = 2.3e^{-0.6t}$ and $y = -2.3e^{-0.6t}$ together with the solution you graphed in (b). Add them to your sketch *as dashed curves*.

What you have just seen is an example of what is called **underdamped vibration**. There is damping, but it is small enough to allow the mass to move up and down while the vibration decays.

2. (a) Solve the initial value problem consisting of Equation (1) with $m = 5$, $\beta = 50$ and $k = 80$, and initial conditions $y(0) = 2$, $y'(0) = 6$.

   (b) Sketch the graph of your solution over a time period long enough to show what is happening over time. Make your vertical scale such that the general shape of the graph can be seen.

   (c) Compare and contrast your solution to this IVP with the solution to the IVP from Exercise 1, in terms of amplitude over time and oscillation.

For your answer to 2(c) you should have noted that the solution of the IVP in Exercise 1 oscillates as it decays, and the solution of the IVP in Exercise 2 does not. We say that the situation from Exercise 2 is **overdamped vibration**. The initial conditions do not affect whether the vibration will be underdamped or overdamped (can you see why not?), so the only difference is the value of β.

3. (a) Determine a value of β that is the "dividing line" between equations have solutions that oscillate as they decay and those that do not oscillate. Explain how you determined it.

   (b) Solve the IVP with $m = 5$, $k = 80$ and the value of β you obtained in (a), along with the initial conditions from Exercise 2. Graph the solution using some technology and sketch the graph.

   (c) Suppose now that $m = 5$ and $k = 80$. How would you expect the solution to (1) to behave if β was slightly smaller than the value obtained in (a), and why. How would you expect the solution to behave for β slightly larger than the value obtained in (a)? **Answer these questions in complete sentences.**

For your answer to 3(c) you should have said that if β is a little less than the value that you found in (a) the solution will behave like that in Exercise 1, and if β is a little more than that value the solution will behave like that in Exercise 2. The situation in Exercises 3(a) and 3(b) is called the **critically damped** case, meaning that if there is just a little less damping the behavior will be oscillatory, but if the damping has that critical value or greater, the motion will not be oscillatory.

4. Describe a method for determining from the coefficients m, β and k whether the solution will be underdamped, overdamped, or critically damped.

5. Sketch the graph of the solution to a 2nd-order, constant-coefficient homogeneous ODE with the given damping conditions and initial conditions.

   (a) Underdamped, $y(0) < 0,\ y'(0) > 0.$

   (b) Overdamped, $y(0) < 0,\ y'(0) < 0.$

   (c) Overdamped, $y(0) > 0,\ y'(0) < 0.$ (I think there may be a couple different appearances possible here - we'll investigate this more later.)

   (d) Undamped (not *under*damped, but *un*damped), $y(0) > 0,\ y'(0) > 0.$

6. Consider a circuit like that shown in Figure 3.1, but without the voltage source $E = 0$. The components are a 0.15 henry inductor, a 500 ohm resistor and a 2×10^{-5} farad capacitor, the initial charge on the capacitor is 1.3×10^{-6} coulombs and there is no initial current. (This can be achieved by having an open switch in the circuit and closing the switch at time zero.)

   (a) Is the system overdamped, underdamped, or critically damped?

   (b) Use equation (3) to determine the charge on the capacitor at any time t. Round all constants to three significant figures and give correct units with your answer.

   (c) If you didn't already in part (b), put your answer in $y = Ce^kt\sin(\omega t + \phi)$ form.

   (d) Give the current in the circuit at any time t, again giving units with your answer and rounding to two significant digits.

3.4 Particular Solutions, Part One

Performance Criteria:

3. (h) Find a particular solution to a second order linear, constant coefficient ODE using the method of undetermined coefficients.

In the previous section we found out how to solve equations of the form

$$ay'' + by' + cy = f(t) \tag{1}$$

when $f(t) = 0$, but in practice we are usually interested in situations where $f(t) \neq 0$. For those situations we will use a technique called the **method of undetermined coefficients** to find a solution. To do this we (again!) substitute a guess, which we will call the trial particular solution, into the ODE. This trial solution will contain one or more unknown constants, whose values can be determined by finding the result obtained when the trial solution is substituted into the left hand side of the ODE and then setting it equal to the known right hand side $f(t)$. Like terms are then equated, and the constants determined. The resulting solution is called the **particular solution** to (1). Later we will see how the particular solution is combined with the homogeneous solution to give the general solution to (1).

Let's consider the ODE

$$y'' + 9y = 5e^{-2t}.$$

If y had the form $y = Ae^{-2t}$ for some constant A, then y'' would have the same form. Perhaps there is then a choice of A for which $y'' + 9y$ will equal $5e^{-2t}$. The next example shows us that is in fact the case.

◇ **Example 3.4(a):** Find a value A for which $y = Ae^{-2t}$ is a solution to $y'' + 9y = 5e^{-2t}$.

Solution: First we observe that

$$y = Ae^{-2t} \quad \Longrightarrow \quad y' = -2Ae^{-2t} \quad \Longrightarrow \quad y'' = 4e^{-2t}.$$

Substituting these into the left side of (1) we get

$$y'' + 9y = 4Ae^{-2t} + 9Ae^{-2t} = 13Ae^{-2t}.$$

We now set this equal to the right hand side of the ODE to get $13Ae^{-2t} = 5e^{-2t}$. The only way that this can be true is if the coefficients of e^{-2t} are equal: $13A = 5 \Rightarrow A = \frac{5}{13}$. Therefore $y = \frac{5}{13}e^{-2t}$ is a solution to $y'' + 9y = 5e^{-2t}$.

A particular solution to a differential equation is any solution that does not contain arbitrary constants, so $y = \frac{5}{13}e^{-2t}$ is a particular solution to the ODE $y'' + 9y = 5e^{-2t}$. We sometimes subscript the dependent variable with the letter p to indicate a particular solution. For the above we would then write $y_p = \frac{5}{13}e^{-2t}$. This distinction is made because, as we will soon see, there are other solutions as well.

