Ordinary Differential Equations

for Engineers and Scientists

Gregg Waterman
Oregon Institute of Technology

Contents

B Review of Calculus and Algebra

B.1 Review of Differentiation

Performance Criteria:

B. 1. Apply the rules of differentiation, along with provided formulas, to find the derivatives of functions.

The purpose of this appendix is to remind you of how to find the derivatives of some functions of the sorts that we will encounter as we go on. For this we will usually take the independent variable to be x and the dependent variable y; the statement $y = f(x)$ indicates that y is a function of x; recall also the notations $y' = f'(x) = \dfrac{dy}{dx}$ for the derivative. Let's begin by giving the derivatives of some common functions, and some basic rules of taking derivatives. The notation $(\quad)'$ means the derivative of the quantity in the parentheses, and k represents an arbitrary constant.

Derivatives of Some Functions

$$(k)' = 0 \qquad (x)' = 1 \qquad (x^n)' = nx^{n-1} \qquad (e^x)' = e^x$$

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x \qquad (\ln x)' = \frac{1}{x}$$

For the following, k again represents an arbitrary constant, and $f(x)$, $g(x)$, $u = u(x)$ and $v = v(x)$ are functions of the independent variable x.

Derivative Rules

$$[kf(x)]' = kf'(x) \qquad [u \pm v]' = u' \pm v'$$

$$(uv)' = uv' + vu' \qquad \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

$$f[g(x)]' = f'[g(x)]g'(x)$$

You'll recall the third and fourth items above as the **product rule** and **quotient rule**, and the last item is the **chain rule**. The first two rules above tell us that the derivative is something called a **linear operator**. This is the same idea as a linear transformation, for those of you who have had linear algebra. Now we'll see how to use some of the rules and derivatives of common functions to find derivatives of some combinations of those functions.

Derivatives of Polynomials

You'll recall that to find the derivative of a polynomial like $f(x) = 5x^3 - \frac{3}{4}x^2 - 7x + 2$ we simply use the various basic derivatives and rules as follows: Multiply the coefficient of each term by the corresponding power of x, and decrease the power by one, remember that the derivative of a constant is zero.

◇ **Example B.1(a):** Find the derivative of $f(x) = 5x^3 - \frac{3}{4}x^2 - 7x + 2$.

$$f(x) = 5x^3 - \frac{3}{4}x^2 - 7x + 2 \qquad \Longrightarrow \qquad f'(x) = 15x^2 - \frac{3}{2}x - 7$$

Second Derivatives

For certain applications we will need to take the derivative of the derivative of a function. You'll recall that such a derivative is called the **second derivative** (so the kind of derivative you did above is called a **first derivative**). The notation is this: If the first derivative is y' or $f'(x)$, then the second derivative is y'' or $f''(x)$. If the first derivative is $\frac{dy}{dx}$, then the second derivative is $\frac{d^2y}{dx^2}$. Note the different placement of the "exponents" in the numerator and denominator of this expression - there is a reason for this, but it's not too exciting, so I won't go into that here.

Other Applications of the Power Rule

Since we can write things like $\frac{1}{x^2}$ and $\sqrt{x}$ using negative and fractional exponents, we can use the power rule to find their derivatives. Here are some examples:

◇ **Example B.1(b):** Find the derivative of $y = \frac{5}{x^3}$

$$y = \frac{5}{x^3} = 5x^{-3} \quad \Rightarrow \quad \frac{dy}{dx} = -15x^{-4} = \frac{15}{x^4}$$

◇ **Example B.1(c):** Find the derivative of $f(x) = \frac{\sqrt[3]{x}}{4}$

$$f(x) = \frac{\sqrt[3]{x}}{4} = \frac{1}{4}x^{\frac{1}{3}} \quad \Rightarrow \quad f'(x) = \frac{1}{12}x^{-\frac{2}{3}} = \frac{1}{12\sqrt[3]{x^2}}$$

Note that the four in the denominator is a constant, but it is really $\frac{1}{4}$, not 4.

