Ordinary Differential Equations

for Engineers and Scientists

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Contents

6 The Laplace Transform and Differential Equations

Learning Outcomes:

1. Use the Laplace transform to solve second order, linear, constant coefficient IVPs.

   Performance Criteria:

   (a) Find the Laplace transform of a function from a table.

   (b) Find the inverse Laplace transform of a function from a table.

   (c) Use partial fractions or completing the square to find the inverse Laplace transform of a function from a table.

   (d) Use the Laplace transform to solve an IVP with a second order, linear, constant coefficient ODE.

   (e) Find the Laplace transform of a function directly from the definition.

   (f) Graph piecewise defined functions, with their definitions given either with or without using the unit step function. Use the unit step function to write a piecewise defined function as one equation.

   (g) Find the Laplace transform of a piecewise defined function.

   (h) Find inverse Laplace transforms that result in piecewise defined functions of t.

   (i) Use the Laplace transform to solve an IVP with a second order, linear, constant coefficient ODE and piecewise defined forcing function.

   (j) Use the Laplace transform to solve an IVP with a second order, linear, constant coefficient ODE and impulse defined forcing function.

   (k) Find Laplace transforms of functions defined by convolution.

   (l) Determine a convolution whose Laplace transform is a given function of s.

   (m) Express solutions to IVPs for second order, linear, constant coefficient ODEs in terms of convolutions.

6.1 Introduction to the Laplace Transform

Performance Criteria:

6. (a) Find the Laplace transform of a function from a table.
   (b) Find the inverse Laplace transform of a function from a table.

In Section 3.2 we introduced the idea of an **operator** as a "function that acts on functions" to create other functions. The simplest example, that we are all familiar with, is the derivative operator. It "takes in" a function f and "gives out" the derivative function f'. The most important property of the derivative is that it is a **linear operator**. That is, for any functions y_1 and y_2 and constants a and b,

$$\frac{d}{dx}(ay_1 + by_2) = a\frac{dy_1}{dx} + b\frac{dy_2}{dx}.$$

If we know this along with derivatives of a few basic functions, like those given in the table below, we can find the derivatives of many functions.

$f(x)$	$f'(x) = \frac{d}{dx}[f(x)]$	$f(x)$	$f'(x) = \frac{d}{dx}[f(x)]$
1	0	$\cos ax$	$-a\sin ax$
x^n	nx^{n-1}	e^{ax}	ae^{ax}
$\sin ax$	$a\cos ax$	$\ln x$	$\frac{1}{x}$

Even though we may have forgotten (or never known!) where the above derivatives come from, we can still apply the derivatives in the table along with linearity of the derivative to compute derivatives.

◇ **Example 6.1(a):** Find the derivative of $f(x) = 5x^3 - 7\cos 4x + 2e^{-5x} + 4$.

By linearity, and the fact that $4 = 4(1)$, we have

$$\frac{d}{dx}[f(x)] = 5\frac{d}{dx}(x^3) - 7\frac{d}{dx}(\cos 4x) + 2\frac{d}{dx}(e^{-5x}) + 4\frac{d}{dx}(1)\,.$$

We then use the derivatives in the above table to obtain

$$\begin{aligned} f'(x) \quad = \quad \frac{d}{dx}[f(x)] \quad &= \quad 5(3x^2) - 7(-4\sin 4x) + 2(5e^{-5x}) + 4(0) \\ &= \quad 15x^2 + 28\sin 4x + 10e^{-5x} \end{aligned}$$

We can also use the same table to find **anti-derivatives** of functions, also (unfortunately) called integrals or, more properly, **indefinite integrals**. The indefinite integral is also linear, expressed by

$$\int (ay_1 + by_2)\,dx = a\int y_1\,dx + b\int y_2\,dx$$

Using this fact, the table from the previous page, and a trick we can then find many anti-derivatives.

◇ **Example 6.1(b):** Find the anti-derivative of $\sin 3x$.

Symbolically, we wish to find $\int \sin 3x\,dx$. In order to use the table, we need to get one of the forms in the second or fourth column, so that we can simply read off the anti-derivative from the first or third column. Thus we need to have the form in the top row of the fourth column, $-a\sin ax$. That is easy enough to do, we simply multiply $\sin 3x$ by $\left(-\frac{1}{3}\right)(-3) = 1$ and then apply the linearity of the anti-derivative to take the $-\frac{1}{3}$ out of the indefinite integral:

$$\int \sin 3x\,dx = \int \left(-\tfrac{1}{3}\right)(-3)\sin 3x\,dx = -\tfrac{1}{3}\int -3\sin 3x\,dx = -\tfrac{1}{3}\cos 3x$$

There is a special class of operators called **transforms**, toward which we will now direct our attention. Note that if f is a function with independent variable x, then f' also has independent variable x. What distinguishes transforms from other operators is that the independent variable for the transform of a function is different than the independent variable for the original function. The two most well known transforms are the **Laplace transform** and the **Fourier transform**, both of which are extremely useful in engineering applications. As indicated by the title of the chapter, we will focus here on the Laplace transform.

