Ordinary Differential Equations

for Engineers and Scientists

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Contents

4 More on Second Order Differential Equations

Learning Outcome:

4. Understand independence of solutions to ODEs, and know how to use reduciton of order to find second solutions. Understand the nature of solutions to second order linear, constant coefficient ODEs and IVPs modeling spring-mass systems or RLC circuits, including resonance and beats.

Performance Criteria:

(a) Demonstrate that two functions f and g are dependent by giving nonzero constants c_1 and c_2 for which $c_1f(x) + c_2g(x) = 0$.

(b) Use the Wronskian to determine whether two solutions to a second order linear ODE are independent.

(c) Given one solution to a second order homogeneous ODE, use reduction of order to find a second solution.

(d) Determine the particular solution to a differential equation of the form $ay'' + by' + cy = f(t)$ when the homogeneous solution has the same form as $f(t)$.

(e) Determined whether a forced, undamped system will exhibit resonance, beats, or neither. Determine the solution for such a system.

(f) For a spring-mass system or electric circuit, demonstrate an understanding of the relationships between

- the physical situation (presence and type of damping and/or forcing)
- the form of the ODE, including the function f
- the the analytic and graphical nature of the solution (in particular, the presence and appearance of transient and steady-state parts of the solution)

The bulk of our efforts in Chapter 3 were focused on solving second order ODEs of the form

$$ay'' + by' + cy = f(t). \tag{1}$$

There are three issues that came up that we put off at the time:

- When solving the homogeneous equation $ay'' + by' + cy = 0$ we usually found two "different" solutions, but when solving an equation like $y'' + 6y' + 9y = 0$ we only found one solution, $y = e^{-3t}$.

- In some cases when we attempted to find a particular solution to (1) the "standard" guess for a trial particular solution failed to give us a result.

- We neglected to address situation in which $b = 0$ and $f(t) \neq 0$, which we call forced, undamped vibration.

Regarding the first item, there are two questions we will address:

(1) What do we mean by "different" solutions?

(2) In addition to the solution $y = e^{-3t}$ to $y'' + 6y' + 9y = 0$ that we found using the auxiliary equation, we also saw that $y = te^{-3t}$ is a solution. How is such a solution found?

The first question above addresses the concept of **linear indepndence** of solutions. If you have had a course in linear algebra you should be familiar with the idea in that context. This is addressed in Section 4.1. The second question is answered by a method called **reduction of order**, which we'll see in Section 4.2.

In Section 4.3 we will return to the finding of particular solutions. When one of the solutions to the homogeneous equation $ay'' + by' + cy = 0$ associated with the ODE

$$ay'' + by' + cy = f(t). \tag{1}$$

has the same form as $f(t)$, our previously used guesses for particular solutions will not yield a result, so we must modify our trial particular solution in a way described in Sectoin 4.3.

Finally, we'll go back to undamped systems, but with nonzero forcing functions $f(t)$, which are often sine or cosine functions. This will give rise to two phenomena called **beats** and **resonance**. These things will be studied in Section 4.4.

4.1 Linear Independence of Solutions

Performance Criteria:

4. (a) Demonstrate that two functions f and g are dependent by giving nonzero constants c_1 and c_2 for which $c_1f(x) + c_2g(x) = 0$.

   (b) Use the Wronskian to determine whether two solutions to a second order linear ODE are independent.

We begin with two questions:

(1) When solving the ODE $y'' + 3y' + 2y = 0$, we assumed a solution of the form $y = e^{rt}$ for some constant r and found that r must equal -1 or -2. We then assumed that every solution to the ODE is of the form $y = C_1e^{-t} + C_2e^{-2t}$. How do we know that this is the case?

(2) When solving $y'' + 2y' + y = 0$ we found only one solution, $y = e^{-t}$. We then demonstrated that $y = te^{-t}$ is also a solution, and we assumed that the general solution to the ODE is $y = C_1e^{-t} + C_2te^{-t}$. How might one know or find the second solution without it being given?

In this section we will develop some language and see some theorems that answer the first question, and in the next section we'll see a way to use **reduction of order** (see Chapter 2 exercises) that gives an answer to the second question.

Linearly Independent Solutions

Linearly Independent Functions

Two functions f and g are **linearly dependent** on an interval $[a, b]$ if there exist two *non-zero* constants c_1 and c_2 for which

$$c_1f(x) + c_2g(x) = 0 \qquad \text{for every } x \text{ in } [a, b]. \tag{1}$$

If (1) is true only when both c_1 and c_2 are zero, then f and g are **linearly independent** on $[a, b]$.

The expression $c_1f(x) + c_2g(x)$ above is called a **linear combination** of f and g.

◇ **Example 4.1(a):** Show that the two functions $y = e^{-2t}$ and $y = e^{-t}$ are linearly independent for all values of t.

Solution: Suppose that $c_1e^{-2t} + c_2e^{-t} = 0$ for some c_1 and c_2. Then $e^{-2t}(c_1 + c_2e^t) = 0$, but e^{-2t} is never zero so it must be the case that $c_1 + c_2e^t = 0$, which implies that $c_1 = -c_2e^t$. Because e^t is not constant, this can only be true if $c_1 = c_2 = 0$. Therefore $y = e^{-2t}$ and $y = e^{-t}$ are linearly independent.

There may be times that it is difficult to tell, using the definition above, whether two functions are linearly independent. In those cases we can use a new function created from the two functions, called the **Wronskian**, to determine whether the functions are linearly independent.

