Ordinary Differential Equations

for Engineers and Scientists

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Contents

5	Boundary Value Problems		137
	5.1	Deflection of Horizontal Beams	139
	5.2	Second-Order Boundary Value Problems, Eigenfunctions and Eigenvalues	144
	5.3	Eigenvalue Problems, Deflection of Vertical Columns	151
	5.4	The Heat Equation in One Dimension	159
	5.5	Chapter 5 Summary	163
D	Solu D.5	tions to Exercises Chapter 5 Solutions	208 208

Learning Outcome:

5. Set up and solve boundary value problems.

Performance Criteria:

- (a) Solve a boundary value problem for the deflection of a horizontal beam.
- (b) Give the boundary conditions for a horizontal beam.
- (c) Predict the shape of the deflection curve for a horizontal beam that is supported in a given manner.
- (d) Determine whether a function is an eigenfunction of a differential operator. If it is, give the corresponding eigenvalue.
- (e) Give eigenfunctions of the first or second derivative, for a given eigenvalue.
- (f) Solve a boundary value problem for eigenvalues and the corresponding eigenfunctions.
- (g) Give the boundary conditions for a vertical column.
- (h) Find the buckling modes (non-trivial solutions) for a vertical column.
- (i) Find the critical loads for a vertical column.
- (j) Give the pinning conditions resulting in each of the buckling modes of a vertical column.

All of the applications that we have studied so far have involved some quantity that is a function of time; that is, time has been the independent variable. Arbitrary constants have arisen in the process of solving the associated ODEs, and we have used given initial conditions to determine the values of the constants. In this chapter we look at **deflection** (bending) of horizontal beams and vertical columns. For horizontal beams the deflection is a function of the distance along the beam or column. The independent variable is then a one dimensional position variable, as discussed in Section 1.4. As seen in Section 1.7, we use boundary conditions, rather than initial conditions, to determine the values of the arbitrary constants.

We will designate the variable x to denote the distance along the beam (or column) from one end or the other. Due to the weight of the beam there will be some deflection y off of the horizontal line the beam would follow if it had no weight. The deflection will be different at different points along the beam, so y = y(x). That is, the amount of deflection depends on where one is looking along the length of the beam. The solution function is obtained from a fourth order ODE having four boundary conditions. Solving such problems is relatively straightforward.

The situation will be significantly different for vertical columns, in a way that might be somewhat surprising. We'll see that such a column will remain straight as more and more weight is added to it until, at some weight (called the **first critical load**), it suddenly deflects ("buckles"). It will then either deform or break as a the load is increased. However, if we prevent the middle of the column from deflecting, each half will deflect at a load (called the **second critical load**) that is *four times* the first critical load.

Solving the boundary value problems associated with vertical columns requires solving what we call an **eigenvalue problem**, which is more nuanced that the boundary value problems associated with

horizontal beams. We will devote two sectoins of this chapter to eigenvalue problems and vertical columns. We then conclude the chapter with a look at perhaps the simplest application of partial differential equation, heat distribution in a rod. The method of solution leads us two two types of ODEs, one of which is an eigenvalue problem.

Performance Criteria:

- 5. (a) Solve a boundary value problem for the deflection of a horizontal beam.
 - (b) Give the boundary conditions for a horizontal beam.
 - (c) Predict the shape of the deflection curve for a horizontal beam that is supported in a given manner.

In this section we will take a look at the differential equations associated with beams that are suspended horizontally in some way. The beams themselves will not be horizontal over their entire lengths, because the force of gravity will cause some bending. The first thing to understand is the mathematical setup. Suppose that we have a beam of length 10 feet. We put the cross-sectional center of its left end at the origin of an x-y coordinate plane, and the cross-sectional center of its right end at the point (10,0). The longitudinal axis of symmetry of the beam then runs along the x-axis from x = 0 to x = 10; see the figure below and to the left.



Now the beam will deflect (a fancy term for "sag") in some way, due to any weight it is supporting, *including its own weight*. The shape it takes will depend on the manner in which it is supported (we will get into that soon), but one possibility is shown in the figure above and to the right. The points along what was the original axis of symmetry of the beam now follow the graph of a function, which we will call y(x). Note that the domain of the function is just the interval [0, 10]. Our goal will be to find the mathematical equation of the function.

Let us still consider a 10 foot beam, but we will represent it with just the curve described by the deflection of the longitudinal axis of symmetry. (*From now on, when we talk about the beam, we really mean the deflected original longitudinal axis of symmetry of the beam.*) Suppose also that the ends of the beam are what we call **embedded**. This means that they are not only supported at both ends, but the ends are also held horizontal by being "clamped" somehow. A good image to keep in mind is a beam that is stuck into two opposing walls of a structure. See the diagram to the right.



The theory behind obtaining a differential equation to model a horizontal beam is beyond the scope of this class. Suffice it to say that it involves ideas from the area of statics, like the "bending moment" of the beam, and the properties of the material from which the beam is built. The differential equation itself is fourth order:

$$EI\frac{d^4y}{dx^4} = w(x) \tag{1}$$

Here E is Young's modulus of elasticity for the material from which the beam is made, I is the moment of inertia of a cross-section of the beam and w(x) is the load per unit length of the beam. If the beam has uniform cross-section and the only weight that it is supporting is its own weight, then w(x) is a constant. We will consider only that situation.

Let's consider the situation shown in Figure 5.1(a), where both ends are embedded. Because the ODE (1) is fourth order, we will need *four* boundary conditions to determine all of the constants that will arise in solving it. We first recognize that because the two ends are supported, there will be no deflection at either end. Therefore y(0) = y(10) = 0. This will be the case for any horizontal beam that is supported at both ends. Next we consider the fact that the ends of the beam are embedded horizontally into a wall. The embedding causes both ends to be horizontal right at the points where they leave the walls they are embedded in, so the slope of the beam is zero at those points. Mathematically we express this by y'(0) = y'(10) = 0. When we put the ODE together with these boundary conditions we get a boundary value problem. Suppose that for our ten foot beam beam E = 10, I = 5 and w(x) = 100. The boundary value problem that we have is then

$$50\frac{d^4y}{dx^4} = 100, \qquad y(0) = 0, \ y'(0) = 0, \qquad y(10) = 0, \ y'(10) = 0$$
(2)

We solve this by simply taking a succession of antiderivatives and finding constants along the way, when we are able to.

♦ Example 5.1(a): Solve the boundary value problem (2) above.

Solution: We begin by dividing both sides by 50 to get $\frac{d^4y}{dx^4} = 2$. Our task now is to keep integrating both sides until we find y = y(x). Integrating once gives $\frac{d^3y}{dx^3} = 2x + C_1$, and integrating again gives $\frac{d^2y}{dx^2} = x^2 + C_1x + C_2$. Next we find that $\frac{dy}{dx} = \frac{1}{3}x^3 + \frac{1}{2}C_1x^2 + C_2x + C_3$, and applying the initial condition y'(0) = 0 gives $C_3 = 0$. Substituting this value and integrating one more time we get $y = \frac{1}{12}x^4 + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_4$, and applying the boundary condition y(0) = 0 results in

$$y = \frac{1}{12}x^4 + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2.$$
 (3)

We now apply the initial condition y(10) = 0 to get $0 = \frac{10,000}{12} + \frac{1000}{6}C_1 + \frac{100}{2}C_2$, and the initial condition y'(10) = 0 to get $0 = \frac{1000}{3} + \frac{100}{2}C_1 + 10C_2$. To solve this system we multiply the first equation by 12 and the second by 6 to get the system to the left below, which can be solved in the manner shown in the other steps:

$$2000C_{1} + 600C_{2} = -10,000 \implies 20C_{1} + 6C_{2} = -100$$

$$300C_{1} + 60C_{2} = -2000 \implies -30C_{1} - 6C_{2} = 200$$

$$-10C_{1} = 100$$

$$C_{1} = -10$$

Substituting this value for C_1 and solving for C_2 gives us $C_2 = -\frac{40}{3}$. Putting these values into (3), the solution to the IVP is $y = \frac{1}{12}x^4 - \frac{5}{3}x^3 - \frac{20}{3}x^2$.