NOTE: Some authors will use the terms operator and transform interchangeably, but we will distinguish them from each other in the way described.

Suppose that we have a function f with independent variable t. We will denote its Laplace transform by F, with independent variable s. The act of changing f to F is done by the Laplace transform itself, which we denote by $\mathscr{L}$. Thus we can write

$$\mathscr{L}\{f\} = F \qquad \text{and} \qquad \mathscr{L}\{f(t)\} = F(s)$$

depending on whether we are considering just the transform of the function f or the action of the function and its transform on the independent variables t and s, respectively. One might wonder what exactly $\mathscr{L}$ does to f to get F. That will be revealed in Section 6.3, but we really don't need to know how transforms of specific functions are obtained in order to find Laplace transforms, just as we don't need to know how the derivatives in the table on the previous page are obtained. For the time being we need two things, some Laplace transforms of common functions and the linearity of the Laplace transform. Here is a table of some Laplace transforms:

Table of Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$	$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$
1	$\dfrac{1}{s} \quad (s > 0)$	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2} \quad (s > 0)$
t^n	$\dfrac{n!}{s^{n+1}} \quad (s > 0)$	$e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2 + \omega^2} \quad (s > a)$
e^{at}	$\dfrac{1}{s-a} \quad (s > a)$	$e^{at} \cos \omega t$	$\dfrac{s-a}{(s-a)^2 + \omega^2} \quad (s > a)$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}} \quad (s > a)$	$y'(t)$	$sY(s) - y(0)$
$\sin \omega t$	$\dfrac{\omega}{s^2 + \omega^2} \quad (s > 0)$	$y''(t)$	$s^2 Y(s) - sy(0) - y'(0)$

The linearity of the Laplace transform is described precisely below. That is followed by some examples of finding Laplace transforms.

Linearity of the Laplace Transform

Let f and g be functions of the variable t with Laplace transforms F and G with independent variable s. If a and b are any two constants

$$\mathscr{L}\{af(t) + bg(t)\} = a\mathscr{L}\{f\}(s) + b\mathscr{L}\{g\}(s) = aF(s) + bG(s).$$

◇ **Example 6.1(c):** Find the Laplace transform of $f(t) = t^3$.

Looking in the table, we see that the transform of t^n is $\dfrac{n!}{s^{n+1}}$. In our case $n = 3$, so the transform is

$$F(s) = \frac{3!}{s^{3+1}} = \frac{6}{s^4}\,.$$

◇ **Example 6.1(d):** Find the Laplace transform of $g(t) = e^{-5t}$.

The transform of e^{at} is $\dfrac{1}{s-a}$. In our case $a = -5$, giving us the transform

$$G(s) = \frac{1}{s-(-5)} = \frac{1}{s+5}\,.$$

◇ **Example 6.1(e):** Find the Laplace transform of $f(t) = \cos 2t$.

Here we have $\omega = 2$, so the transform is

$$F(s) = \frac{s}{s^2 + 2^2} = \frac{s}{s^2 + 4}\,.$$

So far we have simply obtained our transforms from the table. Now we contemplate some examples in which we utilize the linearity of the Laplace transform.

◇ **Example 6.1(f):** Find the Laplace transform of $3t^2 - 5t + 2$.

Using the $\mathscr{L}$ notation we have

$$\begin{aligned} \mathscr{L}\{3t^2 - 5t + 2\} &= 3\mathscr{L}\{t^2\} - 5\mathscr{L}\{t\} + 2\mathscr{L}\{1\} \\ &= 3 \cdot \frac{2}{s^3} - 5 \cdot \frac{1}{s^2} + 2 \cdot \frac{1}{s} \\ &= \frac{6}{s^3} - \frac{5}{s^2} + \frac{2}{s} \end{aligned}$$

Along with the operation of differentiation, we are equally interested in anti-differentiation, or the process of determining indefinite integrals. Analogously, we need to reverse the process of the Laplace transform. Doing so is referred to as finding the **inverse Laplace transform** of a function. As with differentiation, the process of inverting the operator is trickier than applying it in the "forward direction." We'll do a few simpler examples here, and follow with some special techniques in the next section. Perhaps the most basic technique is the one for getting a desired numerator, as illustrated in the following example.

◇ **Example 6.1(g):** Find the inverse Laplace transform $f(t)$ of $F(s) = \dfrac{5}{s^3}$.