The Chain Rule

This is perhaps the most confusing (but also most important!) of the derivative rules. Let's use an example to try to explain it; suppose that $y = (x^5 + 2x - 4)^8$. Now if you knew $x = 3$ and wanted to compute y it would be sort of a two step process. First you would compute $3^5 + 2(3) - 4$, then you would take the result to the eighth power. To do the derivative, you do the derivative of the last part first

$$[(\ stuff\ )^8]' = 8(\ stuff\ )^7\,,$$

and multiply by the derivative $(x^5 + 2x - 4)' = 5x^4 + 2$:

◇ **Example B.1(d):** Find the derivative of $y = (x^5 + 2x - 4)^8$.

$$y' = 8(x^5 + 2x - 4)^7 \cdot (x^5 + 2x - 4)' = 8(x^5 + 2x - 4)^7(5x^4 + 2)$$

In the notation of the rule in the box on the previous page, $g(x) = x^5 + 2x - 4$ and $f(u) = u^8$.

◇ **Example B.1(e):** Find the derivative of $f(t) = 5.6\sin(4.2t - 1.3)$.

$$f'(t) = 5.6[\cos(4.2t - 1.3)] \cdot (4.2t - 1.3)' = 5.6[\cos(4.2t - 1.3)] \cdot (4.2) = 23.6\cos(4.2t - 1.3)$$

In this last example there is the additional constant 5.6 that is just multiplied by the result of the derivative of the rest of the function. Note that the result of the derivative of $4.2t - 1.3$ is brought to the front and multiplied by the original constant at the end of the process. More on this in a bit.

In light of what you have seen, we can revise some of our basic derivatives some. Again, u is a function of x: $u = u(x)$.

Derivatives of Some Functions

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx} \qquad \frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx} \qquad \frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}$$

Proper Manners(!) for Writing Functions

You will often have functions that are products of constants, powers of x and exponential, trig or log functions. In the case of such a function, the correct order to write the factors is

constant, power of x, exponential function, trigonometric function

Occasionally you will have a logarithmic function, but these will not generally occur with trigonometric functions; a logarithmic function goes where the trigonometric function is listed above.

The Product Rule

Since $(u + v)' = u' + v'$, one might hope that $(uv)' = u'v'$. A simple example shows that this is not true:

◇ **Example B.1(f):** Let $u(x) = x^2$ and $v(x) = x^3$, and find $[u(x)v(x)]'$ and $u'(x)v'(x)$.

We see that

$$[u(x)v(x)]' = [x^2x^3]' = (x^5)' = 5x^4 \qquad \text{and} \qquad u'(x)v'(x) = (2x)(3x^2) = 6x^3.$$

Therefore $[u(x)v(x)]'$ is NOT equal to $u'(x)v'(x)$!

So how do we find the derivative of a product of two functions? Well, we use the product rule, which says that if $u = u(x)$ and $v = v(x)$ are two functions of x, then

$$(uv)' = uv' + vu' \qquad \text{or} \qquad \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

◇ **Example B.1(g):** Find the derivative of $y = 3x^2 \sin 5x$.

$$\begin{aligned} y' &= (3x^2)(\sin 5x)' + (\sin 5x)(3x^2)' \\ &= (3x^2)(5\cos 5x) + (\sin x)(6x) \\ &= 15x^2 \cos 5x + 6x \sin x \end{aligned}$$

The Quotient Rule

The quotient rule is similar to the product rule. For two functions u and v of x,

$$\left(\frac{u}{v}\right)' = \frac{v\,u' - u\,v'}{v^2}$$

◇ **Example B.1(h):** Find the derivative of $f(x) = \dfrac{e^{2x}}{x^5}$

$$\begin{aligned} f'(x) &= \frac{(x^5)(e^{2x})' - (e^{2x})(x^5)'}{(x^5)^2} \\ &= \frac{(x^5)(2e^{2x}) - (e^{2x})(5x^4)}{x^{10}} \\ &= \frac{2x^5e^{2x} - 5x^4e^{2x}}{x^{10}} \end{aligned}$$