The Wronskian of Two Functions

The **Wronskian** W of two functions f and g is

$$W(x) = f(x)g'(x) - f'(x)g(x).$$

Those of you who have had linear algebra may recognize the Wronskian as the determinant of the 2×2 matrix

$$\left[\begin{array}{cc} f(x) & g(x) \\ f'(x) & g'(x) \end{array}\right]$$

Our interest is in determining whether two solutions to an ODE are linearly independent. Note that any linear second order homogeneous ODE with independent variable t can (almost always, and definitely in the case that p and q are constant) be written in the form

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

The following tells how the Wronskian is used to determine whether two solutions to an equation of the form (2) are linearly independent.

The Wronskian and Linearly Independent Solutions

Two solutions f and g of (2) are linearly independent on the interval (a, b) if there exists some point x in the interval for which $W(x) \neq 0$.

◇ **Example 4.1(b):** Show that the two solutions $y = \sin 3t$ and $y = \cos 3t$ of $y'' + 9y = 0$ are linearly independent for all values of t.

Solution: The Wronskian of these two functions is

$$W(t) = (\sin 3t)(\cos 3t)' - (\cos 3t)(\sin 3t)' = -\sin^2 3t - \cos^2 3t = -1,$$

which is clearly not zero for *any* value of t. Therefore $y = \sin 3t$ and $y = \cos 2t$ are linearly independent for all values of t.

We conclude with why we are interested in all of this.

General Solutions to $y'' + p(t)y' + q(t)y = 0$

If y_1 and y_2 are linearly independent solutions to $y'' + p(t)y' + q(t)y = 0$, then the general solution is

$$y = C_1y_1 + C_2y_2$$

for arbitrary constants C_1 and C_2.

Note that the above says that the general solution is a linear combination of the two solutions y_1 and y_2.

In the exercises you will show that $y = e^{-t}$ and $y = te^{-t}$ are linearly independent solutions to $y'' + 2y' + y = 0$, so the above tells us that the general solution is then $y = C_1e^{-t} + C_2te^{-t}$. In the next section we'll find out where the solution $y = te^{-t}$ comes from.

Section 4.1 Exercises

To Solutions

1. For each pair of functions, give nonzero constants c_1 and c_2 for which $c_1f(x)+c_2g(x) = 0$ for all real numbers x *if possible*. Note that when this can be done, the two functions are dependent.

   (a) $f(x) = 3x^2 - 5$, $g(x) = 2x + 1$

   (b) $f(x) = 4x + 2$, $g(x) = 2x + 1$

   (c) $f(x) = 3e^{5x}$, $g(x) = -2e^{5x}$

   (d) $f(x) = 3e^{5x}$, $g(x) = e^{2x}$

2. Use the facts that $\cos(-x) = \cos x$ and $\sin(-x) = -\sin(x)$ for the following.

   (a) Give nonzero constants c_1 and c_2 such that $c_1 \cos x + c_2 \cos(-x) = 0$. Are $\cos x$ and $\cos(-x)$ linearly independent?

   (b) Repeat part (a) for $\sin x$ and $\sin(-x)$.

3. For each pair of functions in Exercise 1 that you could not find nonzero constants c_1 and c_2 for which $c_1f(x) + c_2g(x) = 0$, give the Wronskian and one value of x for which it is not zero.

4. Find the Wronskian for $y_1 = e^{kt}$ and $y_2 = te^{kt}$ (where k is any nonzero constant) and give a value of t for which it is not zero. What does this tell us about the functions y_1 and y_2?

5. Use the Wronskian to determine whether e^x and e^{-x} are linearly independent.

4.2 Reduction of Order

Performance Criteria:

4. (c) Given one solution to a homogeneous second order ODE, use reduction of order to find a second solution.

Recall the questions with which we begin the previous section:

(1) When solving the ODE $y'' + 3y' + 2y = 0$, we assumed a solution of the form $y = e^{rt}$ for some constant r and found that r must equal -1 or -2. We then assumed that every solution to the ODE is of the form $y = C_1e^{-t} + C_2e^{-2t}$. How do we know that this is the case?

(2) When solving $y'' + 2y' + y = 0$ we found only one solution, $y = e^{-t}$. We then demonstrated that $y = te^{-t}$ is also a solution, and we assumed that the general solution to the ODE is $y = C_1e^{-t} + C_2te^{-t}$. How might one know or find the second solution without it being given?

The result in the box at the bottom of page 122 answers the first question. In this section we take up the second question.

Reduction of order is a method for finding a second solution to a second order differential equation when one solution is already known. Our main interest in this is finding the second solution when we have repeated roots, so we will not go into the method in excessive detail. Perhaps the best way to introduce the method is through an example. The two key ideas are these:

- We will assume that if $y_1 = y_1(t)$ is a solution, then the second solution has the form $y_2(t) = u(t)y_1(t)$ for some function u. We then substitute y_2 into the ODE, which results in a new ODE for u.
- The new ODE for u will contain u'' and u' terms, but no u term. If we let $v(t) = u'(t)$ then $v'(t) = u''(t)$, and making these two substitutions we get a first order equation in v. (This is where the name *reduction of order* comes from - we've reduced a second order equation to a first order equation.) We solve that for v, then solve $u'(t) = v(t)$ to get u.

Now let's get to that example!

◇ **Example 4.2(a):** Use the solution $y_1(t) = e^{-t}$ and reduction of order to find a second solution to $y'' + 3y' + 2y = 0$.