Here we want to use the second entry in the table, which tells us that the Laplace transform of t^n is $\dfrac{n!}{s^{n+1}}$. We can see that

$$F(s) = \frac{5}{s^3} = 5 \cdot \frac{1}{s^{2+1}} = \frac{5}{2} \cdot \frac{2!}{s^{2+1}}\,.$$

(The last step results from multiplying the top and bottom of the fraction by two.)The fact that the Laplace transform is linear results in the inverse Laplace transform being linear as well. Thus we can simply multiply the inverse transform of $\dfrac{2!}{s^{2+1}}$ by $\frac{5}{2}$ to get $f(t) = \frac{5}{2}t^2$.

◇ **Example 6.1(h):** Find the inverse Laplace transform $y(t)$ of $Y(s) = \dfrac{5s - 2}{s^2 + 9}$.

First we can apply the useful identity $\dfrac{a \pm b}{c} = \dfrac{a}{c} \pm \dfrac{b}{c}$ and a bit of basic algebra/arithmetic to get

$$Y(s) = \frac{5s}{s^2 + 9} - \frac{2}{s^2 + 9} = 5 \cdot \frac{s}{s^2 + 3^2} - 2 \cdot \frac{1}{s^2 + 3^2} = 5 \cdot \frac{s}{s^2 + 3^2} - \frac{2}{3} \cdot \frac{3}{s^2 + 3^2}\,.$$

We can see from the table that the inverse transform of $\dfrac{s}{s^2+3^2}$ is $\cos 3t$, and the inverse transform of $\dfrac{3}{s^2+3^2}$ is $\sin 3t$. Therefore

$$y(t) = 5\cos 3t - \tfrac{2}{3}\sin 3t\,.$$

The following example illustrates again the use of the linearity of the inverse Laplace transform along with a useful algebraic technique.

◇ **Example 6.1(i):** Find the inverse Laplace transform $g(t)$ of $G(s) = \dfrac{3}{s} + \dfrac{7}{2s+5}$.

We note that

$$G(s) = 3\cdot\frac{1}{s} + 7\cdot\frac{1}{2(s+\frac{5}{2})} = 3\cdot\frac{1}{s} + \frac{7}{2}\cdot\frac{1}{s-(-\frac{5}{2})}\,.$$

From the table we then have

$$g(t) = 3(1) + \tfrac{7}{2}e^{-\frac{5}{2}t} = 3 + \tfrac{7}{2}e^{-\frac{5}{2}t}\,.$$

Our goal is to be able to use the Laplace transforms to solve some of the differential equations we have worked with in the past. At this point we don't have enough techniques for finding inverse Laplace transforms to solve a variety of ODEs, but we can solve a few very simple ones. To do this we need the following, which can also be found in the table a few pages back.

Laplace Transforms of the Derivatives of a Function

When solving an initial value problem for the independent variable $y = y(t)$, we have

$$\mathscr{L}\{y(t)\} = Y(s), \qquad\qquad \mathscr{L}\{y'(t)\} = sY(s) - y(0),$$

$$\mathscr{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

Note that the Laplace transform of the first derivative requires only one initial condition, and the transform of the second derivative requires two initial conditions.

Let's now look at a very simple example of solving an IVP using the Laplace transform. In the exercises you will do two others, but most IVPs will have to wait until we develop some more sophisticated methods for finding inverse transforms.

◇ **Example 6.1(j):** Solve the IVP $y' + 3y = 0$, $y(0) = 2$ using the Laplace transform.

We begin by applying the Laplace transform to both sides of the ODE:

$$\mathscr{L}\{y' + 3y\} = \mathscr{L}\{0\}.$$

The Laplace transform of zero is zero, and we can use linearity of the transform on the left side to get

$$\mathscr{L}\{y'\} + 3\mathscr{L}\{y\} = 0.$$

We now use the above to get

$$sY(s) - 2 + 3Y(s) = 0$$

and we then solve for $Y(s)$ to get

$$Y(s) = \frac{2}{s+3} = 2 \cdot \frac{1}{s-(-3)}.$$

The inverse transform of this is the solution to the IVP, $y(t) = 2e^{-3t}$.

It should be clear from this example that solving an IVP using the Laplace transform is essentially an algebraic process.

Section 6.1 Exercises

1. Determine the Laplace transform of each of the following functions. Give your answers labeled with correct notation. (The Laplace transform of $f(t)$ should be labeled as $F(s)$, for example.)

   (a) $f(t) = 3t^2 - 7t + 2$ (b) $g(t) = 3e^{-2t}$ (c) $f(t) = 2\sin 5t$

   (d) $g(t) = t^5 - 3t^2$ (e) $f(t) = 7t^2e^{-5t}$ (f) $g(t) = 5e^{-3t}\cos 2t$

2. (a) Determine the Laplace transform of $f(t) = 2\sin 7t - 5\cos 7t$.

   (b) Your answer to (a) should consist of two separate expressions with common denominators. Combine them into one.

3. (a) Determine the Laplace transform of $g(t) = -3e^{-t} + 4e^{-5t}$.

   (b) Your answer to (a) should consist of two separate expressions. Get a common denominator and combine them into one.