Use your calculator or an online grapher like *Desmos* to graph the solution from x = 0 to x = 10, using a y scale that allows you to actually see the deflection of the beam. Does the result surprise you? (It should!) One annoying feature of the differential equation is that it is based on taking *down*

to be the positive direction. To see what the actual shape of the beam will be, multiply your solution by -1, then graph it. Now the result should look something like Figure 5.1(a).

Suppose again that we have a 10 foot beam, but now it is supported by a fulcrum at each end, and each end is free to pivot around the fulcrum. We will call the beam **simply supported** in this case. Civil engineers might call this "pinned-pinned." See the diagram to the right. In this case, we will have y(0) = y(10) = 0, just like the embedded case. However, we can see that we will not have y'(0) = 0 or y'(10) = 0.



So how do we get two more boundary conditions? Note that the downward force of gravity in the interior and the upward force of the supports at the ends bend the beam into a concave up shape. (Remember concavity from differential calculus?) The upward concavity here means that it must be the case that y''(x) > 0 for values of x between, but not equal to 0 and 10. However, there are no opposing forces to bend the beam right at its ends. Thus there is no concavity right at the ends of the beam, resulting in the conditions y''(0) = y''(10) = 0.

The final situation we will consider for now is a beam that is embedded at the left end and free at the right end, as shown. The left end of the beam is embedded, so we know the values y(0) = y'(0) = 0. We know neither the displacement nor the slope of the right end, but what we do know there is that there is no concavity there, so y''(10) = 0. This gives us three boundary conditions, but of course we need four. The last condition comes from some theory we won't go into here, but it is y'''(10) = 0.



Let us now summarize the the possible boundary conditions for a horizontal beam:

- At an embedded end both y and y' are zero.
- At a simply supported end y and y'' are zero.
- At a free end y'' and y''' are zero.

I expect you be able to give any of those conditions - you should be able to figure all of them out each time you need them, without memorization, with the possible exception of the third derivative just discussed.

Section 5.1 Exercises

To Solutions

1. For each beam pictured below, list the boundary conditions. Assume that the height of the left end of each is zero.





- 2. Suppose that you have a beam that is 8 feet long. For each of the scenarios given, determine first whether it would make any sense physically to have the beam supported in the manner given. If not, explain why. If it does make sense, give the four boundary conditions.
 - (a) Left end embedded horizontally, right end simply supported.
 - (b) Left end simply supported, right end free.
 - (c) Both ends free.
- 3. Find the deflection function y = y(x) for an eight foot beam that is embedded at both ends, carrying a constant load of w(x) = 150 pounds per foot. Suppose also that E = 30 and I = 80, in the appropriate units.
 - (a) Give the appropriate boundary value problem (differential equation plus boundary conditions).
 - (b) Solve the differential equation and apply the boundary conditions in order to determine the constants. What is the final solution?
 - (c) Graph your solution on the appropriate x interval, using a y scale that allows you to actually see the deflection of the beam. Remember to multiply the right side of your solution from (b) by negative one so that it appears the same way that the beam will.
 - (d) Where do you believe the maximum deflection should occur? Find the deflection there you need not give units with your answer, since I have been somewhat vague about the units of the constants E and I.
- 4. (a) Sketch a graph of the deflection of a beam that is embedded at its left end and free at its right end.
 - (b) Suppose that the beam is 10 feet long, with values of w_0 , E and I of 100, 10 and 5, respectively. Solve the boundary value problem.
 - (c) Graph your solution and compare with your sketch in part (a). Of course they should be the same.
 - (d) What is the maximum deflection of the beam, and where does it occur?
- 5. Repeat parts (a)-(d) of Exercise 4 for a 10 foot beam that is simply supported on both ends.

- Repeat steps (a)-(c) of Exercise 4 for an eight foot beam, with the same parameters as in Exercise 3, that is embedded on the left end and simply supported on the right end. Then do the following:
 - (d) It was intuitively clear where the maximum deflection occurred for the two previous situations, but it is not so clear in this case. Take a guess as to about where you think it should occur for this case. Then use the graph on your calculator, along with the trace function, to determine where the maximum deflection occurs, and how much it is.
- 7. The graphs below are those of some fourth degree polynomials. The points labeled A, D, E, H, I and L are maxima for their respective functions, and the points labeled B, C, F, G, J and K are inflection points. For each of the following boundary situations, give the endpoints of a section of graph that has the shape the deflection curve would take. *Assume that both ends of the beam are supported at the same level, and that the dashed lines are horizontal.*
 - (a) Simply supported at the left end, embedded at the right end.
 - (b) Embedded at both ends.
 - (c) Simply supported at both ends.



Performance Criteria:

- 5. (d) Determine whether a function is an eigenfunction of a differential operator. If it is, give the corresponding eigenvalue.
 - (e) Give eigenfunctions of the first or second derivative, for a given eigenvalue.

We begin by returning to a boundary value problem that we saw in Section 1.7. It is similar to a sort of problem that comes up often in applications. The main thing that distinguishes this from an initial value problem is that the idependent variable is postion, x, rather than time. Another difference we will usually see in boundary value problems is that we are given values of the function at two different values of the independent variable, in this case at zero and π .

Example 5.2(a): Solve the boundary value problem

$$y'' + \frac{1}{4}y = 0,$$
 $y(0) = 3, y(\pi) = -4.$

Solution: The auxiliary equation for the differential equation is $r^2 + \frac{1}{4} = 0$, which has the solution $r = \pm \frac{1}{2}i$. This gives us the solution

$$y = C_1 \sin \frac{1}{2}x + C_2 \cos \frac{1}{2}x \tag{1}$$

to the differential equation. To find the values of the constants we apply the boundary conditions y(0) = 3, $y(\pi) = -4$. For the boundary condition y(0) = 3 we substitute x = 0 and y = 3 into (3) to get

$$3 = C_1 \sin \frac{1}{2}(0) + C_2 \cos \frac{1}{2}(0).$$

This gives us $C_2 = 3$. Substituting $x = \pi$ and y = -4 into (1) gives us $C_1 = -4$. Therefore the solution to the boundary value problem is $y = -4\sin\frac{1}{2}x + 3\cos\frac{1}{2}x$.

Differential Operators, Again

Recall from Section 3.5 that a mathematical object that "works on" a function to produce another function is called an **operator**, and the derivative is probably the simplest example of an operator. Of course the second derivative is an operator as well. In that section we also showed that we can combine derivatives to get other operators.

♦ **Example 5.2(b):** The second derivative $\frac{d^2}{dx^2}$ is an operator. You should be quite familiar with its action:

$$\frac{d^2}{dx^2} \left(5x^3 + 7x^2 - 2x + 4\right) = 30x + 14$$

 Example 5.2(c): We can create new operators by forming something called a *linear combination* of derivatives. As an example, we can define an operator

$$D = 3\frac{d^2}{dt^2} + 5\frac{d}{dt} - 4$$

by its action on a function y = y(t):

$$D(y) = 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} - 4y.$$

So, for example, if $y = e^{-2t}$,

$$D(e^{-2t}) = 3\frac{d^2}{dt^2}(e^{-2t}) + 5\frac{d}{dt}(e^{-2t}) - 4e^{-2t} = 12e^{-2t} - 10e^{-2t} - 4e^{-2t} = -2e^{-2t}$$

You saw such operators when we studied second order linear ODEs.

♦ Example 5.2(d): You may know that when we multiply the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ times the vector $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ we get $A\vec{\mathbf{u}} = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} (-4)(1) + (-6)(3) \\ (3)(1) + (5)(3) \end{bmatrix} = \begin{bmatrix} -22 \\ 18 \end{bmatrix}$ Similarly, for the vector $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $A\vec{\mathbf{v}} = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

We can think of A as an operator that acts on vectors with two components to create other vectors with two components.

Recall that the derivative operator is what we call a **linear operator**. What this means is that if f and g are functions, and c is a constant, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx}(x) + \frac{dg}{dx}(x) \qquad \text{ and } \qquad \frac{d}{dx}[cf(x)] = c\frac{df}{dx}(x)$$

This behavior is not unique. If A is a matrix, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are vectors, and c is a scalar (constant),

$$A(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = A\vec{\mathbf{u}} + A\vec{\mathbf{v}}$$
 and $A(c\vec{\mathbf{u}}) = cA\vec{\mathbf{u}}$

Linear operators have these two properties, of "distributing over addition" and "passing through constants." (This is where the language "linear" in *linear algebra* comes from.) Many operators used in applications are linear operators.