Section B.1 Exercises

To Solutions

1. Find the derivative of each polynomial:

   (a) $y = 3x^5 - 7x^4 + x^2 - 3x + 2$

   (b) $h(t) = -16t^2 + 23.7t + 3.5$

   (c) $f(x) = \frac{2}{3}x^4 + 5x^3 - \frac{1}{8}x^2 + 3$

   (d) $s(t) = t^3 - 2t^2 + 3t - 5$

2. Find the second derivative of each of the functions in Exercise 1.

3. Find the derivative of each of the following. Give your final answers without negative or fractional exponents.

   (a) $f(x) = \dfrac{3}{x^2}$

   (b) $g(t) = \dfrac{t^2}{6} - \dfrac{6}{t^2}$

   (c) $y = \frac{1}{5}\sqrt{x}$

   (d) $h(x) = \dfrac{4}{\sqrt[3]{x}}$

4. Find the second derivatives of the functions from 3(a) and (b).

5. Find the derivative of each of the following, utilizing the suggestions provided.

   (a) $g(x) = \sqrt{16 - x^2}$ - If you write the square root as an exponent you will have something very much like the first example above.

   (b) $y = 3\cos\left(\frac{\pi}{2}t\right)$

   (c) $A(t) = 500e^{-0.3t}$ - This is a two-step process (again, ignoring the constant of 500), with the first step being $-0.3t$ and the second step being the exponential. Remember that the derivative of e^x is e^x.

6. Find the derivative of each of the following.

   (a) $y = 5e^{2x}$ (b) $x = 4\sin(3t)$ (c) $g(x) = \dfrac{2}{(x^2 - 4x)^7}$

   (d) $s(t) = \cos(\frac{2}{5}t)$ (e) $y = e^{x^2}$

7. For each of the following, put the product in the correct order.

   (a) $e^{5t^3} \cdot 15t^3$ (b) $3[-\sin(5x - 2)] \cdot 5$ (c) $4\sin(5t + 3) \cdot (e^{-2t})$

8. For each of the following, multiply and put the final product in the correct order.

   (a) $4\sin(5t + 3) \cdot (-2e^{-2t})$ (b) $7e^{-t}\cos(3t - 1) \cdot 2t$ (c) $(3\ln x)(4x^3)$

9. Use the product rule to find the derivative of each of the following.

   (a) $f(t) = t^2 e^{7t}$ (b) $y = 3x\cos 2x$ (c) $h(t) = 2e^{-3t}\sin \pi t$

10. Find the derivative of each of the following.

    (a) $f(x) = \dfrac{3\sin 2t}{e^{7t}}$ (b) $y = \dfrac{4e^{-5t}}{t^2}$ (c) $g(x) = \dfrac{\cos 6x}{2x^3}$

11. Use the quotient rule and the fact that $\tan x = \dfrac{\sin x}{\cos x}$ to determine the derivative of $\tan x$.

B.2 Review of Integration

Performance Criteria:

B. 2. Apply rules of integration, along with formulas, to find indefinite integrals of functions.

In this course we need on occasion to find anti-derivatives, or indefinite integrals, of functions. As you learned in integral calculus, doing so is often a challenging endeavor! Fortunately we only need to be able to find indefinite integrals of a handful of types of functions in our study of differential equations. We can use some simple formulas, found on the formula sheet (Appendix A), that are derived from methods you learned before, like substutution and integration by parts. We begin with linearity of the indefinite integral and the most basic antiderivative:

Linearity of the Indefinite Integral

Let c be any constant, and let f and g be functions.

$$\int c\,f(x)\,dx = c\int f(x)\,dx, \qquad \int [f(x)\pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Indefinite Integral of x^n

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}+C \text{ if } n\neq -1, \qquad \int \frac{1}{x}\,dx = \ln|x|+C,$$

where C is an arbitrary constant.