4. Find the inverse Laplace transform of $F(s) = \dfrac{2}{5s+1}$ by first pulling the 2 off the top, factoring 5 out of the bottom and pulling it out also. Then find the appropriate form in the table.

5. Find the inverse Laplace transform of $G(s) = \dfrac{5}{s^2+9}$. Again, begin by pulling the 5 off the top. You will then need a value on top to match the form of the transform of the sine function. Put that number on top, while at the same time multiplying by its reciprocal "out front." Then apply the table.

6. Find the inverse Laplace transform of $F(s) = \dfrac{2s-1}{s^2+9}$ by first applying the (underrated and often very useful) fact that $\dfrac{a \pm b}{c} = \dfrac{a}{c} \pm \dfrac{b}{c}$. Then apply the techniques of the previous two exercises.

7. Determine the inverse Laplace transform of each of the following.

(a) $F(s) = \dfrac{3}{s+5}$ (b) $G(s) = \dfrac{5}{s^2+16}$ (c) $F(s) = \dfrac{7}{s^4}$

(d) $G(s) = \dfrac{3s}{s^2+4}$ (e) $F(s) = \dfrac{4}{s^2+5}$ (f) $G(s) = \dfrac{2}{3s+5}$

(g) $F(s) = \dfrac{5s-3}{s^2+2}$ (h) $G(s) = \dfrac{6}{4s+1}$ (i) $F(s) = \dfrac{2s+1}{s^2+25}$

8. We know that if we launch a projectile upward from an initial height of five feet, at an initial upward velocity of 80 feet per second, the height y in feet is given by $y = -16t^2 + 80t + 5$. This can be obtained by solving the initial value problem

$$y'' = -32, \qquad y(0) = 5,\ y'(0) = 80.$$

(See Section 0.2.) In this exercise you will solve the IVP using the Laplace transform.

(a) Apply the Laplace transform to both sides of the equation, keeping in mind that $-32 = -32(1)$.

(b) Solve for $Y(s)$, which will be the sum of three fractions. Rather than dividing by s^2 at some point, you might find it easier to multiply by $\dfrac{1}{s^2}$.

(c) Find the inverse Laplace transform of $Y(s)$. The result is the solution $y(t)$ - make sure it is what we expected!

9. The IVP for an undamped, unforced second order system is

$$y'' + 9y = 0, \qquad y(0) = -2,\ y'(0) = 1.$$

Solve the IVP using the Laplace transform. The process is fairly similar to that shown in Example 6.1(i). Is your solution what we should expect for an undamped, unforced system?

6.2 Inverse Laplace Transforms - Special Techniques

Performance Criteria:

6. (c) Use partial fractions or completing the square to find the inverse Laplace transform of a function from a table.

The use of Laplace transforms in solving initial value problems can involve numerous algebraic manipulations. We will not go into great detail in that regard, but in this section we see two common and useful techniques, partial fractions and completing the square. We begin with a fairly straightforward example of each, then dig into completing the square a bit more.

◇ **Example 6.2(a):** Find the inverse Laplace transform $f(t)$ of $F(s) = \dfrac{3s-5}{s^2+5s+6}$.

Factoring the denominator and performing a partial fraction decomposition (see Appendix B.3), we get

$$F(s) = \frac{3s-5}{(s+3)(s+2)} = 14\,\frac{1}{s+3} - 11\,\frac{1}{s+2}\,.$$

From the table we then have

$$f(t) = 14e^{-3t} - 11e^{-2t}\,.$$

◇ **Example 6.2(b):** Find the inverse Laplace transform $g(t)$ of $G(s) = \dfrac{5}{s^2+6s+13}$.

First we should try factoring the denominator of the expression, as in the previous example, but that is not possible in this case. Instead we will complete the square in the denominator as follows. First we take half of the coefficient of s (half of 6 is 3) and square the result, getting 9. We then add and subtract 9 in the denominator, creating the trinomial s^2+6s+9, which factors to $(s+3)(s+3) = (s+3)^2$. We then proceed to obtain the form for the Laplace transform of $e^{at}\sin\omega t$:

$$\begin{aligned} G(s) &= \frac{5}{s^2+6s+13} \\ &= \frac{5}{s^2+6s+9-9+13} \\ &= \frac{5}{(s^2+6s+9)+(-9+13)} \\ &= \frac{5}{(s+3)^2+4} \\ &= \frac{5}{2}\frac{2}{(s+3)^2+2^2}\,. \end{aligned}$$

We can then use partial fractions (see Appendix B.3) to split the expression, obtaining From the table we then have

$$f(t) = \tfrac{5}{2}e^{-3t}\sin 2t\,.$$

We next look at an example in which the numerator is the variable s, and which is in the form obtained after completing the square in the denominator. After that we will see how to obtain the inverse transform of the sort of result typically obtained when solving an initial value problem for an underdamped, unforced system.