Next we will examine the ODE $y'' + 7y' + 10y = 5t^2 - 8$. We might guess that the particular solution will be a fourth degree polynomial, because then y'' would be a second degree polynomial. It turns out that there is no harm in trying a fourth degree polynomial (you will try it in the exercises), but in fact a second degree polynomial is adequate. The next example demonstrates this.

⋄ **Example 3.4(b):** Find the coefficients A, B and C for which $y_p = At^2 + Bt + C$ is a solution to $y'' + 7y' + 10y = 5t^2 - 8$.

Solution: First we compute the needed derivatives:

$$y_p = At^2 + Bt + C \qquad \Longrightarrow \qquad y_p' = 2At + B \qquad \Longrightarrow y_p'' = 2A.$$

Next we substitute these values into the left side of the ODE and *group by powers of* t:

$$\begin{aligned} y_p'' + 7y_p' + 10y_p &= 2A + 7(2At + B) + 10(At^2 + Bt + C) \\ &= 2A + 14At + 7B + 10At^2 + 10Bt + 10C \\ &= 10At^2 + (14A + 10B)t + (2A + 7B + 10C) \end{aligned}$$

Setting this equal to the right hand side of the ODE gives us

$$10At^2 + (14A + 10B)t + (2A + 7B + 10C) = 5t^2 + 0t - 8,$$

and equating coefficients of powers of t (including the constant term) results in the three equations

$$10A = 5, \qquad 14A + 10B = 0, \qquad 2A + 7B + 10C = -8.$$

From the first equation we see that $A = \frac{1}{2}$. Substituting that value into the second equation and solving for B results in $B = -\frac{7}{10}$. Finally, when we substitute the values we have found for A and B into the last equation and solve for C we get $C = -\frac{41}{100}$. The particular solution to $y'' + 7y' + 10y = 5t^2 - 8$ is then

$$y_p = \tfrac{1}{2}t^2 - \tfrac{7}{10}t - \tfrac{41}{100}.$$

It is important to note that even though the right hand side $5t^2 - 8$ contains only a t^2 term and a constant term, we need to include a t term in our guess for the particular solution. If we had instead guessed a particular solution of the form $y_p = At^2 + B$ and substituted it into the ODE we would have obtained the two equations

$$10A = 5 \qquad \text{and} \qquad 14A = 0,$$

both of which cannot be true at the same time!

So far we have found a particular solution to $ay'' + by' + cy = f(t)$ when $f(t)$ is an exponential function (Example 3.4(a)) and a polynomial (Example 3.4(b)). The only other type of function we will consider for $f(t)$ is a trigonometric function; suppose we have the ODE

$$y'' + 4y' + 3y = 5\sin 2t. \tag{2}$$

Based on what we have seen so far, our first inclination might be that we should consider a particular solution of the form $y_p = A\sin 2t$. Let's try it:

$$y_p = A\sin 2t \qquad \Longrightarrow \qquad y_p' = 2A\cos 2t \qquad \Longrightarrow \qquad y_p'' = -4A\sin 2t$$

so

$$\begin{aligned} y_p'' + 4y_p' + 3y_p &= -4A\sin 2t + 8A\cos 2t + 3A\sin 2t \\ &= -A\sin 2t + 8A\cos 2t. \end{aligned}$$

Here we need $A = 0$ because there is no cosine term on the right hand side of (2), but then we'd have no sine term either!

So what do we do? Well, the "trick" is to let y_p have both sine and cosine terms even though the right side of the ODE has only a sine term.

◇ **Example 3.4(c):** Determine values for A and B so that $y_p = A\sin 2t + B\cos 2t$ is the particular solution to $y'' + 4y' + 3y = 5\sin 2t$.

Solution: We see that

$$y_p' = 2A\cos 2t - 2B\sin 2t, \qquad\qquad y_p'' = -4A\sin 2t - 4B\cos 2t.$$

so

$$\begin{aligned} y_p'' + 4y_p' + 3y_p &= (-4A\sin 2t - 4B\cos 2t) \\ &\qquad + 4(2A\cos 2t - 2B\sin 2t) + 3(A\sin 2t + B\cos 2t) \\ &= (-A - 8B)\sin 2t + (8A - B)\cos 2t \end{aligned}$$

Thus, in order for the left hand side to equal the right hand side, it must be the case that $8A - B = 0$ because there is no cosine term in the right hand side, and $-A - 8B = 5$, so that the sine terms are equal. Solving the first equation for B and substituting into the second equation results in $A = -\frac{1}{13}$. Substituting this back into the second equation gives $B = -\frac{8}{13}$. Our particular solution is then $y_p = -\frac{1}{13}\sin 2t - \frac{8}{13}\cos 2t$.

At this point we have the following guesses for a particular solution to a differential equation of the form $ay'' + by' + cy = f(t)$ when using the method of undetermined coefficients:

- If f is a polynomial of degree n, then

$$y_p = A_n t^n + A_{n-1}t^{n-1} + \cdots + A_2 t^2 + A_1 t + A_0$$

  Note that all powers of t less than or equal to the degree of $f(t)$ are included.

- If $f(t) = Ce^{kt}$, then $y_p = Ae^{kt}$.

- If $f(t) = C_1\sin kt + C_2\cos kt$, then $y_p = A\sin kt + B\cos kt$. *Even if one of C_1 or C_2 is zero, the trial y_p must still contain both the sine and cosine terms.*

In Section 4.3 we will see that there is a bit more to be added to this story, but for now the above summarizes what we have seen so far.

Section 3.4 Exercises To Solutions

1. For each of the following, give the form the particular solution must have.

   (a) $y'' + 3y' + 2y = 5t - 1$ (b) $y'' + 6y' + 9y = \cos 2t$

   (c) $y'' + 9y = 2e^{5t}$ (d) $y'' - 4y' - 5y = 6\sin t$

   (e) $2y'' + 3y' + y = 7$ (f) $y'' + 3y = 2t + 3e^{-t}$

2. Determine the particular solution for each of the ODEs in Exercise 1.

3. Suppose that you thought that the ODE $y'' + 3y' + 2y = 5t - 1$ should have a particular solution of $y_p = At^3 + Bt^2 + Ct + D$. (Note that this is the ODE from Exercise 1(a).) Substitute this into the ODE and see what happens for this guess. Does it give you the correct particular solution?