Solution: We begin by assuming $y_2 = u(t)e^{-t}$. Then (using the product rule),

$$y_2' = -u(t)e^{-t} + u'(t)e^{-t} \qquad \text{and} \qquad y_2'' = u(t)e^{-t} - 2u'(t)e^{-t} + u''(t)e^{-t}.$$

Substituting into the ODE we get

$$\begin{aligned}
y_2'' + 3y_2' + 2y_2 &= \left[u(t)e^{-t} - 2u'(t)e^{-t} + u''(t)e^{-t}\right] + 3\left[-u(t)e^{-t} + u'(t)e^{-t}\right] + 2u(t)e^{-t} \\
&= u(t)e^{-t} - 2u'(t)e^{-t} + u''(t)e^{-t} - 3u(t)e^{-t} + 3u'(t)e^{-t} + 2u(t)e^{-t} \\
&= u''(t)e^{-t} + u'(t)e^{-t} \\
&= e^{-t}(u''(t) + u'(t))
\end{aligned}$$

Setting the result equal to zero (because we want $y_2 = u(t)e^{-t}$ to be a solution to $y''+3y'+2y = 0$) and noting that e^{-t} is never zero, we must have $u''(t)+u'(t) = 0$. Here we let $v(t) = u'(t)$, so $v'(t) = u''(t)$ and this last ODE becomes $v'(t)+v(t) = 0$. This equivalent to $v'(t) = -v(t)$, so $v(t) = C_1e^{-t}$.

We now replace $v(t)$ with $u'(t)$ to obtain $u'(t) = C_1e^{-t}$. The solution to this is $u(t) = C_2e^{-t}+C_3$; for reasons to be given later, we can take C_2 to be any non-zero value and C_3 can have any value. We'll take $C_2 = 1$ and $C_3 = 0$. Therefore $y_2(t) = u(t)e^{-t} = e^{-t}e^{-t} = e^{-2t}$. Disregarding the constant (because we will replace it when adding this solution to the one given), we have the second solution $y_2 = e^{-2t}$.

Forming the linear combination of the given solution and the one that we found using it, we get $y = ae^{-t} + be^{-2t}$ for constants a and b. We now examine the way that the constants C_2 and C_3 were handled in the above. Let's see what would have happened if we had not let $C_2 = 1$ and $C_3 = 0$. In that case we would have had

$$y_2 = u(t)y_1(t) = (C_2e^{-t} + C_3)e^{-t} = C_2e^{-2t} + C_3e^{-t}.$$

When we then form a linear combination of y_1 and y_2 using constants A and B, we'll get

$$\begin{aligned} y &= Ae^{-t} + B(C_2e^{-2t} + C_3e^{-t}) \\ &= Ae^{-t} + BC_2e^{-2t} + BC_3e^{-t} \\ &= (A + BC_3)e^{-t} + BC_2e^{-2t} \\ &= ae^{-t} + be^{-2t}, \end{aligned}$$

where $a = A + BC_3$ and $b = BC_2$. If we keep the constants C_2 and C_3, they essentially get "absorbed" into the constants for the linear combination of the two solutions.

In Example 4.2(a) there was no need to use reduction of order to determine a second solution $y_2 = e^{-2t}$ from the first solution $y_1 = e^{-2t}$; we could arrive at both solutions via the auxiliary equation method. However, the above example demonstrates how the method works. In the exercises you will encounter ODEs for which you will again be asked to find a second solution by this method when it is unnecessary, but you will also use it for situations where the second solution (and maybe the first as well) can't be obtained by methods we have used so far. You will also use reduction of order to find the second solution to $y'' + 2y' + y = 0$, knowing the first solution $y_1 = e^{-t}$, which is obtained by the auxiliary equation method.

Section 4.2 Exercises To Solutions

1. Consider the ODE $y'' + 8y' + 15y = 0$.

   (a) Given that one solution is $y_1 = e^{-5t}$, use reduction of order to find another solution.

   (b) Use the auxiliary equation to find both solutions, to check your answer to (a).

2. Using the auxiliary equation method with $y'' + 2y' + y = 0$, we get the single solution $y_1 = e^{-t}$. Use reduction of order to obtain the second solution $y_2 = te^{-t}$.

3. Given that one solution to $2x^2y'' + xy' - 3y = 0$ is $y_1 = \frac{1}{x}$, find a second solution y_2.

4. Given that one solution to $x^2y'' + 2xy' - 2y = 0$ is $y_1 = x$, find a second solution y_2.

4.3 Particular Solutions, Part Two

Performance Criteria:

4. (d) Determine the particular solution for a differential equation of the form $ay'' + by' + cy = f(t)$ when the homogeneous solution has the same form as $f(t)$.

At this point you have seen the entire process for solving initial value problems for second order, linear, constant coefficient differential equations. In this section we see one difficulty that can arise, and how to handle such situations. We begin with an example.

◇ **Example 4.3(a):** Determine the values of A and B for which

$$y_p = A\sin 3t + B\cos 3t$$

is the particular solution to the ODE $y'' + 9y = 2\sin 3t$.

Solution: As usual, we begin by finding the derivatives of y_p:

$$y_p' = 3A\cos 3t - 3B\sin 3t \qquad \Longrightarrow \qquad y_p'' = -9A\sin 3t - 9B\cos 3t.$$

We then have

$$\text{LHS} = y_p'' + 9y_p = -9A\sin 3t - 9B\cos 3t + 9(A\sin 3t + B\cos 3t) = 0.$$

Thus there are no values of A and B for which $y = A\sin 3t + B\cos 3t$ is a solution to $y'' + 9y = 2\sin 3t$.