⋄ **Example 6.2(c):** Find the inverse Laplace transform $f(t)$ of $F(s) = \dfrac{s}{(s+2)^2+9}$.

We see that we wish to use the formula for the Laplace transform of $e^{at}\cos\omega t$, but we need a numerator of $s+2$ rather than just s. To get that we add and subtract two in the numerator, then split into two fractions.

$$F(s) = \frac{s}{(s+2)^2+9} = \frac{s+2-2}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+9} - \frac{2}{(s+2)^2+9}\,. \tag{1}$$

The first fraction now matches the form for the Laplace transform of $e^{at}\cos\omega t$. By the method of the previous example

$$\frac{2}{(s+2)^2+9} = \frac{2}{3}\frac{3}{(s+2)^2+9}\,.$$

From this and (1) it now follows that

$$f(t) = e^{-2t}\cos 3t - \tfrac{2}{3}e^{-2t}\sin 3t\,.$$

⋄ **Example 6.3(d):** Find the inverse Laplace transform $g(t)$ of $G(s) = \dfrac{5s-1}{(s+2)^2+9}$.

This example will use the method of Example 6.3(c), but you will see that the process is a bit more complicated. There are two obvious ways to go about it, and I will show both. (They both amount to pretty much the same thing.)

$$G(s) = \frac{5s-1}{(s+2)^2+9} = \frac{5(s+2-2)-1}{(s+2)^2+9} = \frac{5(s+2)-11}{(s+2)^2+9} = 5\,\frac{s+2}{(s+2)^2+9} - 11\,\frac{1}{(s+2)^2+9}\,.$$

Applying the techniques of Examples 6.2(b) and (c) now gives us

$$g(t) = 5e^{-2t}\cos 3t - \tfrac{11}{3}e^{-2t}\sin 3t\,.$$

A similar, but slightly different, method is to "split" the fraction right away and then proceed from there:

$$G(s) = \frac{5s-1}{(s+2)^2+9} = 5\,\frac{s}{(s+2)^2+9} - \frac{1}{(s+2)^2+9}\,. \tag{2}$$

At this point it would be tempting to tackle each of the two fractions individually, but let's just do the following instead:

$$\begin{aligned}
5\,\frac{s}{(s+2)^2+9} &= 5\,\frac{s+2-2}{(s+2)^2+9} \\
&= 5\left[\frac{s+2}{(s+2)^2+9} - \frac{2}{(s+2)^2+9}\right] \\
&= 5\,\frac{s+2}{(s+2)^2+9} - 10\,\frac{1}{(s+2)^2+9}
\end{aligned}$$

Section 6.2 Exercises

1. Use partial fractions to find the inverse transforms of each of the following functions. Label each inverse transform as the appropriate function of the variable t.

   (a) $F(s) = \dfrac{2s+3}{s^2+6s+5}$

   (b) $G(s) = \dfrac{3s-1}{s^2+5s+4}$

   (c) $F(s) = \dfrac{4s+7}{s^2+7s+10}$

   (d) $G(s) = \dfrac{2s-5}{s^2+6s+8}$

2. Use completing the square to find the inverse Laplace transform of each of the following.

   (a) $F(s) = \dfrac{3}{s^2+8s+20}$

   (b) $G(s) = \dfrac{s}{s^2+8s+20}$

   (c) $G(s) = \dfrac{5s}{s^2+2s+10}$

   (d) $F(s) = \dfrac{3s-7}{s^2+2s+10}$

   (e) $F(s) = \dfrac{4s+5}{s^2+6s+14}$

   (f) $G(s) = \dfrac{2s+1}{s^2+4s+6}$

6.3 Solving Initial Value Problems

Performance Criteria:

6. (d) Use the Laplace transform to solve an IVP with a second order, linear, constant coefficient ODE.

We have already seen (Example 6.1(j)) how to use the Laplace transform to solve a first order initial value problem. Let's now solve some second order initial value problems.

◇ **Example 6.3(a):** Use the Laplace transform to solve the initial value problem

$$y'' + 4y' + 3y = 0, \qquad y(0) = 2, \qquad y'(0) = -1$$

We apply the Laplace transform to both sides of the ODE, remembering that the Laplace transform of zero is zero, to get

$$[s^2Y(s) - 2s - (-1)] + 4[sY(s) - 2] + 3Y(s) = 0\,.$$

Solving for $Y(s)$ we obtain

$$Y(s) = \frac{2s+7}{s^2+4s+3}\,,$$

which has the partial fraction decomposition

$$Y(s) = \frac{5}{2}\frac{1}{x+1} - \frac{1}{2}\frac{1}{x+3}\,.$$

Taking the inverse transform, we obtain the solution

$$y(t) = \tfrac{5}{2}e^{-t} - \tfrac{1}{2}e^{-3t}\,.$$

◇ **Example 6.3(b):** Use the Laplace transform to solve the initial value problem