We note at this point that any indefinite integral includes the addition of an arbitrary constant, as shown above, but *we will not allways include the constant in future formulas, even though it belongs in all of them.*

Here are a couple of special examples of two of the above things:

◇ **Example B.2(a):** Find the integrals $\int dx$ and $\int k\,dx$, where $k\neq 0$ is a constant.

Solution: $\displaystyle\int dx = \int x^0\,dx = \frac{1}{0+1}x^{0+1}+C = x+C, \quad \int k\,dx = k\int dx = k(x+C) = kx+C$

Note that when we write $k(x+C)=kx+C$, the two constants C are actually different, but it is common to abuse notation this way.

The results of Example B.2(b) and the integral of x^n, $n\neq -1$ are commonly used when finding the integral of a polynomial. You probably don't think of it quite like this, but here is what happens:

◇ **Example B.2(b):** Find $\displaystyle\int (5x^2 - 6x + 4)\,dx$.

Solution:

$$\begin{aligned}\int (5x^2 - 6x + 4)\,dx &= \int 5x^2\,dx - \int 6x\,dx + \int 4\,dx \\ &= 5\int x^2\,dx - 6\int x\,dx + \int 4\,dx \\ &= 5\left(\tfrac{1}{3}x^3\right) - 6\left(\tfrac{1}{2}x^2\right) + 4x + C \\ &= \tfrac{5}{3}x^3 - 3x^2 + 4x + C\end{aligned}$$

It is not necessary that you show all of the above steps when doing such an integral - anything is pretty much alright as long as you arrive at the correct result!

We will occasionally see integral slike the following, which are quite easy *if we use the results from the previous page and negative exponents*.

◇ **Example B.2(c):** Find $\displaystyle\int \frac{7}{x^4}\,dx$.

Solution: Using the fact that $\dfrac{1}{x^4} = x^{-4}$ and noting that when we add one to -4 we get -3, we have the following:

$$\int \frac{7}{x^4}\,dx = 7\int \frac{1}{x^4}\,dx = 7\int x^{-4}\,dx = 7 \cdot \frac{1}{-3}\,x^{-3} + C = -\frac{7}{3x^3} + C$$

Exponential functions are extremely important in applications, and a large part of their importance is the result of the fact that they are essentially their own derivatives and integrals:

Indefinite Integral of e^{ax}

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax} + C$$

◇ **Example B.2(d):** Find $\displaystyle\int e^{-5x}\,dx$.

Solution: $\displaystyle\int e^{-5x}\,dx = \tfrac{1}{-5}\,e^{-5x} + C = -\tfrac{1}{5}\,e^{-5x} + C.$

Note that, unlike the integral of x^{-5}, *the exponent does not change when we integrate an exponential function*. This holds true for the integrals of trigonometric functions as well. When working with applications of differential equations, we usually only need the following.

Indefinite Integrals of Sine and Cosine

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

◇ **Example B.2(e):** Find $\displaystyle\int 3\sin\tfrac{\pi}{2}x\,dx$.

Solution: $\displaystyle\int 3\sin\tfrac{\pi}{2}x\,dx = 3\int \sin\tfrac{\pi}{2}x\,dx = 3\left(-\frac{1}{\frac{\pi}{2}}\cos\tfrac{\pi}{2}x + C\right) = -\tfrac{6}{\pi}\cos\tfrac{\pi}{2}x + C.$

Note that the second C above is really three times the first C!

There are a number of somewhat specialized formulas on the formula sheet that we will use fairly often. We'll give an example of the use of one of them here, and the others will be addressed in the exercises. Here is the formula we'll use in the next example:

$$\int e^{at}\cos bt\ dt = \frac{e^{at}}{a^2+b^2}\left(a\cos bt + b\sin bt\right) + C \tag{1}$$

Note that the variable is t, rather than x. This will be common for us.

◇ **Example B.2(f):** Find and simplify $\displaystyle\int 3e^{-5t}\cos 2t\,dt$.