The problem here is that the homogeneous solution to $y'' + 9y = 2\sin 3t$ is $y_h = C_1\sin 3t + C_2\sin 3t$. Thus we cannot hope to obtain $2\sin 3t$ when applying the operator $D = \dfrac{d^2}{dt^2} + 9$ to $y = A\sin 3t + B\cos 3t$, as the result is always zero. However, we will find that a different guess for y_p will give us the particular solution that we seek:

◇ **Example 4.3(b):** Determine the values of A and B for which

$$y_p = At\sin 3t + Bt\cos 3t$$

is the particular solution to the ODE $y'' + 9y = 2\sin 3t$.

Solution: We *carefully* use the product rule to find the derivatives of y_p:

$$y_p' = 3At\cos 3t + A\sin 3t - 3Bt\sin 3t + B\cos 3t$$

and

$$y_p'' = -9At\sin 3t + 3A\cos 3t + 3A\cos 3t - 9Bt\cos 3t - 3B\sin 3t - 3B\sin 3t.$$

Grouping the like terms of the second derivative gives us

$$y_p'' = -9At\sin 3t - 9Bt\cos 3t - 6B\sin 3t + 6A\cos 3t.$$

Substituting into the left side of the ODE gives us

$$\begin{aligned} y_p'' + 9y_p &= -9At\sin 3t - 9Bt\cos 3t - 6B\sin 3t + 6A\cos 3t + 9(At\sin 3t + Bt\cos 3t) \\ &= -6B\sin 3t + 6A\cos 3t. \end{aligned}$$

In order for this to equal $2\sin 3t$ we must have $A = 0$ and $B = -\frac{1}{3}$, so the particular solution to $y'' + 9y = 2\sin 3t$ is $y_p = -\frac{1}{3}t\cos 3t$.

We already knew the homogeneous solution, so the general solution to $y'' + 9y = 2\sin 3t$ is

$$y = C_1\sin 3t + C_2\cos 3t - \tfrac{1}{3}t\cos 3t.$$

We can now make an amendment to the listing at the end of Section 3.4 to get the overall summary for guesses to use for particular solutions.

Undetermined Coefficients

Consider the constant coefficient ODE $ay'' + by' + cy = f(t)$, and assume y_h contains no terms that are constant multiples of $f(t)$. The trial particular solution y_p is chosen as follows.

- If f is a polynomial of degree n, then

$$y_p = A_nt^n + A_{n-1}t^{n-1} + \cdots + A_2t^2 + A_1t + A_0$$

- If $f(t) = Ce^{kt}$, then $y_p = Ae^{kt}$.
- If $f(t) = C_1\sin kt + C_2\cos kt$, then $y_p = A\sin kt + B\cos kt$. *Even if one of C_1 or C_2 is zero, y_p still contains both the sine and cosine terms.*

When y_h contains any term that is a constant multiple of f, y_p will be as above but multiplied by the smallest power of t for which no terms of y_p are of the same form as any terms of y_h.

◇ **Example 4.3(c):** Find the trial particular solution to $y'' + y' - 6y = 5t - 3$.

Solution: The homogeneous solution is $y_h = C_1e^{-3t} + C_2e^{2t}$, so the trial particular solution is $y_p = At + B$.

◇ **Example 4.3(d):** Find the trial particular solution to $y'' + y' - 6y = 7\cos 5t$.

Solution: The homogenous solution is the same as in Example 4.3(c), so trial particular solution is $y_p = A\sin 5t + B\cos 5t$.

◇ **Example 4.3(e):** Find the trial particular solution to $y'' + y' - 6y = 4e^{2t}$.

Solution: Again the homogeneous solution is $y_h = C_1e^{-3t} + C_2e^{2t}$. $f(t)$ has the same form as one of the terms of the homogeneous solution, the trial particular solution is $y_p = Ate^{2t}$.

We conclude this section by further examining homogeneous and particular solutions to an ODE

$$ay'' + by' + cy = f(t). \tag{1}$$

Let's denote the left side of (1) using the operator notation $D(y)$. We have found that the homogeneous solution consists of a linear combination of two functions $g(t)$ and $h(t)$ that are both, by themselves, solutions to $D(g) = 0$ and $D(h) = 0$. By a linear combination we mean

$$y_h = C_1g(t) + C_2h(t),$$

where C_1 and C_2 are *ANY* constants. When D is applied to the particular solution y_p the result is $D(y_p) = f(t)$. The general solution is

$$y = C_1g(t) + C_2h(t) + y_p(t).$$

Applying D to the solution then gives

$$D(C_1g + C_2h + y_p) = C_1D(g) + C_2D(h) + D(y_p) = 0 + 0 + f(t) = f(t).$$

Note the use of the fact that D is a linear operator in this computation.

◇ **Example 4.3(f):** The general solution to (1) is $y = C_1e^{-2t} + C_2e^{-t} + 4\cos 5t$. Which of the following are solutions to $ay'' + by' + cy = 0$?

(a) $y = 5e^{-t}$ (b) $y = 7e^{-2t} + 4\cos 5t$ (c) $y = 7e^{-2t} + 5e^{-t}$

Solution: The homogeneous solution is the part containing the arbitrary constants, $y_h = C_1e^{-2t} + C_2e^{-t}$. It is a solution to $ay'' + by' + cy = 0$ for all choices of C_1 and C_2, so the functions in (a) and (c) are both solutions. The function in (b) is a solution to (1), but not to $ay'' + by' + cy = 0$ because D applied to $7e^{-2t}$ is zero, but when applied to the particular solution $y_p = 4\cos 5t$ the result is $f(t)$, not zero.