4. You would think that the particular solution to $y'' + 3y' + 2y = 6e^{-t}$ would have the form $y_p = Ae^{-t}$, but that is not the case. In Section 4.3 we will see what our guess for the particular solution should be. For now, try substituting the given particular solution into the ODE to see what happens.

5. Solve each of the following homogeneous ODEs, assuming the independent variable for each is t.

   (a) $y'' + 4y' + 29y = 0$
   (b) $2y'' + 11y' + 5y = 0$
   (c) $y'' + 6y' + 9y = 0$
   (d) $y'' + 3y = 0$

3.5 Differential Operators

Performance Criterion:

3. (i) Evaluate a differential operator for a given function.

An Example

In this section we will begin by exploring a specific second order ODE in order to illustrate some ideas we will capitalize on in order to solve linear, constant coefficient, second order ODEs. The ODE that we will be considering is

$$y'' + 9y = 5e^{-2t}, \tag{1}$$

which we found to have a particular solution of $y_p = \frac{5}{13}e^{-2t}$. (See Example 3.1(a).) This was obtained by substituting a guess of $y = Ae^{-2t}$ for y in $y'' + 9y$ and setting the result equal to $5e^{-2t}$. The following example shows that $y = \frac{5}{13}3^{-2t}$ is not the only solution to (1).

◇ **Example 3.5(a):** Show that $y = C\sin 3t + \frac{5}{13}e^{-2t}$, where C is *any* constant, is a solution to the differential equation (1).

Solution: Taking derivatives we get

$$y' = 3C\cos 3t - \tfrac{10}{13}3^{-2t} \qquad \text{and} \qquad y'' = -9C\sin 2t + \tfrac{20}{13}e^{-2t}.$$

Substituting in to the left hand side of the ODE we get

$$\begin{aligned} y'' + 9y &= -9C\sin 2t + \tfrac{20}{13}e^{-2t} + 9(C\sin 3t + \tfrac{5}{13}e^{-2t}) \\ &= -9C\sin 2t + \tfrac{20}{13}e^{-2t} + 9C\sin 3t + \tfrac{45}{13}e^{-2t}) \\ &= \tfrac{65}{13}e^{-2t} \\ &= 5e^{-2t} \end{aligned}$$

Thus $y = C\sin 3t + \frac{5}{13}e^{-2t}$ is a solution to $y'' + 9y = 5e^{-2t}$.

To examine further what is going on here it is convenient to develop some terminology and notation.

Differential Operators

A function can be thought of as a "mathematical machine" that takes in a number and, in return, gives out a number. There are other mathematical machines that take in things other than numbers, like functions or vectors, usually giving out things like what they take in. These sorts of "machines" are often referred to as **operators**. A simple example of an operator that you are quite familiar with is the derivative operator. When we take the derivative of a function, the result is another function. To indicate the action of a derivative on a function $y = y(t)$ we will write $\dfrac{dy}{dt}$ as

$$\frac{d}{dt}(y),$$

which is like the function notation $f(x)$, with $\frac{d}{dt}$ taking the place of f and y taking the place of x. The derivative operator has a very special property that you should be familiar with from calculus. If a and b are any constants and y_1 and y_2 are functions of t, then

$$\frac{d}{dt}(ay_1 + by_2) = \frac{d}{dt}(ay_1) + \frac{d}{dt}(by_2) = a\frac{d}{dt}(y_1) + b\frac{d}{dt}(y_2). \tag{2}$$

Operators that "distribute over addition" and "pass through constants" like this are called **linear operators**.

You might guess that the second derivative, and other higher derivatives, are linear operators as well, and that is correct. We are particularly interested in operators that are created by multiplying a function and some of its derivatives by constants and adding them all together. It is customary to denote such operators with the letter D, for **differential operator**. An example would be $D = 3\frac{d^2}{dt^2} + 5\frac{d}{dt} - 4$, whose action on a function $y = y(t)$ is defined by

$$D(y) = 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} - 4y. \tag{3}$$

Let's look at a specific example of how this operator works.

◇ **Example 3.5(b):** For the operator D defined by (3), find $D(y)$ when $y = t^2 - 3t$.

Solution:

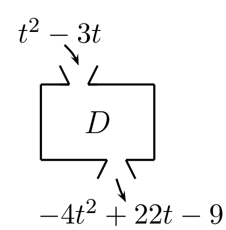
$$\begin{aligned} D(y) &= 3\frac{d^2}{dt^2}(t^2 - 3t) + 5\frac{d}{dt}(t^2 - 3t) - 4(t^2 - 3t) \\ &= 3(2) + 5(2t - 3) - 4(t^2 - 3t) \\ &= 6 + 10t - 15 - 4t^2 + 12t \\ &= -4t^2 + 22t - 9 \end{aligned}$$

When we combine several mathematical objects by multiplying each by a constant and adding (or subtracting) the results, we obtain what is called a **linear combination** of those objects. The operator D defined by (3) is a linear combination of a function and its first two derivatives. (We can think of the function itself as the "zeroth derivative," making all the things being combined derivatives.) You have seen linear combinations in other contexts; any polynomial function like

$$f(x) = 3x^4 - 7x^3 + \frac{1}{3}x^2 - x + 5.83$$

is a linear combination of $1, x, x^2, x^3, x^4, \ldots$. Those of you who have had a course in linear algebra have seen linear combinations of vectors.

Again, we can think of an operator as a machine that takes in a function and gives out some resulting function that is based somehow on the input function. This is illustrated to the right for Example 3.5(b). The next example demonstrates that a differential operator formed as a linear combination of derivatives is a linear operator.



◇ **Example 3.5(c):** Show that the operator D defined by

$$D(y) = 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} - 4y$$

is a linear operator by showing that it satisfies (2).