Solution: First we note that, in the context of formula (1), $a = -5$ and $b = 2$. Pulling the constant 3 out of the integral and applying the formula we get

$$\begin{aligned}\int 3e^{-5t}\cos 2t\,dt &= 3\left[\frac{e^{-5t}}{(-5)^2+2^2}\big(-5\cos 2t + 2\sin 2t\big)\right] + C\\ &= \frac{3e^{-5t}}{29}\big(-5\cos 2t + 2\sin 2t\big) + C\\ &= e^{-5t}\big(-\tfrac{15}{29}\cos 2t + \tfrac{6}{29}\sin 2t\big) + C\end{aligned}$$

Section B.2 Exercises To Solutions

1. We will commonly encounter integrals for which this formula is useful:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|\,ax+b\,| + C$$

Use the formula to compute the following integrals.

(a) $\displaystyle\int \frac{5}{2x+3}\,dx$ (b) $\displaystyle\int \frac{2}{3-5x}\,dx$ (c) $\displaystyle\int \frac{1}{1.6-0.08A}\,dA$

2. Evaluate each of the following indefinite integrals, using the formula sheet as you need. For exercises involving decimals, round to two significant figures. **Be sure to note and use the correct variable, in the case (upper or lower) given**.

(a) $\int (x^2 - 7x + 3)\,dx$

(b) $\int 7 \sin 3t\,dt$

(c) $\int 3te^{-2t}\,dt$

(d) $\int \frac{3}{x^2}\,dx$

(e) $\int \frac{dA}{2.0 - 0.1A}$

(f) $\int 3e^{-t} \sin 5t\,dt$

(g) $\int \frac{3\,dx}{5x - 1}$

(h) $\int 5t^2 e^{-3t}\,dt$

(i) $\int e^{-4t} \cos 3t\,dt$

(j) $\int 5 \cos \frac{\pi}{2} t\,dt$

B.3 Solving Systems of Equations

In this course we will often need to solve systems of two equations in two unknowns, and it is important to be able to do this quickly *and correctly*. There are two methods, the **addition method** and the **substitution method**. Each has its own advantages and disadvantages; both methods will be demonstrated here.

The Addition Method

We'll begin with the addition method, going from the easiest scenario to the most difficult (which still isn't too hard).

◇ **Example B.3(a):** Solve the system $\begin{aligned} 3x - y &= 5 \\ 2x + y &= 15 \end{aligned}$.

The basic idea of the addition method is to add the two equations together so that one of the unknowns goes away. In this case, as shown below and to the left, nothing fancy need be done. The remaining unknown is then solved for and placed back into *either* equation to find the other unknown as shown below and to the right.

$$\begin{aligned} 3x - y &= 5 \\ 2x + y &= 15 \\ \hline 5x &= 20 \\ x &= 4 \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} 3(4) - y &= 5 \\ 12 - y &= 5 \\ 12 &= y + 5 \\ 7 &= y \end{aligned}$$

The solution to the system is $(4, 7)$.

What made this work so smoothly is the $-y$ in the first equation and the $+y$ in the second; when we add the two equations, the sum of these is zero and y "has gone away." In the next two examples we see what to do in slightly more difficult situations.

◇ **Example B.3(b):** Solve the system $\begin{aligned} 3x + 4y &= 13 \\ x + 2y &= 7 \end{aligned}$.

We can see that if we just add the two equations together we get $4x + 6y = 20$, which doesn't help us find either of x or y. The trick here is to multiply the second equation by -3 so that the the first term of that equation becomes $-3x$, the opposite of the first term of the first equation. When we then add the two equations the x terms go away and we can solve for y:

$$\begin{aligned} 3x + 4y &= 13 \\ x + 2y &= 7 \end{aligned} \quad \begin{aligned} &\Longrightarrow \\ &\underset{\text{times } -3}{\Longrightarrow} \end{aligned} \quad \begin{aligned} 3x + 4y &= 13 \\ -3x - 6y &= -21 \\ \hline -2y &= -8 \\ y &= 4 \end{aligned} \quad \longrightarrow \quad \begin{aligned} x + 2(4) &= 7 \\ x + 8 &= 7 \\ x &= -1 \end{aligned}$$