◇ **Example 4.3(g):** The general solution to (1) is $y = C_1e^{-2t} + C_2e^{-t} + 4\cos 5t$. Which of the following are solutions to (1)?

(a) $y = 4\cos 5t$ (b) $y = 8\cos 5t$ (c) $y = 7e^{-2t} + 5e^{-t} + 4\cos 5t$

Solution: We again recognize that the homogeneous solution is $y_h = C_1e^{-2t} + C_2e^{-t}$ and the particular solution is $y_p = 4\cos 5t$. Because the particular solution by itself is a solution to (1), the function in (a) is a solution. Unlike the homogeneous solution, a constant in the particular solution *is not* arbitrary, so the function in (b) is not a solution. (Test it to see for sure?) The function in (c) is a solution, because it is simply the general solution with the arbitrary constants having the specific values $C_1 = 7$ and $C_2 = 5$.

Section 4.3 Exercises To Solutions

1. Below are each of the ODEs from Examples 4.3(c), (d) and (e). In each case, substitute the given trial particular solution from the example into the ODE to determine the value(s) of any constant(s).

   (c) $y'' + y' - 6y = 5t - 3, \qquad y_p = At + B$

   (d) $y'' + y' - 6y = 7\cos 5t, \qquad y_p = A\sin 5t + B\cos 5t$

   (e) $y'' + y' - 6y = 4e^{2t}, \qquad y_p = Ate^{2t}$

2. Suppose that when you were finding the particular solution to $y'' + y' - 6y = 4e^{2t}$ you didn't notice that $4e^{2t}$ was of the same form as one of the terms of y_h. Try a particular solution of $y_p = Ae^{2t}$ and see what happens. It will try to tell you that something is wrong!

3. Solve each of the following IVPs by the process described in the box at the start of Section 3.4.

   (a) $y'' + y' - 6y = 1 + 8t - 6t^2\,, \qquad y(0) = 2\,, \qquad y'(0) = -3$

   (b) $y'' + 7y' + 10y = 6e^{-2t}\,, \qquad y(0) = 2\,, \qquad y'(0) = -11$

   (c) $y'' + 4y = 3\sin 2t\,, \qquad y(0) = \frac{1}{2}\,, \qquad y'(0) = \frac{5}{2}$

4. The functions below are solutions to second order linear, constant coefficient initial value problems. Give the steady-state and transient parts of each.

   (a) $y = -\frac{2}{3}\sin 3t + \frac{5}{3}\cos 3t$

   (b) $y = e^{-3t}(4\sin t + 7\cos t) + \frac{3}{4}\cos 7t$

   (c) $y = \frac{3}{5}\sin 5t - \frac{6}{5}\cos 5t + \frac{7}{2}e^{-2t}$

   (d) $y = 3te^{-5t} - 7e^{-5t} + e^{-t}$

5. Suppose that the ODE
$$ay'' + by' + cy = f(t) \tag{1}$$
has general solution
$$y = e^{-2t}\big(C_1\sin 3t + C_2\cos 3t\big) + 5e^{-2t}.$$
Which of the following are then solutions to (1)?

   (a) $y = 7e^{-2t}\sin 3t + 5e^{-2t}$

   (b) $y = e^{-2t}\big(3\sin 3t - 2\cos 3t\big) + 5e^{-2t}$

   (c) $y = e^{-2t}\big(3\sin 3t - 2\cos 3t\big)$

   (d) $y = e^{-2t}\big(3\sin 3t - 2\cos 3t\big) + 4e^{-2t}$

   Which of the following are solutions to
$$ay'' + by' + cy = 0? \tag{2}$$

   (e) $y = 7e^{-2t}\sin 3t + 5e^{-2t}$

   (f) $y = e^{-2t}\big(3\sin 3t - 2\cos 3t\big)$

   (g) $y = 7e^{-2t}\sin 3t$

   (h) $y = e^{-2t}\big(C_1\sin 3t + C_2\cos 3t\big) + 4e^{-2t}$

4.4 Forced, Undamped Vibration

Performance Criterion:

4. (e) Determined whether a forced, undamped system will exhibit resonance, beats, or neither. Determine the solution for such a system.

Suppose that we have either

- a mass on a spring with no damping, subject to a sinusoidal external force, or
- an inductor and a capacitor (no resistor) in series with a voltage source that is putting out a sinusoidal current.

How would we expect the functions describing the position of the mass or the charge on the capacitor to behave? That is what you will investigate in the exercises for this section. You might want to take a guess as to what you would expect, before the mathematics of the situation answers the question.

Section 4.4 Exercises To Solutions

1. For this exercise and each of the following, work in decimals, rounding all values to the nearest tenth.

   (a) Solve the initial value problem

   $$x'' + 4.84x = 8\cos 5t\ , \qquad x(0) = x'(0) = 0$$

   (b) Graph the solution using some technology, sketch the graph. Be sure to get a viewing window that is appropriate. Put a scale on your graph.

   (c) Discuss the situation of transient and steady-state solutions. Why should we expect this before even solving the differential equation?

2. (a) Solve the initial value problem

   $$x'' + 4.84x = 8\cos 2.2t\ , \qquad x(0) = x'(0) = 0$$

   (b) Graph the solution using some technology, sketch the graph.

   (c) The phenomenon you are observing here is called **resonance**. In either the mechanical or electrical case, as the amplitude gets larger and larger, something will fail - the spring or one of the electrical components. What is it about the situation that is causing this to happen?

Note that the angular frequency of 2.2 that appears in the solution to the IVP comes from the homogeneous equation, so it depends only on the spring-mass or LC system, not on the forcing function. That frequency is sometimes called the **natural frequency** of the system.