$$y'' + 10y' + 29y = 0, \qquad y(0) = 3, \qquad y'(0) = 1$$

Applying the Laplace transform to both sides of the ODE gives

$$[s^2Y(s) - 3s - 1] + 10[sY(s) - 3] + 29Y(s) = 0\,.$$

Solving for $Y(s)$ we obtain

$$Y(s) = \frac{3s+31}{s^2+10s+29}\,.$$

The bottom of this doesn't factor, so we complete the square to get

$$Y(s) = \frac{3s+31}{(s+5)^2+4}\,.$$

The methods of Example 6.3(d) can then be used to get

$$Y(s) = \frac{3(s+5-5)+31}{(s+5)^2+4} = \frac{3(s+5)-15+31}{(s+5)^2+4} = 3\,\frac{s+5}{(s+5)^2+4} + 8\,\frac{2}{(s+5)^2+4}\,.$$

Taking the inverse transform, we obtain the solution

$$y(t) = 3e^{-5t}\cos 2t + 8e^{-5t}\sin 2t\,.$$

Section 6.3 Exercises

1. Use the Laplace transform to solve each initial value problem.

   (a) $y' + 4y = 0, \quad y(0) = 3$

   (b) $y' + 7y = 0, \quad y(0) = -1$

   (c) $y'' + 3y' + 2y = 0, \quad y(0) = 2,\ y'(0) = 5$

   (d) $y'' + 2y = 0, \quad y(0) = 4,\ y'(0) = -1$

   (e) $y'' + 2y' + 10y = 0, \quad y(0) = -3,\ y'(0) = 1$

   (f) $y'' + 16y = 0, \quad y(0) = 1,\ y'(0) = 2$

   (g) $y'' + 6y' + 11y = 0, \quad y(0) = 1,\ y'(0) = -1$

   (h) $y'' + 6y' + 5y = 0, \quad y(0) = -1,\ y'(0) = 2$

6.4 Derivations of Laplace Transforms

Performance Criteria:

6. (e) Find the Laplace transform of a function directly from the definition.

Section 6.4 Exercises

1. Find the Laplace transform of $f(t) = \cos\omega t$ from the definition and by applying the formula $\displaystyle\int e^{at}\cos bt\,dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$. **Show your work, integrating from zero to** R **and then applying a limit to the result.**

2. Find the Laplace transform of $f(t) = \sin\omega t$ by applying the identity $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and using the linearity of the transform and the already known transform of e^{at}. Remember that i is just a constant, and can be treated as such. You will end up with two fractions; get a common denominator and combine them.

6.5 The Unit Step Function and Piecewise Defined Functions

Performance Criteria:

6. (f) Graph piecewise defined functions, with their definitions given either with or without using the unit step function. Use the unit step function to write a piecewise defined function as one equation.

   (g) Find the Laplace transform of a piecewise defined function.

   (h) Find inverse Laplace transforms that result in piecewise defined functions of t.

   (i) Use the Laplace transform to solve an IVP with a second order, linear, constant coefficient ODE and piecewise defined forcing function.

Section 6.5 Exercises

4. Consider the function $f(t) = \begin{cases} t^2 & \text{if } \ 0 \le t < 1 \\ 2-t & \text{if } \ 1 \le t < 2 \\ 0 & \text{if } \ t \ge 2 \end{cases}$

   (a) Sketch a *neat* graph of the function.

   (b) Write the function as a single function using the unit step function $u(t)$.

5. *Neatly* sketch the graph of each of the following:

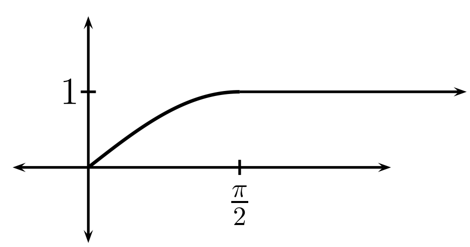
   (a) $f(t) = t - tu(t-2)$

   (b) $g(t) = t + (2-t)u(t-2)$

   (c) $h(t) = t[1-u(t-2)] + (4-t)[u(t-2)-u(t-4)]$

6. Give the Laplace transforms of each of the functions in the previous exercise. Simplify your answers when possible.

7. (a) Use the unit step function $u(t)$ to write the function $f(t)$ whose graph is shown to the right as a single function. The initial part of the graph is the sine function, then the constant function one.



   (b) Give the Laplace transform of the function.

1. Consider the function $f(t) = \begin{cases} 0 & \text{if } \ t < 0 \\ 2 & \text{if } \ 0 \le t < 1 \\ 1 & \text{if } \ 1 \le t < 3 \\ 5 & \text{if } \ t \ge 3 \end{cases}$

   (a) Sketch a graph of the function.

(b) Give a single equation of the function, using step functions.