Eigenfunctions and Eigenvalues

Let's go back to the differential equation $y'' + \frac{1}{4}y = 0$ from Example 5.2(a). Note that we can arrange the differential equation as

$$\frac{d^2y}{dx^2} = -\frac{1}{4}y.$$
(2)

When we seek a solution to this differential equation, the equation tells us that we are looking for a function y = y(x) whose second derivative is one-fourth the function itself. We looked at such equations in Section 1.2, and established by guessing and checking that a function of the form

$$y = C_1 \sin \frac{1}{2}x + C_2 \cos \frac{1}{2}x \tag{3}$$

is a solution for any values of C_1 and C_2 . Of course we now know how to solve (2) using its auxiliary equation, and we know also that *every* solution to (2) must have the form (3). The fact that the action of the second derivative operator the function (3) is to simply multiply the the function by $-\frac{1}{4}$ is something fairly special. That's not the case for most other functions when the second derivative operator "works on" them. Here's an example of a more complicated operator and two functions, one of which has this property that the operator acting on it is the same as multiplying by a number, and the another function for which this is not the case.

 \diamond **Example 5.2(e):** Let L be the differential operator defined on a function y = y(x) by

$$L(y) = (x^2 - 1) \frac{d^2y}{dx^2} + 2x \frac{dy}{dx}$$

Apply this operator to the functions $p(x) = x^2 - 5x + 2$ and $q(x) = 5x^3 - 3x$.

Solution: We see that

$$L(x^{2} - 5x + 2) = (x^{2} - 1)\frac{d^{2}}{dx^{2}}(x^{2} - 5x + 2) + 2x\frac{d}{dx}(x^{2} - 5x + 2)$$

= $(x^{2} - 1)(2) + 2x(2x - 5)$
= $6x^{2} - 10x - 2$

and

$$L(5x^{3} - 3x) = (x^{2} - 1)\frac{d^{2}}{dx^{2}}(5x^{3} - 3x) + 2x\frac{d}{dx}(5x^{3} - 3x)$$
$$= (x^{2} - 1)(30x) + 2x(15x^{2} - 3)$$
$$= 60x^{3} - 36x$$

There is nothing special about the result when the operator L of the previous example is applied to $p(x) = x^2 - 5x + 2$, but we see that

$$L[q(x)] = L(5x^3 - 3x) = 60x^3 - 36x = 12(5x^3 - 3x) = 12q(x).$$

Note that the ultimate effect of L on q is to multiply it by twelve. When an operator operates on a function and the result is to simply multiply the function by a constant, we call the function an **eigenfunction** of the operator:

Eigenfunctions and Eigenvalues

Let A be an operator that operates on functions and let y be a *nonzero* function for which there is a constant λ such that

$$Ay = \lambda y.$$

Then y is an **eigenfunction** of the operator A, with corresponding **eigenvalue** λ . Note that $\lambda = 0$ is allowable, but y = 0 is not.

Example 5.2(f): For the operator L of Example 5.2(e), give an eigenfunction and the corresponding eigenvalue.

Solution: Because L[q(x)] = 12q(x), $q(x) = 5x^3 - 3x$ is an eigenfunction of L with eigenvalue 12.

♦ **Example 5.2(g):** Consider again the second derivative $\frac{d^2}{dx^2}$, and note that

$$\frac{d^2}{dx^2}(\sin\frac{1}{2}x) = -\frac{1}{4}\sin\frac{1}{2}x.$$

The effect of the derivative on $\sin \frac{1}{2}x$ is to simply multiply the function by $-\frac{1}{4}$, so $\sin \frac{1}{2}x$ is an eigenfunction for the operator $\frac{d^2}{dx^2}$, with corresponding eigenvalue $-\frac{1}{4}$.

♦ Example 5.2(h): Note that in Example 5.2(d), the result of A times \vec{v} was simply -1 times \vec{v} . We say that $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is an eigenvector (instead of eigenfunction) for the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$, with eigenvalue -1.

♦ **Example 5.2(i):** Because the derivative of a constant is zero, which is also zero times the function, every nonzero constant function is an eigenfunction of the first derivative operator $\frac{d}{dx}$, with corresponding eigenvalue zero. This emphasizes that even though the zero function isn't allowed as an eigenfunction, eigenfunctions *are* allowed to have eigenvalues of zero.

Eigenfunction Problems

Here we see how eigenfunctions are important to us in our study of differential equations. The differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0 \tag{4}$$

is called **Legendre's Differential Equation** and arises when modeling steady-state heat distribution in a solid medium using polar coordinates. If we let $n(n + 1) = \lambda$, move the term $n(n + 1)y = \lambda y$ to the right side, apply the derivative outside the brackets on the left (product rule!) and negate both sides, (4) becomes

$$(x^{2}-1)\frac{d^{2}y}{dx^{2}} + 2x\frac{dy}{dx} = \lambda y.$$
(5)

If we then let L be the operator of Example 5.2(e) defined by

$$L(y) = (x^{2} - 1)\frac{d^{2}y}{dx^{2}} + 2x\frac{dy}{dx}.$$

then (5) becomes the eigenfunction/eigenvalue equation

$$L(y) = \lambda y. \tag{6}$$

Solving (4) then is equivalent to finding eigenfunctions and eigenvalues of the operator L. This is what we mean by solving the eigenvalue problem (5).

We now make some observations related to Example 5.2(g), where we saw that $y = \sin \frac{1}{2}x$ is an eigenfunction for the second derivative with eigenvalue $-\frac{1}{4}$, and Example 5.2(a). First, for any constant C,

$$\frac{d^2}{dx^2}(C\sin\frac{1}{2}x) = -\frac{1}{4}C\sin\frac{1}{2}x = -\frac{1}{4}(C\sin\frac{1}{2}x).$$
(7)

This indicates that any constant multiple of an eigenfunction is also an eigenfunction, with the same eigenvalue. This holds for eigenvectors as well; we don't really think of such multiples as new eigenfunctions or eigenvectors. We also see that

$$\frac{d^2}{dx^2}(\cos\frac{1}{2}x) = -\frac{1}{4}\cos\frac{1}{2}x,\tag{8}$$

showing that an operator can have more than one eigenfunction (beyond just constant multiples) with the same eigenvalue. This also holds for matrices and eigenvectors.

Finally, combining (7) and (8) gives us that any function of the form

$$y = C_1 \sin \frac{1}{2}x + C_2 \cos \frac{1}{2}x$$

is an eigenfunction of the second derivative with eigenvalue $-\frac{1}{4}$. This indicates that solving the differential equation $y'' + \frac{1}{4}y = 0$ amounts to finding eigenfunctions of the second derivative with eigenvalue $-\frac{1}{4}$.

Section 5.2 Exercises

To Solutions

- 1. In this exercise you will be considering the first derivative operator $\frac{d}{dx}$.
 - (a) The function $y = e^{3x}$ is an eigenfunction for the operator. What is the corresponding eigenvalue?
 - (b) Give the eigenfunction of the operator that has eigenvalue -5.
 - (c) Based on your answers to parts (a) and (b), what is the general form of any eigenfunction of the first derivative operator and what is the corresponding general eigenvalue?