3. (a) Solve the initial value problem

   $$x'' + 4.84x = 8\cos 2t\ , \qquad x(0) = x'(0) = 0$$

   (b) Graph your solution from $t = 0$ to $t = 75$. Sketch the graph.

(c) The phenomenon you are observing here is called **beats** - in electronics this is **amplitude modulation**. All I know about this is that the AM in AM radio stands for amplitude modulation (FM is frequency modulation)! Ask your local EET instructor for details. Look carefully at how this initial value problem compares with the other two. What do you suppose it is that is causing the beats?

(d) A trig identity can help us get a little better insight into the solution. You should be able to write your solution in the form $x(t) = A(\cos\omega_0 t - \cos\omega t)$. Use the identity

$$\cos u - \cos v = 2\sin\left(\frac{v-u}{2}\right)\sin\left(\frac{u+v}{2}\right)$$

to rewrite your solution. The new form of the solution is trying to talk to you. Can you see what it is trying to tell you?

(e) Graph $y = 19\sin(0.1t)$ and $y = -19\sin(0.1t)$ together with the graph of the solution, and sketch what you see. Can you *now* see what the solution to (d) is trying to tell you?

4.5 Chapter 4 Summary

Performance Criteria:

4. (f) For a spring-mass system or electric circuit, demonstrate an understanding of the relationships between

    - the physical situation (presence and type of damping and/or forcing)
    - the form of the ODE, including the function f
    - the the analytic and graphical nature of the solution (in particular, the presence and appearance of transient and steady-state parts of the solution)

In this section we will attempt to summarize all that we have seen in Chapters 3 and 4. In particular, we want to recognize from an ODE, the solution to an ODE, or the graph of the solution to an ODE whether it models a situation

- in which the system is undamped, under-damped, critically damped or over-damped
- with or without an external forcing function
- for which the solution has transient or steady-state parts, or both
- resulting in or exhibiting resonance or beats

The type of differential equation that we are talking about here is one of the form

$$ay'' + by' + cy = f(t) \tag{1}$$

where the coefficients a, b and c are constant parameters based on the physical properties of the system we are considering:

- In a spring-mass system a is the mass, b is the coefficient of damping, and c is the spring constant.
- In an electric circuit, a is the inductance, b is the resistance and c is the reciprocal of the capacitance.

It is clear that without a mass and a spring there is no spring-mass system, so for that situation neither a nor c can be zero. For the electric circuit situation it is reasonable to consider a system with only a resistance and inductance, but that can be treated as a first order ODE in a manner you have already seen. For an RLC circuit of the sort we wish to consider, none of the values a, b or c are zero, although we will consider the case $b = 0$ as a theoretical possibility.

The Left Side of the ODE

The left hand side of the ODE describes the system itself. In the case of a spring-mass system, it is the spring, the mass, and the damping. In an electric circuit it is the resistor, inductor and capacitor. *The system doesn't cause motion or current, it just shapes it by the way it reacts to the forcing function and/or initial conditions.* There will not be any motion or current unless there are nonzero initial conditions, a forcing function f or E, or both. What is of real concern to us on the left hand side of the equation (1) is the role of b, which controls the damping. Here is a summary of that:

- When $b = 0$ the system is undamped.

- When $b^2 - 4ac < 0$ the system is under-damped. Oscillation will occur, but any oscillation due to the initial conditions will decay. (The solution will have a transient part if either of the initial conditions is nonzero.) Any steady-state behavior will be due to the forcing function f.

- When $b^2 - 4ac > 0$ the system will be over-damped, and there will be no oscillation due to the system itself. The system will again have a transient part, and any steady-state part of the solution is again due to the external forcing function f.

- When $b^2 - 4ac = 0$ the system is critically damped, and the solution will behave similarly to the over-damped situation. The transient part will have a te^{-kt} term $(k > 0)$, but still decays over time because e^{-kt} decays faster than t grows.

Any quick investigation of an ODE of the form (1) should perhaps begin by observing whether a damping term is present. If it is, computation of $b^2 - 4ac$ should follow to determine which of the last three cases above we are dealing with.

Initial Conditions

In order to solve a second order ODE, we must have two initial conditions. For a spring-mass system the meaning of the initial conditions is pretty straightforward. The initial position tells us whether the mass is raised or pulled down at time zero. In either case, there is potential energy due to either gravity (for $y(0) > 0$) or the spring (for $y(0) < 0$) that is converted to kinetic energy of motion when the mass is let go. $y'(0)$ is the initial velocity imparted to the mass; it is negative if the initial velocity is downward, and positive if the initial velocity is upward. For an RLC circuit, $q(0)$ is the initial charge on the capacitor, which has electric potential that can cause current, and $i(0) = q'(0)$ is the initial current.

The effect of initial conditions is not lasting, unless the system is undamped. For any damped system, the initial conditions will lead to a transient part of the solution. For an undamped system, initial conditions will lead to a steady-state part of the solution.

The Forcing Function f

The function f, on the right hand side of (1), is the forcing function that is imposed on the system. In the case of the spring-mass system it might be something like effect of bumps in the road for a shock absorber, or perhaps the effect of some vibration added by a motor. For an electrical circuit, it is the voltage source that is supplying the circuit. Often f will be, in reality, a periodic function made up of sine or cosine functions. For this reason it is sufficient to understand the behavior of the system when f is a single trig function.

f provides input to the system over time, unlike the initial conditions, which only supply input right at time zero. It, of course, leads to the particular solution to (1). At this point we will only consider decaying exponential or trigonometric forcing functions.