(c) Find the Laplace transform of the function.

2. Let $g(t)$ be the function that is zero until time one and then ramps up (linearly) to a value of six at time $t = 5$. Repeat the parts of Exercise 1 for this function.

3. The point of this exercise is to change the expression $4t - t^2$ into a function of $(t - 4)$.

   (a) Replace t everywhere with $[(t - 4) + 4]$. Perform the operations of multiplying by four and squaring *without breaking up or "foiling out" the* $(t-4)$. See the notes or the video on Laplace transforms with step functions for how the squaring part goes.

   (b) Combine any like terms, still not breaking up or "foiling out" the $(t - 4)$. Your final result should be something of the form $a(t - 4) - (t - 4)^2$ for some constant a.

   (c) Your result from (b) is $f(t - 4)$ for some function f. Give the function.

4. (a) Sketch the graph of the function $h(t) = (4t - t^2)[u(t) - u(t - 4)]$.

   (b) Find the Laplace transform of the function. You will need to distribute the $4t - t^2$ to both step functions. The first part will be ready to go, and the result of the previous exercise should be useful for the second part.

5. Find the inverse Laplace transform of each of the following using $\mathscr{L}^{-1}[e^{-cs}F(s)] = f(t - c)\,u(t - c)$.

   (a) $G(s) = \dfrac{3e^{-2s}}{s^2}$

   (b) $G(s) = \dfrac{7e^{-5s}}{s - 2}$

   (c) $G(s) = \dfrac{2se^{-3s}}{s^2 + 25}$

   (d) $G(s) = e^{-s}\left(\dfrac{1}{s^3} - \dfrac{1}{s^2} + \dfrac{1}{s}\right)$

6. In this exercise you will be solving the IVP $2y'' + 2y' + 5y = g(t), \quad y(0) = 0, \quad y'(0) = 0,$ where $g(t)$ is two from $t = 3$ to $t = 18$ and zero at all other times.

   (a) Graph $g(t)$.

   (b) Give $g(t)$ as a single function, and substitute it into the ODE.

   (c) Take the Laplace transform of both sides of the ODE and solve for $Y(s)$.

   (d) Use Wolfram Alpha to determine $y(t)$. The result you get will be really crazy, but there will be two graphs of the solution displayed. Sketch a reasonably neat and large version of the second graph, but starting only at zero.

7. In this exercise you will solve the IVP $y'' + 4y = h(t), \quad y(0) = 0, \quad y'(0) = 0,$ where $h(t)$ is the function from Exercise 2 of Assignment 7, that has value zero until time one, then "ramps up" (linearly) to value six at five seconds, then stays six from then on. Recall that that function can be expressed using step functions as

$$h(t) = \tfrac{3}{2}(t - 1)u(t - 1) - \tfrac{3}{2}(t - 5)u(t - 5).$$

   (a) Take the Laplace transform of both sides of the ODE and solve for $Y(s)$.

   (b) Use Wolfram Alpha to determine $y(t)$. Sketch a reasonably neat and large version of the second graph of the solution, but starting only at zero.

6.6 The Dirac Delta Function

Performance Criteria:

6. (j) Use the Laplace transform to solve an IVP with a second order, linear, constant coefficient ODE and impulse defined forcing function.

Section 6.6 Exercises

1.

6.7 The Laplace Transform and Convolution

Performance Criteria:

6. (k) Find Laplace transforms of functions defined by convolution.
   (l) Determine a convolution whose Laplace transform is a given function of s.
   (m) Express solutions to IVPs for second order, linear, constant coefficient ODEs in terms of convolutions.

Section 6.7 Exercises

1. Use the convolution rule to give the Laplace transform of each of the following functions. Label each transform with its name, of course!

   (a) $f(t) = \displaystyle\int_0^t (t-\tau)^3 e^{-5\tau}\, d\tau$ (b) $g(t) = \displaystyle\int_0^t e^{2(t-\tau)}\cos(3t)\, d\tau$ (c) $h(t) = \displaystyle\int_0^t \sin 2\tau\, \cos 3(t-\tau)\, d\tau$

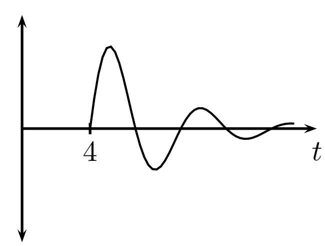
2. Give the inverse Laplace transform of each of the following as a convolution.

   (a) $F(s) = \dfrac{5s}{(s+2)(s^2+9)}$ (b) $G(s) = \dfrac{6}{(s-5)(s^2+9)}$ (c) $H(s) = \dfrac{7}{s^4(s+3)}$

3. Give the inverse Laplace transform of $F(s) = \dfrac{s\,G(s)}{s^2+5}$ as a convolution involving the function $g(t)$.

4. Find the solution to the initial value problem $y'' + 9y = g(t),\ y(0) = -2,\ y'(0) = 1$. Your solution will contain a convolution involving the unknown function $g(t)$.