Solution: To determine whether D is linear we need to apply it to $ay_1 + by_2$:

$$\begin{aligned}
D(ay_1+by_2) &= 3\frac{d^2}{dt^2}(ay_1+by_2)+5\frac{d}{dt}(ay_1+by_2)-4(ay_1+by_2)\\
&= 3\left(a\frac{d^2y_1}{dt^2}+b\frac{d^2y_2}{dt^2}\right)+5\left(a\frac{dy_1}{dt}+b\frac{dy_2}{dt}\right)-4ay_1-4by_2\\
&= 3a\frac{d^2y_1}{dt^2}+3b\frac{d^2y_2}{dt^2}+5a\frac{dy_1}{dt}+5b\frac{dy_2}{dt}-4ay_1-4by_2\\
&= 3a\frac{d^2y_1}{dt^2}+5a\frac{dy_1}{dt}-4ay_1+3b\frac{d^2y_2}{dt^2}+5b\frac{dy_2}{dt}-4by_2\\
&= a\left(3\frac{d^2y_1}{dt^2}+5\frac{dy_1}{dt}-4y_1\right)+b\left(3\frac{d^2y_2}{dt^2}+5\frac{dy_2}{dt}-4y_2\right)\\
&= aD(y_1)+bD(y_2)
\end{aligned}$$

Because $D(ay_1+by_2) = aD(y_1)+bD(y_2)$, D is a linear operator.

At the second line above we have applied the fact that the first and second derivative are linear operators. With a bit of thought it should be clear that this, along with the distributive property, is what makes a linear combination of derivatives a linear operator.

Back to the Example

We return now to considering the ODE

$$y'' + 9y = 5e^{-2t}, \tag{1}$$

for which we have shown that both

$$y = \tfrac{5}{13}e^{-2t} \qquad \text{and} \qquad y = C\sin 3t + \tfrac{5}{13}e^{-2t} \tag{4}$$

are solutions. If we now let D be the operator defined by $D(y) = y'' + 9y$, then (4) says that

$$D\Big(\tfrac{5}{13}e^{-2t}\Big) = 5e^{-2t} \qquad \text{and} \qquad D\Big(C\sin 3t + \tfrac{5}{13}e^{-2t}\Big) = 5e^{-2t}.$$

Looking a little more closely, we see that

$$D\big(C\sin 3t\big) = \frac{d^2}{dt^2}\big(C\sin 3t\big) + 9\big(C\sin 3t\big) = -9C\sin 3t + 9C\sin 3t = 0.$$

This explains why D applied to the sum of $C\sin 3t$ and $\frac{5}{13}e^{-2t}$ is a solution; by linearity of differential operators,

$$D\Big(y_1 + \tfrac{5}{13}e^{-2t}\Big) = D\big(y_1\big) + D\Big(\tfrac{5}{13}e^{-2t}\Big) = 0 + 5e^{-2t} = 5e^{-2t}$$

for any function y_1 for which $D\big(y_1\big) = 0$ (like $C\sin 3t$, for example). This gives us the following:

Let D be a linear differential operator with independent variable t. If y_2 is a solution to $D(y_2) = f(t)$ and y_1 is a solution to $D(y_1) = 0$, then it is also the case that $D(y_1 + y_2) = f(t)$.

The general form of equation that we are most interested in is

$$ay'' + by' + cy = f(t), \tag{5}$$

where a, b and c are constants. Our goal is to find a **general solution** to this equation, meaning a solution that encompasses all possible solutions. Such a solution will consist of a particular solution to (5) that contains no arbitrary constants plus the family of *all* possible solutions to

$$ay'' + by' + cy = 0. \tag{6}$$

The solution to (5) without arbitrary constants is of course the **particular solution** to the equation, and the family of all possible solutions to (6) is the **homogeneous solution**. We saw in Section 3.1 how to find the homogeneous solution y_h and in Section 3.2 we saw how to find the particular solution y_p. The general solution is then $y = y_h + y_p$.

As an example, we know that the ODE

$$y'' + 9y = 5e^{-2t}$$

has homogeneous solution $y_h = C_1 \sin 3t + C_2 \cos 3t$ and particular solution $y_p = \frac{5}{13}e^{-2t}$, so the general solution is

$$y = y_h + y_p = C_1 \sin 3t + C_2 \cos 3t + \tfrac{5}{13}e^{-2t}.$$

For reasons you will see in Section 4.3, we will always find the homogeneous solution first and then the particular solution.

Section 3.5 Exercises

To Solutions

1. Let the differential operator D be defined on a function $y = y(t)$ by

$$D(y) = \frac{d^2}{dt^2}(y) + 3\frac{d}{dt}(y) + 2y.$$

   Find $D(y)$ for each of the following functions y.

   (a) $y = t^2 + 7t$ (b) $y = 5e^{-2t}$ (c) $y = 5\cos 2t$ (d) $y = \frac{5}{2}t - \frac{17}{4}$

2. What does your answer to Exercise 1(d) tell us about the ODE $y'' + 3y' + 2y = 5t - 1$?

3. Although you don't realize why at this point, your answer to Exercise 1(b) is somewhat special.

   (a) Does the same thing happen for $y = Ce^{-2t}$, where C is some constant other than 5? If so, for what value or values of C?

   (b) Does the same thing happen if $y = e^{kt}$, where k is some constant other than -2? If so, for what value or values of k?

4. Let the differential operator D be defined on a function $y = y(t)$ by

$$D(y) = \frac{d^2}{dt^2}(y) + 6\frac{d}{dt}(y) + 9y.$$

Find $D(y)$ for each of the following functions y.

(a) $y = e^{-3t}$ (b) $y = te^{-3t}$ (c) $y = 5e^{-3t} - 2te^{-3t}$

5. Let S be an operator on functions. S is linear if

$$S(af + bg) = aS(f) + bS(g), \tag{7}$$

where a and b are any constants and f and g are any functions of the sort that S can act on. (7) is equivalent to the two separate conditions that

$$S(af) = aS(f) \qquad \text{and} \qquad S(f + g) = S(f) + S(g), \tag{8}$$

where a, f and g are as before. That is, if both conditions in (8) hold for S, then it is a linear operator. Now let's define a specific operator S by $S(f(t)) = f(t) + 3$ for any function $f(t)$.