The solution to the system of equations is $(-1, 4)$. Note that we could have eliminated y first instead of x:

$$\begin{aligned} 3x + 4y &= 13 \\ x + 2y &= 7 \end{aligned} \quad \begin{aligned} &\Longrightarrow \\ &\underset{\text{times } -2}{\Longrightarrow} \end{aligned} \quad \begin{aligned} 3x + 4y &= 13 \\ -2x - 4y &= -14 \\ \hline x &= -1 \end{aligned} \quad \longrightarrow \quad \begin{aligned} -1 + 2y &= 7 \\ 2y &= 8 \\ y &= 4 \end{aligned}$$

Three things need to be pointed out at this time:

- There is no need for the equations to have the forms

$$\begin{array}{rcl} ax+by & = & e \\ cx+dy & = & f \end{array}.$$

  All that is necessary is that the unknown to be eliminated is on the same side of both equations.

- The unknowns in this course will usually be denoted by letters other than x and y, like C_1, C_2, A and B.

- Often *both* equations must be multiplied by a value in order to eliminate an unknown.

The following example illustrates both of these points.

◇ **Example B.3(c):** Solve the system $\begin{array}{rcl} 4 & = & 3A-5B+1 \\ 0 & = & -5A-3B-2 \end{array}$.

In this case we will chose to eliminate A because its coefficients already have opposite signs in the two equations. We'll multiply the first equation by 5 and the second by 3 so that the coefficients of A will be the same, but with opposite signs. When we then add the two equations the A terms go away and we can solve for B:

$$\begin{array}{rclcrcl} 4 & = & 3A-5B+1 & \overset{\text{times } 5}{\Longrightarrow} & 20 & = & 15A-25B+5 \\ 0 & = & -5A-3B-2 & \underset{\text{times } 3}{\Longrightarrow} & 0 & = & -15A-\ \ 9B-6 \\ \hline & & & & 20 & = & -34B-1 \\ & & & & 21 & = & -34B \\ & & & & -\frac{21}{34} & = & B \end{array}$$

Ordinarily we would substitute this value into one of the original equations and solve for A at this point. However, when one of the unknowns is a messy fraction it is often easier to repeat the same procedure, but eliminate the other unknown. Let's multiply the first equation by 3 and the second by -5:

$$\begin{array}{rclcrcl} 4 & = & 3A-5B+1 & \overset{\text{times } 3}{\Longrightarrow} & 12 & = & 9A-15B+3 \\ 0 & = & -5A-3B-2 & \underset{\text{times } -5}{\Longrightarrow} & 0 & = & 25A+15B+10 \\ \hline & & & & 12 & = & 34A+13 \\ & & & & -1 & = & 34A \\ & & & & -\frac{1}{34} & = & A \end{array}$$

The solution to the system is $A=-\frac{1}{34}$, $B=-\frac{21}{34}$.

Let's summarize the steps for the addition method, which you've seen in the above examples.

The Addition Method

To solve a system of two linear equations by the addition method,

(1) Multiply each equation by something as needed in order to make the coefficients of one unknown the same but opposite in sign in the two equations.

(2) Add the two equations and solve the resulting equation for whichever unknown remains.

(3) Substitute that value into either original equation and solve for the other unknown *OR* repeat steps (1) and (2) for the other unknown.

The Substitution Method

We will now describe the substitution method, then give an example of how it works.

The Substitution Method

To solve a system of two linear equations by the substitution method,

1) Pick one of the equations in which the coefficient of one of the unknowns is either one or negative one. Solve that equation for that unknown.

2) Substitute the expression for that unknown into *the other* equation and solve for the unknown.

3) Substitute that value into the equation from (1), or into either original equation, and solve for the other unknown.

◇ **Example B.3(d):** Solve the system of equations $\begin{aligned} x - 3y &= 6 \\ -2x + 5y &= -5 \end{aligned}$ using the substitution method.