5. Here is an interesting **optional** problem: The graph of the solution to the IVP of Exercise 2 of Assignment 23 is shown to the right; recall that the forcing function for that exercise was a unit impulse at time four. Determine another impulse that can be added to the existing forcing function so that the solution goes to zero after exactly one complete cycle. Check your answer by solving the new IVP and looking at the graph given by Wolfram Alpha when you take the inverse transform. **A special prize will be awarded to anyone who can solve this by Friday and give an explanation (to the class) of how their solution was obtained.**



D Solutions to Exercises

D.6 Chapter 6 Solutions

Section 6.1 Solutions

1. (a) $F(s) = \frac{6}{s^3} - \frac{7}{s^2} + \frac{2}{s}$ (b) $G(s) = \frac{3}{s+2}$ (c) $F(s) = \frac{10}{s^2+25}$

   (d) $G(s) = \frac{120}{s^6} - \frac{18}{s^3}$ (e) $F(s) = \frac{14}{(s+5)^3}$ (f) $G(s) = \frac{5(s+3)}{(s+3)^2+4}$

2. (a) $F(s) = \frac{14}{s^2+49} - \frac{5s}{s^2+49}$ (b) $F(s) = \frac{-5s+14}{s^2+49}$

3. (a) $G(s) = \frac{-3}{s+1} + \frac{4}{s+5}$ (b) $G(s) = \frac{s-11}{(s+1)(s+5)} = \frac{s-11}{s^2+6s+5}$

4. $F(s) = \frac{2}{5} \cdot \frac{1}{s+\frac{1}{5}}$, $f(t) = \frac{2}{5}e^{-\frac{1}{5}t}$

5. $G(s) = \frac{5}{3} \cdot \frac{3}{s^2+3^2}$, $g(t) = \frac{5}{3}\sin 3t$

6. $F(s) = \frac{2s}{s^2+9} - \frac{1}{s^2+9} = 2 \cdot \frac{s}{s^2+9} - \frac{1}{3} \cdot \frac{3}{s^2+9}$, $f(t) = 2\cos 3t - \frac{1}{3}\sin 3t$

7. (a) $f(t) = 3e^{-5t}$ (b) $g(t) = \frac{5}{4}\sin 4t$ (c) $f(t) = \frac{7}{6}t^2$

   (d) $g(t) = 3\cos 2t$ (e) $f(t) = \frac{4}{\sqrt{5}}\sin\sqrt{5}t$ (f) $g(t) = \frac{2}{3}e^{-\frac{5}{3}t}$

   (g) $f(t) = 5\cos\sqrt{2}t - \frac{3}{\sqrt{2}}\sin\sqrt{2}t$ (h) $g(t) = \frac{3}{2}e^{-\frac{1}{4}t}$ (i) $f(t) = 2\cos 5t + \frac{1}{5}\sin 5t$

Section 6.2 Solutions

1. (a) $f(t) = \frac{7}{4}e^{-5t} + \frac{1}{4}e^{-t}$ (b) $g(t) = \frac{13}{3}e^{-4t} - \frac{4}{3}e^{-t}$

   (c) $f(t) = \frac{13}{3}e^{-5t} - \frac{1}{3}e^{-2t}$ (d) $g(t) = \frac{13}{2}e^{-4t} - \frac{9}{2}e^{-2t}$

2. (a) $f(t) = \frac{3}{2}e^{-4t}\sin 2t$ (b) $g(t) = e^{-4t}\cos 2t - 2e^{-4t}\sin 2t$

   (c) $g(t) = 5e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t$ (d) $f(t) = 3e^{-t}\cos 3t - \frac{10}{3}e^{-t}\sin 3t$

   (e) $f(t) = 4e^{-3t}\cos\sqrt{5}t - 2e^{-3t}\sin\sqrt{5}t$ (f) $g(t) = 2e^{-2t}\cos\sqrt{3}t - \frac{3}{\sqrt{3}}e^{-2t}\sin\sqrt{3}t$

Section 6.3 Solutions

1. (a) $y = 3e^{-4t}$ (b) $y = -e^{-7t}$

   (c) $y = 9e^{-t} - 7e^{-2t}$ (d) $y = 4\cos\sqrt{2}t - \frac{1}{\sqrt{2}}\sin\sqrt{2}t$

   (e) $y = -\frac{2}{3}e^{-t}\sin 3t - 3e^{-t}\cos 3t$ (f) $y = \frac{1}{2}\sin 4t + \cos 4t$

   (g) $y = \sqrt{2}e^{-3t}\sin\sqrt{2}t + e^{-3t}\cos\sqrt{2}t$ (h) $y = -\frac{3}{4}e^{-t} - \frac{1}{4}e^{-5t}$