- 2. Now consider the second derivative operator $\frac{d^2}{dx^2}$. There are three general forms of the eigenfunctions for this operator, depending on whether the eigenvalues are positive, negative or zero.
 - (a) Give a *specific* function that has eigenvalue zero; that is, the second derivative of the function is zero.
 - (b) Give the most general form of function whose eigenvalue is zero.
 - (c) Give two different functions (neither of them being a multiple of the other) that are eigenfunctions of the second derivative with eigenvalue -4.
 - (d) Give two different functions, neither of them being a multiple of the other, that are eigenfunctions of the second derivative with eigenvalue -3.
 - (e) Give two different functions, neither of them being a multiple of the other, that are eigenfunctions of the second derivative with eigenvalue $-\lambda^2$, where λ is a positive real number.
 - (f) Give the eigenfunctions that will have eigenvalue nine; there are two of them!
 - (g) Give the general form of the eigenfunctions of the operator that have positive eigenvalues, and give the general eigenvalue.
- 3. Let *D* be the operator $D = \frac{d^2}{dt^2} + 2\frac{d}{dt} 3$, whose action on a function y = y(t) is defined by $Dy = \frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y$.
 - (a) Show that $y = e^{-2t}$ is an eigenfunction for this operator, and determine the corresponding eigenvalue.
 - (b) In general, any function of the form e^{kt} is an eigenfunction for D. Determine the general eigenvalue.
 - (c) Give two values of k for which e^{kt} is an eigenfunction of D with eigenvalue zero.
 - (d) Give two values of k for which e^{kt} is an eigenfunction of D with eigenvalue five.
- 4. A very important ODE in many applications is $y'' + \lambda^2 y = 0$. Note that this can be rearranged to get $y'' = -\lambda^2 y$, which says that any y that is a solution to the differential equation is an eigenfunction with eigenvalue $-\lambda^2$.
 - (a) Give the eigenfunctions of the second derivative with eigenvalue $-\lambda^2$.
 - (b) (Challenge) Let S be the set of functions y = f(x) that have continuous second derivatives on the interval $[0, 2\pi]$ and for which $f(0) = f(2\pi) = 0$. Determine *ALL* eigenfunctions of the second derivative with eigenvalue $-\lambda^2$ that are in S.

5. Let L be the differential operator defined on a function y = y(x) by

$$L(y) = x \frac{d^2y}{dx^2} + (1-x) \frac{dy}{dx}.$$

Determine which of the following are eigenfunctions of L. For those that are, give the corresponding eigenvalue.

- (a) $y = x^2 + 3x$ (b) y = 1 - x(c) $y = x^3 - 9x^2 + 18x - 6$ (d) y = 2x + 3(e) $y = x^2 - 4x + 2$
- 6. Example 5.2(d) showed that the function $P_3(x) = 5x^3 3x$ is an eigenfunction of the operator L defined by

$$L(y) = (x^{2} - 1)\frac{d^{2}y}{dx^{2}} + 2x\frac{dy}{dx},$$

with eigenvalue 12. The function $P_3(x)$ is called a Legendre polynomial. There are more Legendre polynomials, each of which is an eigenfunction for L - here are a few of them:

$$P_1(x) = x,$$
 $P_2(x) = 3x^2 - 1,$ $P_4(x) = 35x^4 - 30x^2 + 3,$ $P_5(x) = 63x^5 - 70x^3 + 15x.$

- (a) Determine the corresponding eigenvalue for each of the eigenfunctions given above.
- (b) Make a table of n values for n = 1, 2, 3, 4, 5 and the corresponding eigenvalues.
- (c) There should be a pattern to the eigenvalues, but you may find it difficult to figure out. Give it a try, and take a guess as to what the eigenvalue is for n = 7. The eigenfunction is $P_7 = 429x^7 693x^5 + 315x^3 35x$ apply L to it to check your guess for the eigenvalue.
- (d) What is the eigenvalue for the *n*th Legendre polynomial $P_n(x)$?

Performance Criteria:

- 5. (f) Solve a boundary value problem for eigenvalues and the corresponding eigenfunctions.
 - (g) Give the boundary conditions for a vertical column.
 - (h) Find the buckling modes (non-trivial solutions) for a vertical column.
 - (i) Find the critical loads for a vertical column.
 - (j) Give the pinning conditions resulting in each of the buckling modes of a vertical column.

When solving the equation

$$\frac{d^2y}{dx^2} = -\frac{1}{4}y$$

we see that we are looking for eigenfunctions of the second derivative with eigenvalue $-\frac{1}{4}$. In many applications we are looking for eigenfunctions without knowing what the eigenvalue is. This seems like an impossible problem to solve, but it isn't, as we see in the following example.

• **Example 5.3(a):** Solve the boundary value problem

$$y'' + \lambda^2 y = 0, \quad y(0) = 0, \ y'(2\pi) = 0,$$
 (1)

where λ is a positive value to be determined.

Solution: The auxiliary equation for the differential equation is $r^2 + \lambda^2 = 0$, which leads to $r^2 = -\lambda^2$, so $r = \pm \lambda i$. The solution to the ODE is then

$$y = C_1 \sin \lambda x + C_2 \cos \lambda x.$$

Applying the first boundary condition y(0) = 0 gives us $C_2 = 0$, so the solution is

$$y = C_1 \sin \lambda x.$$

From this we can compute $y' = C_1 \lambda \cos \lambda x$, and applying the second boundary condition gives us

$$C_1 \lambda \cos 2\pi \lambda = 0.$$

There are three possibilities here: $C_1 = 0$, $\lambda = 0$, or $\cos 2\pi\lambda = 0$. The first two result in a solution of y = 0 for the boundary value problem. This is a valid solution, but quite uninteresting! For this reason we will refer to it as the *trivial solution*. To get a non-trivial solution it must be the case that $\cos 2\pi\lambda = 0$. Now $\cos \theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$ Thus we have

$$2\pi\lambda = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$\lambda = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \dots$$
(2)

Changing C_1 to just C, the non-trivial solutions to the boundary value problem (1) are then

$$y = C \sin \frac{1}{4}x, \ y = C \sin \frac{3}{4}x, \ y = C \sin \frac{5}{4}x, ...,$$
 (3)

each corresponding to one of the values of λ determined in (2).

As stated, each of the solutions (3) to the BVP (1) corresponds to a particular value of λ . Let's verify one of those solutions.

♦ **Example 5.3(b):** Verify that $y = C \sin \frac{5}{4}x$ is a solution to the boundary value problem

$$y'' + \lambda^2 y = 0, \quad y(0) = 0, \ y'(2\pi) = 0$$
 (1)

when $\lambda = \frac{5}{4}$.

Solution: The differential equation in this case is $y'' + \frac{25}{16}y = 0$. We see that

$$y = C\sin\frac{5}{4}x \implies y' = \frac{5}{4}C\cos\frac{5}{4}x \implies y'' = -\frac{25}{16}C\sin\frac{5}{4}x, \tag{4}$$

SO

$$y'' + \frac{25}{16}y = -\frac{25}{16}C\sin\frac{5}{4}x + \frac{25}{16}(C\sin\frac{5}{4}x) = 0,$$

showing that $y = C \sin \frac{5}{4}x$ is a solution to the differential equation.

We must now show that $y = C \sin \frac{5}{4}x$ satisfies the boundary conditions. Note that we found y' in (4).

$$y(0) = C \sin \frac{5}{4}(0) = 0$$
 and $y'(2\pi) = \frac{5}{4}C \cos \frac{5}{4}(2\pi) = \frac{5}{4}C \cos \frac{5\pi}{2} = 0$

Because $y = C \sin \frac{5}{4}x$ satisifies both the differential equation and boundary conditions, it is a solution to the BVP (1) when $\lambda = \frac{5}{4}$.

We note two ways our solution to the boundary value problem from Example 5.3(a) differs from the initial value problems we've solved, and the particular boundary value problems that we have solved up to now:

- There are infinitely many solutions to this boundary value problem, each corresponding to a specific choice of λ.
- There is an arbitray constant whose value we cannot determine from the information given.

For the particular application that we will look at in this section, we consider only one of the solutions

$$y = C \sin \frac{1}{4}x, \ y = C \sin \frac{3}{4}x, \ y = C \sin \frac{5}{4}x, ...,$$
 (3)

at a time. For other applications we need at some point to consider instead the arbitray **linear combination** of solutions

$$y = C_1 \sin \frac{1}{4}x + C_2 \sin \frac{3}{4}x + C_3 \sin \frac{5}{4}x + \dots$$
 (5)

When these applications occur (in the solving of partial differential equations), there is additional information that can be used to determine the values of all these constants. Some of you may recognize (5) as a **Fourier series**.

We now examine Example 5.3(a) in the context of eigenvalues and eigenfunctions. Note that the differential equation from the BVP can be written

$$\frac{d^2}{dx^2}(y) = -\lambda^2 x,$$

which is saying we are looking for functions that are eigenfunctions of the second derivative with eigenvalues $-\lambda^2$. This is called an **eigenvalue problem**, and it is typical that we need to find both

the eigenvalues and eigenfuctions. (Any of you who have had linear algebra may recall that in that course you had to find both eigenvalues and eigenvectors.) The result of Example 5.3(a) gives us the eigenvalues (using the results of (2))

$$-\lambda^2 = -\frac{1}{16}, -\frac{9}{16}, -\frac{25}{16}, ...,$$

with corresponding eigenfunctions

$$y = C \sin \frac{1}{4}x, \ y = C \sin \frac{3}{4}x, \ y = C \sin \frac{5}{4}x, \dots$$

We now look at one application of these ideas that some of you may have encountered in a strengths of materials course.