(a) What is $S(\cos t)$? $S(t^2 + 5t - 1)$?

(b) What is $S(4\cos t)$? What is $4S(\cos t)$? Are both results the same? What does that tell us about S, in terms of linearity?

(c) Find $S(\cos t + e^{2t})$ and $S(\cos t) + S(e^{2t})$.

3.6 Initial Value Problems and Forced, Damped Vibration

Performance Criteria:

3. (j) Solve a second order linear, constant coefficient IVP.
   (c) Set up and solve second order initial value problems modeling spring-mass systems and RLC circuits.
   (k) Identify the transient and steady-state parts of the solution to a damped system with forced vibration.

Now that we know how to find the general solution to $ay'' + by' + cy = f(t)$ we are ready to solve initial value problems whose ODEs are of this form. Here is how the process goes:

Solving the IVP $\quad ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y'_0$

1) Find the homogeneous solution y_h to $ay'' + by' + cy = 0$.
2) Use the method of undetermined coefficients to find the particular solution y_p of the ODE $ay'' + by' + cy = f(t)$.
3) Construct the general solution $y = y_h + y_p$.
4) Apply the initial conditions to the general solution to find the values of the arbitrary constants.

We have already covered the first three steps of the above. It is very important to remember that *the initial conditions are applied to the general solution* to find the values of the arbitrary constants. A common mistake by students is to find the values of the constants based on only the homogeneous solution - this is incorrect.

Let's look at an example:

◇ **Example 3.6(a):** Solve the IVP

$$y'' + 4y' + 3y = 5\sin 2t, \quad y(0) = 2, \quad y'(0) = -1$$

Solution: We must first solve the homogeneous equation $y'' + 4y' + 3y = 0$. The roots of the auxiliary equation are $r_1 = -1$ and $r_2 = -3$ (of course it doesn't matter which is which), so the homogeneous solution is $y_h = C_1 e^{-t} + C_2 e^{-3t}$. Our trial particular solution is $y_p = A\sin 2t + B\cos 2t$. This gives us

$$y'_p = 2A\cos 2t - 2B\sin 2t, \qquad y''_p = -4A\sin 2t - 4B\cos 2t.$$

so

$$\begin{aligned}\text{LHS} &= (-4A\sin 2t - 4B\cos 2t) + 4(2A\cos 2t - 2B\sin 2t) + 3(A\sin 2t + B\cos 2t)\\ &= (-A - 8B)\sin 2t + (8A - B)\cos 2t\end{aligned}$$

Thus, in order for the left hand side to equal the right hand side, it must be the case that $8A - B = 0$ because there is no cosine term in the right hand side, and $-A - 8B = 5$, so that the sine terms are equal. Solving the first equation for B and substituting into the second equation results in $A = -\frac{1}{13}$. Substituting this back into the second equation gives $B = -\frac{8}{13}$. Our particular solution is then $y_p = -\frac{1}{13}\sin 2t - \frac{8}{13}\cos 2t$, and the general solution is

$$y = y_h + y_p = C_1 e^{-t} + C_2 e^{-3t} - \tfrac{1}{13}\sin 2t - \tfrac{8}{13}\cos 2t$$

The derivative of the general solution is

$$y' = -C_1 e^{-t} - 3C_2 e^{-3t} - \tfrac{2}{13}\cos 2t + \tfrac{16}{13}\sin 2t$$

Applying the initial conditions gives us the two equations

$$C_1 + C_2 - \tfrac{8}{13} = 2 \qquad \text{and} \qquad -C_1 - 3C_2 - \tfrac{2}{13} = -1$$

Adding these and solving for C_2 gives us $C_2 = -\frac{23}{26}$. Substituting this into either equation gives $C_1 = \frac{91}{26}$. The solution to the IVP is then

$$y = \tfrac{91}{26}e^{-t} - \tfrac{23}{26}e^{-3t} - \tfrac{1}{13}\sin 2t - \tfrac{8}{13}\cos 2t$$

It is often the case that finding the constants C_1 and C_2 comes down to solving a system of two equations in two unknowns, as it did here.

Section 3.6 Exercises To Solutions

1. You may have found the particular solution to each of the following ODEs in Exercise 2 of Section 3.4. Give the general solution to each.

   (a) $y'' + 3y' + 2y = 5t - 1$ (b) $y'' + 6y' + 9y = 5\cos 3t$

   (c) $y'' + 9y = 2e^{5t}$ (d) $y'' - 4y' - 5y = 6\sin t$

   (e) $2y'' + 3y' + y = 7$ (f) $y'' + 3y = 2t + 3e^{-t}$

2. Solve each of the following IVPs by the process described in the box at the start of the section.

   (a) $y'' + 9y = 4\sin t\,, \quad y(0) = 2\,, \quad y'(0) = 4$

   (b) $y'' + 4y' + 4y = 5e^{3t}\,, \quad y(0) = 0\,, \quad y'(0) = 0$

   (c) $y'' - 10y' + 25y = 30t + 3\,, \quad y(0) = 2\,, \quad y'(0) = 8$

3. Solve each Euler equation.

   (a) $x^2y'' - 6y = 0$ (b) $4x^2y'' + 4xy' - y = 0$

4. Consider the second order initial value problem

$$y'' + 2y' + 10y = 9.4\sin t\ , \quad y(0) = 5\ , \quad y'(0) = 0,$$

   which could model either a spring-mass system or an RLC circuit. In this exercise you will find the solution to this initial value problem, and you will investigate its behavior. Recall the process for solving such an equation:

- Find the homogeneous solution y_h. It will contain two constants.
- Use undetermined coefficients (guessing) to find the particular solution y_p to the equation.
- Add y_c and y_p to find the general solution to the equation.
- Apply the two initial conditions to determine the values of the unknown constants. **Be sure to conclude by writing your final solution to the IVP**.

(a) Carry out the above process for the given IVP. **Give all numbers as decimals, rounded to the tenth's place.**

(b) Graph the solution on your calculator or an online tool like *Desmos*, for $t = 0$ to $t = 10$. Sketch your graph.