Solving the first equation for x, we get $x = 3y + 6$. We now replace x in the second equation with $3y + 6$ and solve for y. Finally, that result for y can be substituted into $x = 3y + 6$ to find x:

$$\begin{aligned} -2(3y+6)+5y &= -5 \\ -6y-12+5y &= -5 \\ -y-12 &= -5 \\ -y &= 7 \\ y &= -7 \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} x-3(-7) &= 6 \\ x+21 &= 6 \\ x &= -15 \end{aligned}$$

The solution to the system of equations is $(-15, -7)$.

When solving a system of three equations in three unknowns, matrix methods are usually employed. However, we will encounter certain systems of three equations in three unknowns for which the substitution method yields a solution rather easily. The next example illustrates this.

◇ **Example B.3(e):** Solve the system of equations

$$\begin{aligned} 9A &= 3 \\ 12A + 8B &= 0 \\ 3A + 6B + 12C &= 8 \end{aligned}$$

using the substitution method.

We divide both sides of the first equation by 9 to obtain $A = \frac{1}{3}$. Substituting this into the second equation we get

$$\begin{aligned} 12(\tfrac{1}{3}) + 8B &= 0 \\ 4 + 8B &= 0 \\ 8B &= -4 \\ B &= -\tfrac{1}{2} \end{aligned}$$

We can now substitute $A = \frac{1}{3}$ and $B = -\frac{1}{2}$ into the third equation to get

$$\begin{aligned} 3(\tfrac{1}{3}) + 6(-\tfrac{1}{2}) + 12C &= 8 \\ 1 - 3 + 12C &= 8 \\ 12C &= 10 \\ C &= \tfrac{5}{6} \end{aligned}$$

The solution to the system is $A = \frac{1}{3}$, $B = -\frac{1}{2}$, $C = \frac{5}{6}$.

Section B.3 Exercises

To Solutions

1. Solve each of the following systems by both the addition method and the substitution method.

   (a) $\begin{aligned} 2x + y &= 13 \\ -5x + 3y &= 6 \end{aligned}$ (b) $\begin{aligned} 2x - 3y &= -6 \\ 3x - y &= 5 \end{aligned}$ (c) $\begin{aligned} x + y &= 3 \\ 2x + 3y &= -4 \end{aligned}$

2. Solve each of the following systems by the addition method.

   (a) $\begin{aligned} 7x - 6y &= 13 \\ 6x - 5y &= 11 \end{aligned}$ (b) $\begin{aligned} 5x + 3y &= 7 \\ 3x - 5y &= -23 \end{aligned}$ (c) $\begin{aligned} 5x - 3y &= -11 \\ 7x + 6y &= -12 \end{aligned}$

3. Consider the system of equations $\begin{aligned} 2x - 3y &= 4 \\ 4x + 5y &= 3 \end{aligned}$.

   (a) Solve for x by using the addition method to eliminate y. Your answer should be a fraction.

   (b) Ordinarily you would substitute your answer to (a) into either equation to find the other unknown. However, dealing with the fraction that you got for part (a) could be difficult and annoying. Instead, use the addition method again, but eliminate x to find y.

4. Consider the system of equations $\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{3}y &= 2 \\ \tfrac{1}{4}x + \tfrac{2}{3}y &= 6 \end{aligned}$. The steps below indicate how to solve such a system of equations.

(a) Multiply both sides of the first equation by the least common denominator to "kill off" all fractions.

(b) Repeat for the second equation.

(c) You now have a new system of equations without fractional coefficients. Solve that system by the addition method.

5. Solve each of the following systems of equations.Each is of the sort that arise in solving various initial value and boundary value problems.