- A decaying exponential function is itself transient, so in a mathematical sense when f is such a function it leads to a transient part of the solution. (That part of the solution is the particular solution.) For an undamped system such a forcing function will also act to cause a steady-state part of the solution as well, even in the absence of initial values. (In that case it acts, in a sense, like an initial velocity.)

- When f is a periodic function like a trig function, it will provide input to the system forever. Because of this, it will usually lead to a periodic steady-state part of the solution. The one exception is for an undamped system, where we find the following behaviors:

- When the frequency of the forcing function is significantly different from the natural frequency (sometimes called the **resonant frequency**) of the system, the result is a steady-state solution with two parts, one with the resonant frequency and one with the frequency of the forcing function.
- When the frequency of the forcing function is the same as the resonant frequency of the system, the forcing function will cause part the solution to be trigonometric functions with linearly increasing amplitude. This is the condition we call **resonance**.
- When the frequency of the forcing function is close to the resonant frequency of the system, it will cause vibration with increasing amplitude when the forcing function is in phase with the vibration. Eventually the forcing function will become out of phase with the natural vibration, canceling it out. It will then get back in phase, then out, over and over. The result is the phenomenon called **beats**.

Exercises on the next page.

Chapter 4 Exercises To Solutions

1. Here are some equations of the sort we have been discussing:

(i) $y'' + 3y' + 2y = 0$

(ii) $y'' + 4y = \sin(9t)$

(iii) $y'' + 10y' + 25y = 0$

(iv) $y'' + 25y = 3.1\sin(5t)$

(v) $y'' + 9y = 0$

(vi) $y'' + 5y' + 7y = 7.4\sin(2.4t)$

(vii) $y'' + 3y' + 5y = 0$

(viii) $y'' + 16y = 7\cos(3.8t) - 4\sin(3.8t)$

(a) Which equations model an undamped system? Which model an under-damped system? Critically damped? Over-damped?

(b) Which equations will have solutions with a transient part? Which will have solutions with a steady-state part?

(c) Which equations will have solutions that exhibit resonance? Which will have solutions exhibiting beats?

2. Consider the following *solutions* to differential equations of the type we have been discussing (second order, constant coefficient).

(i) $y = C_1e^{-t} + C_2e^{-3t}$

(ii) $y = C_1e^{-2t} + c_2te^{-2t}$

(iii) $y = C_1\cos(5.1t) + C_2\sin(5.1t)$

(iv) $y = e^{-1.2t}[C_1\cos(5t) + C_2\sin(5t)]$

(v) $y = e^{-0.4t}[C_1\cos(2t) + C_2\sin(2t)] - 1.3\cos(7t)$

(vi) $y = C_1\cos(3t) + C_2\sin(3t) + 0.13\cos(8t) - 1.46\sin(8t)$

(vii) $y = C_1\cos(3t) + C_2\sin(3t) + 0.13t\cos(3t) - 1.46t\sin(3t)$

(viii) $y = A\sin(0.1t)\sin(6.1t)$

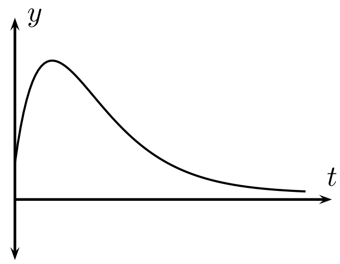
(a) Identify the transient and steady-state parts of each solution. (Some may not have both.)

(b) Which solutions are for differential equations of the form $ay'' + by' + cy = 0$?

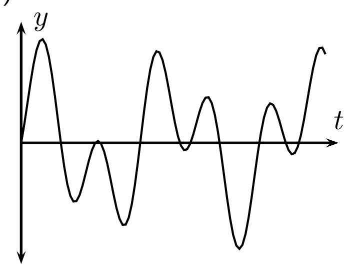
(c) Which solutions are for undamped systems? Which are for under-damped systems? Critically damped? Over-damped?

3. Below are some graphs of solutions to ODEs of the form $ay'' + by' + cy = f(t)$, where either, or both, of b or $f(t)$ may be zero.

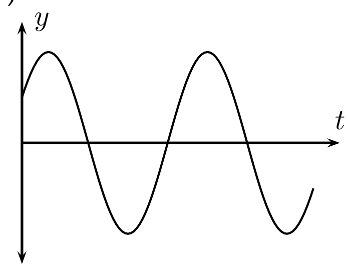
(i)



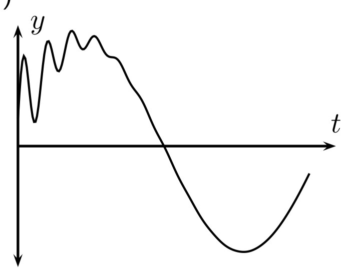
(ii)



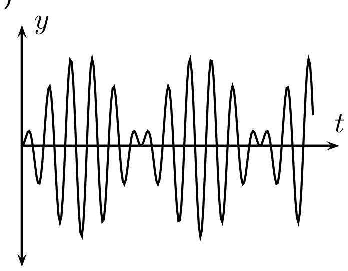
(iii)



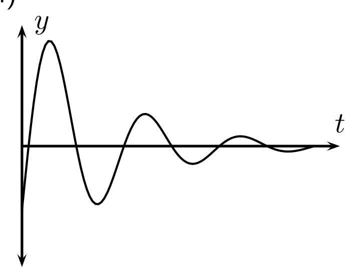
(iv)



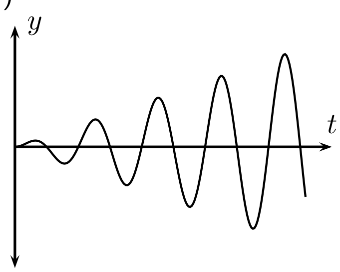
(v)



(vi)



(vii)



(a) Which graphs are for solutions to undamped systems? Under-damped systems? Critically or over-damped systems? (You should not be able to tell the graphs for critically damped or over-damped apart.)