(c) Look carefully at your solution (the equation itself, not the graph). Recall that the transient part of a solution is any part that goes to zero as time goes on. Any part that does not go to zero over time is called the steady-state part of the solution, or just the steady-state solution. Give the transient and steady-state parts of your solution, telling clearly which is which.

(d) Graph the steady-state solution together with the complete solution that you already graphed. The complete solution should approach the steady state solution as time goes on. Add the graph of the steady-state solution to your graph from (b).

3.7 Chapter 3 Summary

With the exception of the methods for solving Euler equations, this chapter was primarily concerned with solving initial value problems of the form

$$ay'' + by' + cy = f(t), \qquad y(0) = y_0,\ y'(0) = y_0', \tag{1}$$

with a, b and c being constants, $a \neq 0$. Here is the procedure for solving the initial value problem (1):

Solving the IVP $ay'' + by' + cy = f(t),\ y(0) = y_0,\ y'(0) = y_0'$

1) Find the homogeneous solution y_h to $ay'' + by' + cy = 0$. *There is a flowchart on the next page, outlining the process for finding the homogeneous solution.*

2) Use the method of undetermined coefficients (see below) to find the particular solution y_p of the ODE $ay'' + by' + cy = f(t)$.

3) Construct the general solution $y = y_h + y_p$.

4) Apply the initial conditions to the general solution to find the values of the arbitrary constants.

Undetermined Coefficients

Consider the constant coefficient ODE $ay'' + by' + cy = f(t)$, and assume y_h contains no terms that are constant multiples of $f(t)$. The trial particular solution y_p is chosen as follows.

- If f is a polynomial of degree n, then

$$y_p = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_2 t^2 + A_1 t + A_0$$

- If $f(t) = Ce^{kt}$, then $y_p = Ae^{kt}$.

- If $f(t) = C_1 \sin kt + C_2 \cos kt$, then $y_p = A \sin kt + B \cos kt$. *Even if one of C_1 or C_2 is zero, y_p still contains both the sine and cosine terms.*

We will see in the next chapter how the method of undetermined coefficients needs to be modified when y_h contains any term that is a constant multiple of f.

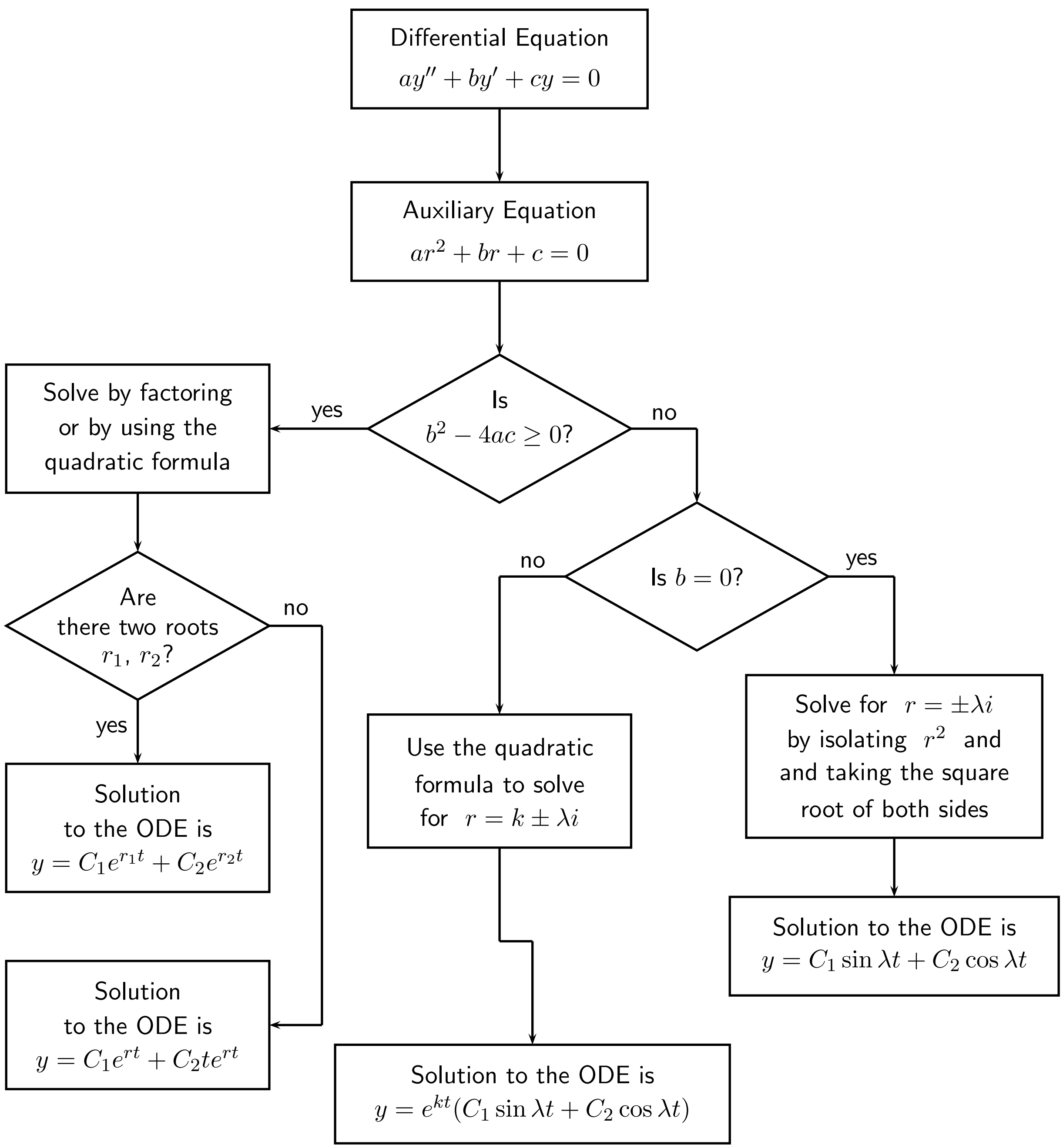
A couple of additional comments are in order:

- The homogeneous solution has the form $y_h = C_1 g(t) + C_2 h(t)$, where C_1 and C_2 are arbitrary constants and g and h are "different" functions (in a sense that will be made more precise in the next chapter) that are each solutions to the homogeneous ODE $ay'' + by' + cy = 0$ by themselves. *Every possible solution to the homogeneous ODE looks like $y_h = C_1 g(t) + C_2 h(t)$ for the same functions g and h.*

- The particular solution to the ODE in (1) contains no arbitrary constants, which is why it is called "particular." Another method for finding the particular solution is called **variation of parameters**. The interested reader can find an explanation of this method on the internet or in any introductory differential equations text.