Deflection of Vertical Columns

Now we will examine the behavior of a vertical column when a load is applied directly downward on its top, as shown in the first picture to the right. We will begin by considering columns that are **pinned**; this means we will allow the top and bottom of the column to be at angles other than vertical. We will require the top and bottom of the column to be vertically aligned with each other - later we will consider a situation where we will relax this condition. So, for example, when enough force is applied downward the column will deflect horizontally as shown in the second picture to the right.



We will set up a coordinate system as shown below and to the right, with x indicating the distance upward from the bottom of the column and y = y(x) representing the *horizontal* deflection of the column at any point x. The differential equation governing the deflection of the column is

$$EI\frac{d^2y}{dx^2} = -Py,\tag{1}$$

where the parameters E and I are again the modulus of elasticity and cross-sectional moment of inertia of the column. They are properties that *could* vary along the column (with the variable x), but this would be unusual. We'll only consider columns where they do not

change. P is the (positive) force exerted downward on the top of the column, and we will look at the effects of different values of P, but for purposes of solving the ODE it is a constant. (It is a parameter rather than a variable, but its value is to be determined when solving the ODE.) Because the ODE is second order we will need two conditions to determine the solution. If the length of the column is denoted by L, we have the boundary conditions y(0) = 0 and y(L) = 0. If we rearrange the equation and combine it with the boundary conditions we get the boundary value problem (BVP)



$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0, \qquad y(0) = 0, \quad y(L) = 0.$$
(2)

We will see that solving this boundary value problem is somewhat different than solving the kind of BVPs we saw for horizontal beams, in the previous section. This is because we can rearrange the ODE (1) to get the eigenvalue problem

$$\frac{d^2y}{dx^2} = -\frac{P}{EI}y.$$
(3)

Note that y is an eigenfunction of the second derivative, with eigenvalue $-\frac{P}{EI}$, where P, E and I are all positive. As we now know very well, y must be of the form

$$y = C_1 \sin ax + C_2 \cos ax$$

for some constant *a* yet to be determined. Let's now go through the details for a specific case:

Example 5.3(c): Solve the boundary value problem

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0, \qquad y(0) = 0, \quad y(20) = 0, \tag{4}$$

with E = 800, I = 150 and P > 0.

Solution: Substituting the values for E and I into the ODE gives us

$$\frac{d^2y}{dx^2} + \frac{P}{120,000} \ y = 0 \tag{5}$$

Using our methods from Chapter 3, this equation has auxiliary equation $r^2 + \frac{P}{120,000} = 0$ and, because P > 0, its roots are $r = \pm i \sqrt{\frac{P}{120,000}}$. The solution to (4) is then

$$y = C_1 \sin \sqrt{\frac{P}{120,000}} \ x + C_2 \cos \sqrt{\frac{P}{120,000}} \ x$$

Applying the boundary condition y(0) = 0 gives us $C_2 = 0$, so the solution is then

$$y = C \sin \sqrt{\frac{P}{120,000}} \ x$$

(I've omitted the subscript for simplicity.) Now here is where things start to get interesting! The other boundary condition tells us that

$$0 = C \sin \sqrt{\frac{P}{120,000}} \,(20) \tag{6}$$

which, in turn, tells us that either C = 0 or $\sin \sqrt{\frac{P}{120,000}} (20) = 0$. The first possibility gives us the "trivial" solution y = 0 - it satisfies the differential equation and boundary conditions, but it isn't particularly interesting! Considering the second possibility, $\sin \theta = 0$ for $\theta = 0, \pi, 2\pi, 3\pi, ..., n\pi, ...$ so, for $C \neq 0$, (6) will be true for

$$\sqrt{\frac{P}{120,000}} (20) = 0, \ \pi, \ 2\pi, \ 3\pi, \ \dots, \ n\pi, \ \dots,$$
(7)

the first of which also gives us the trivial solution y = 0. Therefore, in theory at least (we'll talk later about what all this means from a practical point of view), we can only have nonzero solutions to the BVP if

$$\sqrt{\frac{P}{120,000}} = \frac{\pi}{20}, \frac{2\pi}{20}, \frac{3\pi}{20}, \dots, \frac{n\pi}{20}, \dots$$

This gives us the nonzero solutions

$$y = C \sin \frac{\pi}{20} x, \ y = C \sin \frac{2\pi}{20} x, \ y = C \sin \frac{3\pi}{20} x, \ \dots, \ y = C \sin \frac{n\pi}{20} x, \ \dots$$

to the boundary value problem.

This isn't really the end of the story, but we need to pause to catch our breath and develop some terminology before resuming. At this point what we know about the boundary value problem

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0, \qquad y(0) = 0, \quad y(20) = 0, \tag{4}$$

is that y = 0 is a solution, called the *trivial solution* (because it is mathematically uninteresting), and we only have nontrivial solutions for discrete values of $\sqrt{\frac{P}{120,000}}$; those solutions are the ones the example concluded with. The non-trivial solutions are called **buckling modes** (for reasons you'll soon see). The first non-trivial solution is called the **first buckling mode**, the second is the **second buckling mode**, and so on. The only values of P for which we can have nontrivial solutions are those that satisfy

$$\sqrt{\frac{P}{120,000}} (20) = \pi, \, 2\pi, \, 3\pi, \, \dots, \, n\pi, \, \dots$$
 (7)

Solving for P gives us

$$P = 120000 \left(\frac{\pi}{20}\right)^2, \ 120000 \left(\frac{2\pi}{20}\right)^2, \ 120000 \left(\frac{3\pi}{20}\right)^2, \ \dots, 120000 \left(\frac{n\pi}{20}\right)^2, \ \dots$$
$$= 300\pi^2, \ 300(2\pi)^2, \ 300(3\pi)^2, \ \dots, 300(n\pi)^2, \ \dots$$
$$= 300\pi^2, \ 4(300\pi^2), \ 9(300\pi^2), \ \dots, n^2(300\pi^2), \ \dots$$

These values of P are called **critical loads**. Like the buckling modes, they are numbered, so $300\pi^2$ is the **first critical load**, $4(300\pi^2)$ is the **second critical load**, etc. The word "load" refers to the load held up by the column. Note that the second critical load is four (two squared) times the first critical load, the third critical load is nine (three squared) times the first critical load, and so on.

What happens physically is this: When there is no load on the column it is perfectly straight (the solution y = 0), and it remains that way as we increase the load, until the first critical load is reached. At that point the column will deflect sideways, taking the shape of the curve $y = C \sin \frac{\pi}{20} x$, shown at the left below. In reality, as the load increases beyond the first critical load, the deflection will remain the same shape, but with increasing amplitude, until the column fails.

If we were able to prevent the middle point of the column, at x = 10, from deflecting, the column would be able to support the second critical load. Because the column is held with y(10) = 0, the deflection of the column will take the shape of a full period of the sine function, as shown in the middle picture below - this is the second buckling mode. The act of preventing deflection is sometimes called "pinning." If we pin the column at points one-third and two-thirds of the way along its length, the column would be able to support the third critical load, and the shape of the deflection would be given by the third buckling mode, shown to the right below.



Let's revisit the boundary conditions y(0) = y(L) = 0, where L is the length of the column. We should first note that the only requirement we really had was that the top and bottom of the column were vertically aligned. We could just as well have put y(0) = y(L) = 2, as shown in the diagram to the left below; however, the mathematics involved are a bit simpler if we instead use y(0) = y(L) = 0, as we did. Physically we must still insist that the top and bottom be aligned. Without this restriction, any horizontal shifting of the "ceiling" would result in hinging and a collapse. The beginning of such a collapse is indicated by the three diagrams to the right below.