(b) Which graphs are for ODEs of the form $ay'' + by' + cy = 0$?

4. None of the ODEs in Exercise 1 have a solution equation given in Exercise 2, or solution graph given in Exercise 3. However, we *CAN* match up the *FORMS* of the ODEs, solution equations, and graphs of solutions. For example, equation (iii) from Exercise 1 matches with solution (ii) from Exercise 2 and graph (i) from Exercise 3. Find eight other sets of three like this; one graph will have to be used more than once.

C.4 Chapter 4 Solutions

Section 4.1 Solutions Back to 4.1 Exercises

1. (a) linearly independent (b) $c_1 = 1,\ c_2 = -2$
   (c) $c_1 = 2,\ c_3 = 3$ (d) linearly independent
2. (b) $c_1 = 1,\ c_2 = -1$ (c) $c_1 = 1,\ c_3 = 1$
3. (a) $W(x) = -6x^2 - 6x - 10$ Any value of x will give $W(x) \neq 0$.
   (b) $W(x) = -9e^{7x}$ Any value of x will give $W(x) \neq 0$.
4. $W(t) = e^{2kt}$ Any value of t will give $W(t) \neq 0$.
5. $W(t) = -2$ Any value of t will give $W(t) = -2 \neq 0$.

Section 4.2 Solutions Back to 4.2 Exercises

3. $y_2 = x^{\frac{3}{2}}$ 4. $y_2 = \dfrac{1}{x^2}$

Section 4.3 Solutions Back to 4.3 Exercises

1. (a) $y_p = -\frac{5}{6}t - \frac{13}{36}$ (b) $y_p = \frac{35}{986}\sin 5t - \frac{217}{986}\cos 5t$ (c) $y_p = \frac{4}{5}te^{2t}$
3. (a) $y = -\frac{1}{5}e^{-2t} - te^{-2t} + \frac{1}{5}e^{3t}$ (b) $y = -e^{-2t} + 3e^{-5t} + 2te^{-2t}$
   (c) $y = \frac{13}{8}\sin 2t + \frac{1}{2}\cos 2t - \frac{3}{4}t\cos 2t$
4. (a) Steady-state: $-\frac{2}{3}\sin 3t + \frac{5}{3}\cos 3t$ Transient: none
   (b) Steady-state: $\frac{3}{4}\cos 7t$ Transient: $e^{-3t}(4\sin t + 7\cos t)$
   (c) Steady-state: $\frac{3}{5}\sin 5t - \frac{6}{5}\cos 5t$ Transient: $\frac{7}{2}e^{-2t}$
   (d) Steady-state: none Transient: $3te^{-5t} - 7e^{-5t} + e^{-t}$
5. (a) solution (b) solution (c) solution (d) not a solution
   (e) not a solution (f) solution (g) solution (h) not a solution

Section 4.4 Solutions Back to 4.4 Exercises

1. (a) $x(t) = 0.4\cos 2.2t - 0.4\cos 5t$
   (c) We should expect no transient solution. There is no damping, so the homogeneous solution is periodic, hence steady-state. Because the forcing function is periodic with different frequency than the homogeneous solution, it results in a periodic particular solution, so the general solution is then periodic, so steady-state.
2. (a) $x(t) = 1.8t\sin 2.2t$
   (c) What causes the resonance is that the frequency of the forcing function $f(t) = 8\cos 2.2t$ is the same as the natural frequency of the system, which is seen in the homogeneous solution.

3. (a) $x(t) = 9.5 \cos 2t - 9.5 \cos 2.2t$

   (c) The beats are being caused by the fact that the frequency of the forcing function is close to, but not the same as, the natural frequency of the system.

   (d) $x(t) = 19 \sin(0.1t) \sin(2.1t)$ The first sine function (and the factor of 19) acts as a sort of "variable amplitude" for the higher frequency second sine function.

Chapter 4 Exercises Solutions

Back to Chapter 4 Exercises

1. (a) Undamped: ii, iv, v, viii Under-damped: vi, vii

   Critically damped: iii Over-damped: i

   (b) Transient: i, iii, vi, vii Steady-state: ii, iv, v, vi, viii

   (c) Resonance: iv Beats: viii

2. (a) (i) entire solution is transient (ii) entire solution is transient

   (iii) entire solution is steady-state (iv) entire solution is transient

   (v) Transient: $e^{-0.4t}[C_1 \cos 2t + C_2 \cos 2t]$ Steady-state: $-1.3 \cos 7t$

   (vi) entire solution is steady-state (vii) Steady-state: $C_1 \cos 3t + c_2 \sin 3t$

   (viii) entire solution is steady-state

   (b) i, ii, iii, iv

   (c) Undamped: iii, vi, vii, viii Under-damped: iv, v

   Critically damped: ii Over-damped: i

3. (a) Undamped: ii, iii, v, vii Under-damped: iv, vi Critically or over-damped: i, vi

   (b) i, vi

   (c)

Equation	Solution	Graph
i	i	i
ii	vi	ii
iii	ii	i

Equation	Solution	Graph
iv	vii	vii
v	iii	iii
vi	v	iv
vii	iv	vi
viii	viii	v