Here is a flowchart for finding homogeneous solutions:

Solving Second Order, Linear, Constant Coefficient, Homogeneous ODEs



3.8 Chapter 3 Exercises

1. Solve each Euler equation.

   (a) $3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$ (b) $6x^2\dfrac{d^2y}{dx^2} + 11x\dfrac{dy}{dx} + y = 0$

2. In Exercise 4 of the Chapter 2 Exercises we saw the Euler equation

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R = 0 \tag{1}$$

   which arises in the study of the equilibrium distribution of heat in a circular disk. As pointed out before, r and R are two different variables; R is the dependent variable, and is a function of the independent variable r. In Chapter 2 we solved the equation for the case $n = 0$. Solve it for any integer $n \neq 0$.

3. The ODE $ay'' + by' + cy = 0$ is only second order if $a \neq 0$. We saw in Section 3.3 what the solution to the ODE looks like when $b = 0$. In this exercise and the next we will solve, by two different methods, an equation in which $c = 0$

   (a) In this exercise we will solve $2y'' + 3y' = 0$. begin by making the substitution $y' = x$ (so what then is y''?) and solving the resulting first order ODE for x.

   (b) To determine y you will now need to integrate your answer to (a). DO that, remembering that your final solution needs to contain two arbitrary constants.

4. Solve $2y'' + 3y' = 0$ by assuming $y = e^{rt}$ and following a process like that done for the various scenarios in Section 3.1.

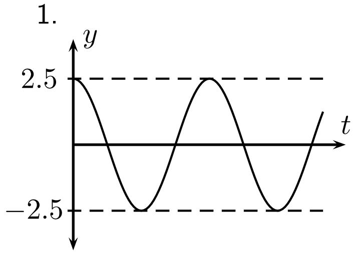
D Solutions to Exercises

D.3 Chapter 3 Solutions

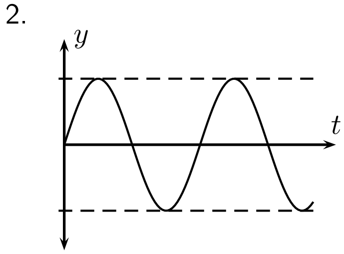
Section 3.1 Solutions Back to 3.1 Exercises

1. (a) $y = C_1 x + C_2 x^4$ (b) $y = C_1 x^{-1} + C_2 x^{-2}$ (c) $y = C_1 x + C_2 x^{\frac{1}{3}}$
2. (a) $y = C_1 e^{3t} + C_2 e^{-t}$ (b) $y = e^{-t}(C_1 \sin 3t + C_2 \cos 3t)$
   (c) $y = C_1 e^{-5t} + C_2 t e^{-5t}$ (d) $y = e^{-3t}(C_1 \sin 2\sqrt{2}\,t + C_2 \cos 2\sqrt{2}\,t)$
   (e) $y = C_1 e^{-t} + C_2 e^{-2t}$ (f) $y = C_1 \sin \sqrt{2}\,t + C_2 \cos \sqrt{2}\,t$
   (g) $y = C_1 e^{-t} + C_2 t e^{-t}$ (h) $y = C_1 \sin 4t + C_2 \cos 4t$
   (i) $y = e^{-1.55t}(C_1 \sin 1.45t + C_2 \cos 1.45t)$
3. (a) $r^2 + 25 = 0$ and $r^2 + 25r = 0$ (b) $y = C_1 + C_2 e^{-25t}$
4. (a) $y = C_1 \sin \lambda t + C_2 \cos \lambda t$ (b) $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$
   (c) $y = e^{kt}(C_1 \sin \lambda t + C_2 \cos \lambda t)$ (d) $y = C_1 e^{rt} + C_2 t e^{rt}$
5. (a) $y = Cx^2$

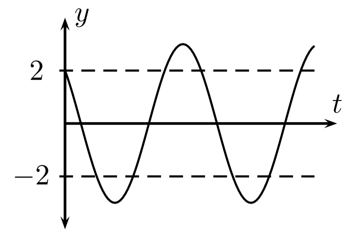
Section 3.2 Solutions Back to 3.2 Exercises

1.


$y(0) = 2.5,\ y'(0) = 0$

2.


$y(0) = 0,\ y'(0) = 3$

3.


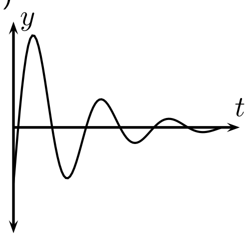
$y(0) = 2,\ y'(0) = -8$

4. (a) $\frac{3}{4}y'' + 15y = 0,\quad y(0) = 2.5,\ y'(0) = 0$
   (b) $y = 2.5 \cos 4.47\ t$ (d) $y = 2.5 \sin(4.47\ t + 1.57)$
   (e) amplitude is 2.5, angular frequency is 4.47, period is 1.41, frequency is 0.71, phase shift is -0.35
5. (a) $y = -1.79 \sin(4.47\,t) + 2.00 \cos(4.47\,t)$
   (c) $y = 2.68 \sin(4.47\,t + 2.30)$
   (d) amplitude is 2.68, angular frequency is 4.47, period is 1.41, frequency is 0.71, phase shift is -0.51
6. (a) $\frac{4}{10}y'' + 4y = 0,\quad y(0) = -4,\ y'(0) = 9$
   (b) $y = 2.85 \sin 3.16\ t - 4.00 \cos 3.16\ t \quad \Longrightarrow \quad y = 4.91 \sin(3.16\ t - 0.95)$
   (c) amplitude 4.91, angular frequency 3.16, period 1.99, frequency 0.50, phase shift 0.30