Now suppose the top and bottom were embedded, rather than being pinned. If the column still has length L, we then have the boundary conditions

$$y'(0) = y'(L) = 0. (5)$$

This gives us two boundary conditions, which is what we should need in order to solve the second order ODE

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0,$$

and leaves us without any conditions on y(0) and y(L). In this situation, we could conceivably allow the "ceiling" to "drift" laterally without collapse, because the column being held in a vertical alignment at the bottom and top would provide enough rigidity to prevent collapse. The first three buckling modes for this situation are shown below; if the ceiling were prevented from drifting, the second mode would then become the first, the fourth would become the second, and so on. You will investigate this situation, along with the corresponding critical loads, in the exercises.



1st buckling mode



2nd buckling mode



3rd buckling mode

Section 5.3 Exercises To Solutions

Give all answers in exact form. When asked for the first ... buckling modes or critical loads, give only nonzero modes or values.

1. For each of the following, boundary values are given to go with the ODE $y'' + \lambda^2 y = 0$, which has the solution

$$y = C_1 \sin \lambda x + C_2 \cos \lambda x.$$

Use the method of Example 5.3(a) to determine the first four nonzero values of λ for which the boundary value problem has a solution, and give the corresponding four solutions.

- (a) y(0) = 0, y(5) = 0 (b) $y'(0) = 0, y'(3\pi) = 0$
- (c) $y(0) = 0, y'(\pi) = 0$ (d) y'(0) = 0, y(7) = 0
- (e) y'(0) = 0, y'(10) = 0 (f) $y(0) = 0, y(5\pi) = 0$
- 2. A 12 foot vertical column is pinned at both ends. For the material it is made of we have E = 500 and, from its design, we have I = 200, both in the appropriate units.
 - (a) Give the first four nonzero buckling modes, showing all steps of solving the IVP to get them.
 - (b) Give the first four nonzero critical loads.
 - (c) How many times larger is the third nonzero critical load than the first nonzero critical load?
- 3. Now suppose we have a 6 foot vertical column with E = 500, I = 200, and both ends pinned.
 - (a) Give the first four nonzero buckling modes, and find the corresponding critical loads.
 - (b) Compare your critical loads with those from the 12 foot length (Exercise 1). How do the corresponding critical loads for the six foot column compare with those for the 12 foot column? Does that make sense intuitively?
 - (c) The first nonzero buckling mode for the six foot column is the same as a six foot section of which buckling mode for the 12 foot column? Draw a picture showing what is going on here.
- 4. Consider now a 12 foot vertical column with E and I values of 500 and 200 again, but this time with both ends embedded. Suppose also that the "ceiling" is not allowed to drift, so the top and bottom of the column are vertically aligned.
 - (a) Give the first three nonzero buckling modes.
 - (b) Give the first three nonzero critical loads.
 - (c) How does the third nonzero critical load compare to the first? (That is, how many times larger is it?)
 - (d) How does the first nonzero critical load compare with the first nonzero critical load for the 12 foot column with pinned ends? (See Exercise 1(a).) How about the other critical loads?

- 5. Repeat the previous exercise, but with the assumption that the ceiling is allowed to drift. (**Hint:** You should be able to use your computations from Exercise 4, rather than re-doing all of them.)
- 6. Repeat parts (a) and (b) of Exercise 2 for a vertical column of length L that is pinned at both ends, with modulus of elasticity E and moment of inertia I. This will give a general form of the buckling modes and critical loads for pinned ends.
- 7. Repeat parts (a) and (b) of Exercise 5 for a vertical column of length L that is embedded at both ends, with the ceiling allowed to drift. Again use a modulus of elasticity E and moment of inertia I. This will give a general form of the buckling modes and critical loads for embedded ends.

5.4 The Heat Equation in One Dimension

In this section we will consider the physically impossible but mathematically convenient situation: We have a metal rod of length L (see picture below) that is perfectly insulated along its length, so that

no heat can enter or escape along its length, but for which heat can enter or leave the ends. At any time t greater than zero and any position x along the rod, the function u(x,t) gives the temperature at that point x and time t. (We will think of the rod as being "infinitely thin," so that



the rod has only one point at each x position. If you are not happy with this, an alternative is to think that if the rod had some thickness the temperature at every point in a cross-sectional slice at some x is the same, so we need not consider the other two space dimensions.) Suppose that at time zero there is some distribution of temperatures along the rod, given by a the function f(x) for $0 \le x \le L$, and suppose also that the ends of the rod are held at temperature zero at all times $t \ge 0$. The function f gives an initial condition for each point x along the length of the rod, and the conditions that the ends are held at temperature zero at length of the rod, and the conditions that the ends are held at temperature zero are boundary conditions.

What we would like to know is whether, and how, we can determine the temperature at any point x with 0 < x < L (we know the temperatures at x = 0 and x = L are always zero), at any time t > 0. Here the dependent variable u depends on the *two* independent variables x and t. Some physical principles concerning heat give us a differential equation for this situation and, due to there being two independent variables, it is a *partial differential equation*. The equation (called the **heat equation**), and the conditions given above can all be stated as

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(L,t) = 0, \qquad u(x,0) = f(x)$$
(1)

Here the conditions u(0,t) = u(L,t) = 0 are boundary conditions and u(x,0) = f(x) is essentially an initial condition (for every point along the rod). Thus we have a problem that is a sort of combination initial value/boundary value problem. But we can really think of it as a boundary value problem for this reason: If we were to think of the Cartesian plane as representing position

x along the horizontal axis and time *t* along the vertical axis we get a picture like the one to the right, where each point in the shaded region represents a point *x* in the rod and some time *t*. Our goal is then to find the temperature at each of those points; in this way we can think of trying to find function values in a region that is bounded by the line from zero to *L* on the *x*-axis and the two "half-infinite" lines from zero to infinity in the *t* direction at x = 0 and x = L. The condition u(x, 0) = f(x) can be thought of as a boundary condition along the bottom, and the conditions u(0,t) = u(L,t) = 0 are boundary conditions along the two sides.



Recall that if we have the function $f(x,y) = x^2y^3$, to get the partial derivative $\frac{\partial f}{\partial x}$ we simply take the derivative of x^2y^3 , treating y (and therefore y^3) as a constant. Similarly, we get $\frac{\partial f}{\partial y}$ by treating x as a constant, so we have

$$rac{\partial f}{\partial x} = 2xy^3$$
 and $rac{\partial f}{\partial y} = 3x^2y^2$

Let's try another.

♦ **Example 5.4(a):** Find
$$\frac{\partial u}{\partial x}$$
 and $\frac{\partial u}{\partial t}$ for $u(x,t) = e^{-2t} \sin 3x$.

Solution: When finding $\frac{\partial u}{\partial x}$ we consider t to be a constant, so e^{-2t} is as well. The derivative is then

$$\frac{\partial u}{\partial x} = 3e^{-2t}\cos 3x.$$

When finding $\frac{\partial u}{\partial t}$, $\sin 3t$ is essentially a constant, so

$$\frac{\partial u}{\partial t} = -2e^{-2t}\sin 3x.$$

Now we'll see that we have a solution to the heat equation!

♦ **Example 5.4(b):** Show that $u(x,t) = e^{-2t} \sin 3x$ satisfies the heat equation $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$.

Solution: We already have

$$\frac{\partial u}{\partial x} = 3e^{-2t}\cos 3x$$
 and $\frac{\partial u}{\partial t} = -2e^{-2t}\sin 3x$

from the previous example. Taking the partial derivative of the first of these with respect to \boldsymbol{x} again gives us

$$\frac{\partial^2 u}{\partial x^2} = -9e^{-2t}\sin 3x$$

so

$$\frac{\partial u}{\partial t} = -2e^{-2t}\sin 3x = \frac{2}{9}\left(-9e^{-2t}\sin 3x\right) = k\frac{\partial^2 u}{\partial x^2}, \qquad k = \frac{2}{9}$$

We will soon see that $u(x,t) = e^{-2t} \sin 3x$ is not the most general solution to the equation.

Those who've had a multivariable calculus course will perhaps recall that the computation of partial derivatives can be significantly more complicated (and therefore difficult) than the ones done above, but if we understand the two examples just given we are ready to understand how

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(L,t) = 0, \qquad u(x,0) = f(x)$$
(1)

is solved. The method for doing it is called **separation of variables**, which is similar in execution, up to a point, to the method of the same name we used to solve separable first order ODEs up to a point. After that point we must proceed differently.