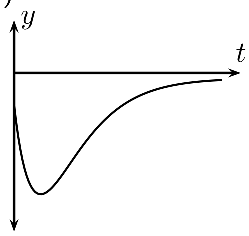
Section 3.3 Solutions Back to 3.3 Exercises

1. (a) $y(t) = e^{-0.6t}(-1.2 \sin 4.0t + 2.0 \cos 4.0t) \quad \Rightarrow \quad y(t) = 2.3e^{-0.6t} \sin(4.0t + 2.1)$
2. (a) $y(t) = \frac{11}{3}e^{-2t} - \frac{5}{3}e^{-8t}$
3. (a) $\beta = 40$ (b) $y(t) = 2e^{-4t} + 14te^{-4t}$

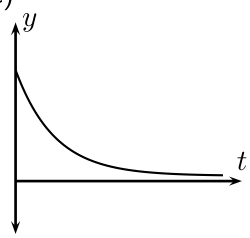
5. (a)



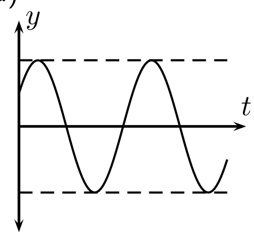
(b)



(c)



(d)



6. (a) The system is underdamped because $R^2 - 4L \cdot \frac{1}{C} < 0$.
   (b) $q(t) = e^{-1670t}(2.18 \times 10^{-5} \sin 1530t + 2.00 \times 10^{-5} \cos 1530t)$ coulombs
   (c) $q(t) = 2.96 \times 10^{-5} e^{-1670t} \sin(1530t + 0.742)$ coulombs
   (d) $i(t) = \dfrac{dq}{dt} = -4.94 \times 10^{-2} \cos(1530t + 0.742)$ amperes

Section 3.4 Solutions Back to 3.4 Exercises

1. (a) $y_p = At + B$ (b) $y_p = A \sin 3t + B \cos 3t$ (c) $y_p = Ae^{5t}$
   (d) $y_p = A \sin t + B \cos t$ (e) $y_p = A$ (f) $y_p = At + B + Ce^{-t}$
2. (a) $y_p = \frac{5}{2}t - \frac{17}{4}$ (b) $y_p = \frac{12}{169} \sin 2t + \frac{5}{169} \cos 2t$ (c) $y_p = \frac{1}{17}e^{5t}$
   (d) $y_p = -\frac{9}{13} \sin t + \frac{6}{13} \cos t$ (e) $y_p = 7$ (f) $y_p = \frac{2}{3}t + \frac{3}{4}e^{-t}$
5. (a) $y = e^{-2t}(C_1 \sin 5\,t + C_2 \cos 5\,t)$ (b) $y = C_1 e^{-5t} + C_2 e^{-\frac{1}{2}t}$
   (c) $y = C_1 e^{-3t} + C_2 t e^{-3t}$ (d) $y = C_1 \sin \sqrt{3}\,t + C_2 \cos \sqrt{3}\,t$

Section 3.5 Solutions Back to 3.5 Exercises

1. (a) $D(y) = 2t^2 + 20t + 23$ (b) $D(y) = 0$
   (c) $D(y) = -10 \cos 2t - 30 \sin 2t$ (d) $D(y) = 5t - 1$
2. $y = \frac{5}{2}t - \frac{17}{4}$ is a particular solution to the ODE $y'' + 3y' + 2y = 5t - 1$.
3. (a) $D(Ce^{-2t}) = 0$ for all values of C.
   (b) $D(e^{kt}) = 0$ only when $k = -2$ or $k = -1$.
4. $D(y) = 0$ in all three cases.
5. (a) $S(\cos t) = \cos t + 3,\;\; S(t^2 + 5t - 1) = t^2 + 5t + 2$
   (b) $S(4 \cos t) = 4 \cos t + 3,\;\; 4S(\cos t) = 4(\cos t + 3) = 4 \cos t + 12$ These are not the same, so S is not linear.
   (c) $S(\cos t + e^{2t}) = \cos t + e^{2t} + 3,\;\; S(\cos t) + S(e^{2t}) = \cos t + 3 + e^{2t} + 3 = \cos t + e^{2t} + 6$

Section 3.6 Solutions Back to 3.6 Exercises

1. (a) $y = C_1e^{-2t} + C_2e^{-t} + \frac{5}{2}t - \frac{17}{4}$
   (b) $y_h = C_1e^{-3t} + C_2te^{-3t} + \frac{5}{18}\sin 3t$
   (c) $y_h = C_1 \sin 3t + C_2 \cos 3t + \frac{1}{17}e^{5t}$
   (d) $y_h = C_1e^{-t} + C_2e^{5t} - \frac{9}{13}\sin t + \frac{6}{13}\cos t$
   (e) $y_h = C_1e^{-\frac{1}{2}t} + C_2e^{-t} + 7$
   (f) $y_h = C_1 \sin \sqrt{3}t + C_2 \cos \sqrt{3}t + \frac{2}{3}t + \frac{3}{4}e^{-t}$
2. (a) $y = \frac{7}{6}\sin 3t + 2\cos 3t + \frac{1}{2}\sin t$
   (b) $y = -\frac{1}{5}e^{-2t} - te^{-2t} + \frac{1}{5}e^{3t}$
   (c) $y = \frac{7}{5}e^{5t} - \frac{1}{5}e^{5t} + \frac{6}{5}t + \frac{3}{5}$
3. (a) $y = C_1x^{-2} + C_2x^3$
   (b) $y = C_1x^{\frac{1}{2}} + C_2x^{-\frac{1}{2}}$
4. (a) $y = e^{-t}(1.4\sin 3t + 5.2\cos 3t) + \sin t - 0.2\cos t$
   (c) Transient part: $e^{-t}(1.4\sin 3t + 5.2\cos 3t)$ Steady-state part: $\sin t - 0.2\cos t$