To begin, we assume that the function u(x,t) is actually a product of a function of x alone and another function of t alone. There is no practical reason to think this might be the case, but the method works, so we'll use it! (This method was invented/discovered in the 1700s by Daniel Bernoulli, part of a family of a number of accomplished mathematicians and scientists. Daniel was also involved in the derivation of the ODE we used for horizontal beams.) If we let X be the function of x and T be the function of t, then u(x,t) = X(x)T(t). (This use of a capital letter for a function and the lower case of the same letter for the independent variable is common practice in the study partial differential equation solution methods.) Now remember that if we are taking the derivative of X(x)T(t) with respect to x, T(t) is treated as a constant, and when taking the derivative with respect to t, X(x) is treated as a constant, so

$$\frac{\partial u}{\partial t} = X(x)T'(t)$$
 and $\frac{\partial^2 u}{\partial x^2} = X''(x)T(t).$

The differential equation in (1) then becomes X(x)T'(t) = kX''(x)T(t). If we divide both sides by kX(x)T(t) we get

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}.$$
(2)

Here is where we deviate from the procedure for solving first order separable equations. (2) needs to be true for all values of x and t, and this is likely only the case if both sides of (2) are equal to some constant (again, we'll see that it works!) that we will call $-\lambda^2$. For reasons we won't go into here, λ is positive. Setting each side equal to $-\lambda^2$ and multiplying by the denominators we get

$$\frac{dT}{dt}(t) = -k\lambda^2 T(t) \qquad \text{and} \qquad \frac{d^2 X}{dx^2}(x) = -\lambda^2 X(x). \tag{3}$$

In addition to this, we also have the boundary conditions X(0) = X(L) = 0 for the second equation. The first equation in (3) tells us that T(t) is an eigenfunction for the first derivative operator, with eigenvalue $-k\lambda^2$, and we know that T(t) is then any constant multiple of $e^{-k\lambda^2 t}$. That is,

$$T(t) = C_1 e^{-k\lambda^2 t}.$$

The second equation says that X(x) is an eigenfunction of the second derivative operator with eigenvalue $-\lambda^2$. Because λ is positive, the eigenfunctions are constant multiples of $\sin \lambda x$ and $\cos \lambda x$, as determined in the previous section, and we have

$$X(x) = C_2 \sin \lambda x + C_2 \cos \lambda x.$$

The general solution to the heat equation then looks like

$$u(x,t) = X(x)T(t) = e^{-k\lambda^2 t} (A\sin\lambda x + B\cos\lambda x)$$
(4)

where $A = C_1 C_2$ and $B = C_1 C_3$.

Let's focus a bit more on the second ODE in (3) and its boundary values X(0) = X(L) = 0. The general solution to the ODE is $X(x) = A \sin \lambda x + B \cos \lambda x$. Applying the condition X(0) = 0 gives us B = 0, so the solution is $X(x) = A \sin \lambda t$. (At this point this story should be starting to feel familiar!) We now consider the boundary condition X(L) = 0, which gives us $0 = A \sin \lambda L$. As before, when considering vertical columns, we don't want to let A = 0, so we must have $\sin \lambda L = 0$. This implies that

$$\lambda L = 0, \ \pi, \ 2\pi, \ 3\pi, \dots \qquad \Longrightarrow \qquad \lambda = 0, \ \frac{\pi}{L}, \ \frac{2\pi}{L}, \ \frac{3\pi}{L}, \dots$$

and the solutions to the boundary value problem (disregarding constants and the zero solution arising from $\lambda = 0$) are

$$\sin\frac{\pi}{L}x, \ \sin\frac{2\pi}{L}x, \ \sin\frac{3\pi}{Lx}, \ldots$$

The solution T then becomes $T(t) = e^{-\frac{k\pi^2}{L^2}t}$, $e^{-\frac{4k\pi^2}{L^2}t}$, $e^{-\frac{9k\pi^2}{L^2}t}$,... depending on λ , so we get a sequence of solutions u(x,t) = X(x)T(t):

$$u(x,t) = e^{-\frac{k\pi^2}{L^2}t} \sin\frac{\pi}{L}x, \ e^{-\frac{4k\pi^2}{L^2}t} \sin\frac{2\pi}{L}x, \ e^{-\frac{9k\pi^2}{L^2}t} \sin\frac{3\pi}{L}x, \dots$$
(5)

Recall that when solving an ODE like y'' + 3y' + 2y = 0 we assumed $y = e^{rt}$ for some constant r. From this we obtain $y = e^{-t}$ or $y = e^{-2t}$, but we saw that the sum of constant multiples of these two, $y = C_1e^{-t} + C_2e^{-2t}$ is the most general solution. By the same reasoning, the most general solution to the PDE we're looking at is an infinite sum of the solutions in (5):

$$u(x,t) = A_1 e^{-\frac{k\pi^2}{L^2}t} \sin\frac{\pi}{L}x + A_2 e^{-\frac{4k\pi^2}{L^2}t} \sin\frac{2\pi}{L}x + A_3 e^{-\frac{9k\pi^2}{L^2}t} \sin\frac{3\pi}{L}x + \dots + A_n e^{-\frac{n^2k\pi^2}{L^2}t} \sin\frac{n\pi}{L}x + \dots$$
(6)

This story goes on quite a bit longer, but let's end it with the following. In order to try to meet the condition u(x,0) = f(x) we must have

$$f(x) = A_1 \sin \frac{\pi}{L} x + A_2 \sin \frac{2\pi}{L} x + A_3 \sin \frac{3\pi}{L} x + \dots + A_n \sin \frac{n\pi}{L} x + \dots$$
(7)

The right hand side of (7) is something called a Fourier series. This brings up the question

In what way (or ways) do we interpret the equal sign in (7), and for what functions f can such interpretation(s) be made?

Attempts to answer this question gave birth to a large amount of mathematics over many years, starting with Joseph Fourier's work in the early 1800s, and with a major result proved as late as 1966. Perhaps some of you will investigate this subject more in later coursework.

5.5 Chapter 5 Summary

- Boundary value problems arise when the independent variable of an ODE is length. Applications include the deflection of horizontal beams and vertical columns along their lengths.
- The differential equation for a horizontal beam is fourth order, and the solution is a fourth order polynomial with four arbitrary constants.
- There are two boundary conditions at each of the two ends of a horizontal beam, giving four conditions used to determine the values of the constants.
- There are three possible end conditions for each end of a horizontal beam:
 - **Embedded:** This is when the end of the beam is "clamped" horizontally. The mathematical conditions for such an end are y = 0 and y' = 0.
 - Simply Supported (Pinned): This is when the beam is held up but allowed to pivot. Mathematically, y = 0 and y'' = 0.
 - Free: This is when an end is completely unsupported, and the other end *must* be embedded. The mathematical conditions for such an end are y'' = 0 and y''' = 0.
- Let A be an operator that operates on functions and $y\,$ a nonzero function. If there is a constant $\lambda\,$ such that

 $Ay = \lambda y$

then y is an **eigenfunction** of the operator A, with corresponding **eigenvalue** λ .

- The ODE for a vertical column is second order, and the solution is either a sine function or a cosine function, depending on the end conditions:
 - When the ends are pinned (hinged) the solution is a sine function.
 - When the ends are embedded the solution is a cosine function.
- Mathematically, there are infinitely many solutions for a vertical column that is pinned at its ends.
 - Each is some multiple of a half period of a sine function beginning at x = 0.
 - The first solution, called the first buckling mode, is a single half-period of the sine function.
 This occurs physically when the column is allowed to deflect over its entire length.
 - Each additional solution (buckling mode) consists of $\frac{n}{2}$ periods of the sine function for n = 2, 3, 4, 5, ... Physically, the solution consisting of $\frac{n}{2}$ periods of the sine function occurs when the the column is pinned along its length at n-1 equally spaced points.
- Mathematically, there are infinitely many solutions for a vertical column that is embedded at its ends.
 - Each is some multiple of a half period of a cosine function beginning at x = 0.
 - In the case that the ends of the column are embedded, it is physically possible that the ceiling can float (move laterally).
 - If the ceiling is allowed to float the first buckling mode is a single half-period of the cosine function. Each additional buckling mode consists of $\frac{n}{2}$ periods of the cosine function for n = 2, 3, 4, 5, ...

- If the ceiling is *NOT* allowed to float the first buckling mode is a single period of the cosine function. Each additional buckling mode consists of n periods of the cosine function for n = 2, 3, 4, 5, ...
- The load that causes the first buckling mode is called the **first buckling load**, and the *n*th buckling load leads to the *n*th buckling load.
- The *n*th buckling load is n^2 times the first buckling load.

D Solutions to Exercises

D.5 Chapter 5 Solutions

Section 5.1 Solutions

Back to 5.1 Exercises

- 1. (a) y(0) = 0, y''(0) = 0, y(12) = 0, y'(12) = 0(b) y(0) = 0, y'(0) = 0, y''(8) = 0, y'''(8) = 0(c) y(0) = 0, y'(0) = 0, y(20) = 0, y''(20) = 0(d) y(0) = 0, y''(0) = 0, y(15) = 0, y''(15) = 0
- 2. Only (a) is possible, y(0) = 0, y'(0) = 0, y(8) = 0, y''(8) = 0
- 3. (a) $(30)(80)\frac{d^4y}{dx^4} = 150, \quad y(0) = y'(0) = y(8) = y'(8) = 0$ (b) $y = \frac{1}{384}x^4 - \frac{1}{24}x^3 + \frac{1}{6}x^2$

2

(d) The maximum deflection should appear at the middle of the beam (x = 4). The deflection there is $\frac{2}{3}$.

4. (b)
$$y = \frac{1}{12}x^4 - \frac{10}{3}x^3 + 50x$$

- (d) The maximum deflection is 2500 at x = 10, the right hand end of the beam.
- 5. (b) $y = \frac{1}{12}x^4 \frac{5}{3}x^3 + \frac{250}{3}x$ (d) The maximum deflection is $\frac{3125}{12}$ at x = 5
- 6. (a) $(30)(80)\frac{d^4y}{dx^4} = 150, \quad y(0) = y'(0) = y(8) = y'' = 0$ (b) $y = \frac{1}{384}x^4 - \frac{5}{96}x^3 + \frac{1}{4}x^2$
 - (d) The maximum deflection is about 1.39 at x = 4.6
- 7. (a) J and L (b) E and H (c) F and G

Section 5.2 Solutions Back to 5.2 Exercises

- 1. (a) The eigenvalue is $\lambda = 3$. (b) $y = e^{-5x}$ (c) Eigenfunction: $y = e^{kx}$ Eigenvalue: k
- 2. (a) y = 3x, y = 5, y = 2x 1, etc. (b) y = Ax + B(c) $y = \sin 2x$, $y = \cos 2x$ (d) $y = \sin \sqrt{3}x$, $y = \cos \sqrt{3}x$

3. (a)
$$D(e^{-2t}) = 4e^{-2t} - 4e^{-2t} - 3e^{-2t} = -3e^{-2t}$$
, the eigenvalue is -3
(b) $D(e^{kt}) = k^2 e^{kt} + 2ke^{kt} - 3e^{kt} = (k^2 + 2k - 3)e^{kt}$, the eigenvalue is $k^2 + 2k - 3$
(c) $k = -3, 1$ (d) $k = -4, 2$

Back to 5.3 Exercises

1. (a) $\lambda = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, ..., \qquad y = C \sin \frac{\pi}{5}x, \ C \sin \frac{2\pi}{5}x, \ C \sin \frac{3\pi}{5}x, \ C \sin \frac{4\pi}{5}x, ...$ (b) $\lambda = \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, ..., \qquad y = C \cos \frac{1}{3}x, \ C \cos \frac{2}{3}x, \ C \cos \frac{3}{3}x, \ C \cos \frac{4}{3}x, ...$ (c) $\lambda = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, ..., \qquad y = C \sin \frac{\pi}{5}x, \ C \sin \frac{2\pi}{5}x, \ C \sin \frac{3\pi}{5}x, \ C \sin \frac{4\pi}{5}x, ...$ (d) $\lambda = \frac{\pi}{14}, \frac{3\pi}{14}, \frac{5\pi}{14}, \frac{7\pi}{14}, ..., \qquad y = C \cos \frac{\pi}{14}x, \ C \cos \frac{3\pi}{14}x, \ C \cos \frac{5\pi}{14}x, \ C \cos \frac{7\pi}{14}x, ...$ (e) $\lambda = \frac{\pi}{10}, \frac{2\pi}{10}, \frac{3\pi}{10}, \frac{4\pi}{10}, ..., \qquad y = C \cos \frac{\pi}{10}x, \ C \cos \frac{2\pi}{10}x, \ C \cos \frac{3\pi}{10}x, \ C \cos \frac{4\pi}{10}x, ...$ (f) $\lambda = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, ..., \qquad y = C \sin \frac{1}{5}x, \ C \sin \frac{2}{5}x, \ C \sin \frac{3}{5}x, \ C \sin \frac{4\pi}{5}x, ...$ 2. (a) $y = C \sin \frac{\pi}{10}x, \ y = C \sin \frac{2\pi}{10}x, \ y = C \sin \frac{4\pi}{10}x, ...$

2. (a)
$$y = C \sin \frac{\pi}{12} x, \ y = C \sin \frac{2\pi}{12} x, \ y = C \sin \frac{3\pi}{12} x, \ y = C \sin \frac{4\pi}{12} x, \ \dots$$

(b)
$$P = \frac{6250}{9}\pi^2$$
, $4(\frac{6250}{9}\pi^2)$, $9(\frac{6250}{9}\pi^2)$, $16(\frac{6250}{9}\pi^2)$, ...

(c) The third critical load is nine times the first critical load.

3. (a)
$$y = C \sin \frac{\pi}{6} x, \ y = C \sin \frac{2\pi}{6} x, \ y = C \sin \frac{3\pi}{6} x, \ y = C \sin \frac{4\pi}{6} x, \ \dots$$

 $P = \frac{25000}{9} \pi^2, \ 4(\frac{25000}{9} \pi^2), \ 9(\frac{25000}{9} \pi^2), \ 16(\frac{25000}{9} \pi^2), \ \dots$

- (b) Each critical load is four times the corresponding critical load for the twelve foot column.
- (c) The first buckling mode for the six foot column is the same as the second buckling mode for the twelve foot column, but is half as high, so it only includes half a period of the sine function, whereas the twelve foot column's second buckling mode has a full period of the sine function.

4. (a)
$$y = C \cos \frac{2\pi}{12} x$$
, $y = C \cos \frac{4\pi}{12} x$, $y = C \cos \frac{6\pi}{12} x$, ...
(b) $P = 4(\frac{6250}{9}\pi^2)$, $16(\frac{6250}{9}\pi^2)$, $36(\frac{6250}{9}\pi^2)$, ...

- (c) The third critical load is nine times the first critical load.
- (d) Each critical load is four times the corresponding critical load for the column with pinned ends.
- 5. (a) $y = C \cos \frac{\pi}{12} x$, $y = C \cos \frac{2\pi}{12} x$, $y = C \cos \frac{3\pi}{12} x$, $y = C \cos \frac{4\pi}{12} x$, ... (b) $P = \frac{6250}{9} \pi^2$, $4(\frac{6250}{9} \pi^2)$, $9(\frac{6250}{9} \pi^2)$, $16(\frac{6250}{9} \pi^2)$, ...
 - (c) The third critical load is (again!) nine times the first critical load.
 - (d) The critical loads are the same as those for the pinned ends.
- 6. (a) $y = C \sin \frac{\pi}{L} x$, $y = C \sin \frac{2\pi}{L} x$, $y = C \sin \frac{3\pi}{L} x$, $y = C \sin \frac{4\pi}{L} x$, ... (b) $P = \frac{EI}{L^2} \pi^2$, $4(\frac{EI}{L^2} \pi^2)$, $9(\frac{EI}{L^2} \pi^2)$, $16(\frac{EI}{L^2} \pi^2)$, ...
- 7. (a) $y = C \cos \frac{\pi}{L} x$, $y = C \cos \frac{2\pi}{L} x$, $y = C \cos \frac{3\pi}{L} x$, $y = C \cos \frac{4\pi}{L} x$, ... (b) $P = \frac{EI}{L^2} \pi^2$, $4(\frac{EI}{L^2} \pi^2)$, $9(\frac{EI}{L^2} \pi^2)$, $16(\frac{EI}{L^2} \pi^2)$, ...