(a) $$\begin{aligned} 4 &= C_1 + C_2 \\ -3 &= -2C_1 - 5C_2 + 7 \end{aligned}$$

(b) $$\begin{aligned} 7A - 2B &= 4 \\ -2A - 7A &= 0 \end{aligned}$$

(c) $$\begin{aligned} 8A &= 4 \\ 10A + 8B &= -3 \\ 4A + 5B + 8C &= 10 \end{aligned}$$

(d) $$\begin{aligned} -3 &= C_1 + C_2 - 3 \\ 2 &= -4C_1 - C_2 + 1 \end{aligned}$$

(e) $$\begin{aligned} 3A - 8B &= -2 \\ -8A - 3B &= 1 \end{aligned}$$

(f) $$\begin{aligned} 0 &= \tfrac{800}{12} + \tfrac{800}{6}C_1 + \tfrac{80}{2}C_2 \\ 0 &= \tfrac{800}{3} + \tfrac{80}{2}C_1 + 8C_2 \end{aligned}$$

B.4 Partial Fraction Decomposition

There are times when we wish to take an expression of the form $\dfrac{Ax+B}{(x-x_1)(x-x_2)}$, where either (but not both) of A or B might be zero, and find two expressions

$$\frac{C}{x-x_1} \qquad \text{and} \qquad \frac{D}{x-x_2} \tag{1}$$

such that

$$\frac{Ax+B}{(x-x_1)(x-x_2)} = \frac{C}{x-x_1} + \frac{D}{x-x_2}. \tag{2}$$

The sum to the right of the equal sign in (2) is called the **partial fraction decomposition** of the expression to the left of the equal sign there. The process of obtaining the right hand side is also called partial fraction decomposition, and we illustrate it in the following example.

◇ **Example B.4(a):** Find two expressions of the form (2) whose sum is $\dfrac{x+17}{x^2+4x-5}$.

First we note that

$$\frac{x+17}{x^2+4x-5} = \frac{x+17}{(x-1)(x+5)},$$

so we are looking for $\dfrac{C}{x-1}$ and $\dfrac{D}{x+5}$ such that

$$\frac{C}{x-1} + \frac{D}{x+5} = \frac{x+17}{x^2+4x-5}.$$

But

$$\frac{C}{x-1} + \frac{D}{x+5} = \frac{C(x+5)}{(x-1)(x+5)} + \frac{D(x-1)}{(x-1)(x+5)} = \frac{Cx+5C+Dx-D}{(x-1)(x+5)}$$

Now if this last expression is to equal $\dfrac{x+17}{(x-1)(x+5)}$, then the numerators of both fractions must be equal (because they both have the same denominator). Note that both fractions have the same denominator, so the two fractions will be equal only if their numerators are equal:

$$Cx+5C+Dx-D = x+17$$

By "grouping like terms," this can be rewritten (be sure you see how) as

$$(C+D)x + (5C-D) = 1x+17,$$

and these will be equal only if $C+D=1$ and $5C-D=17$. Now we have two equations in two unknowns, which we know how to solve (see Appendix B.3). If we add the two equations together we get $6C=18$, so $C=3$. Substituting this into the first equations gives $D=-2$. Thus

$$\frac{x+17}{x^2+4x-5} = \frac{3}{x-1} + \frac{-2}{x+5} = \frac{3}{x-1} - \frac{2}{x+5}.$$

The method of partial fractions has many complications that can arise when the expression to be decomposed has other forms. Those complications are dealt with by varying the above process slightly, but for our needs the above method is sufficient.

Section B.4 Exercises To Solutions

1. Add $\dfrac{3}{x-1}$ and $\dfrac{-2}{x+5}$. Check your answer with the original expression from Example B.3(a).

2. Find the partial fraction decomposition of each expression.

   (a) $\dfrac{4x+7}{x^2+5x+6}$

   (b) $\dfrac{-14}{x^2-3x-10}$

   (c) $\dfrac{11-x}{x^2-x-2}$

   (d) $\dfrac{4x-10}{x^2-1}$

B.5 Series and Euler's Formula

Try the following: Set your calculator for radians (as you always should in this course) and find $\sin(0.05)$, $\sin(0.1)$ and $\sin(0.5)$. You should get (to five places past the decimal) 0.04998, 0.09983 and 0.47943, respectively. These numbers are fairly close to the numbers that you were finding the sine of, with the one closest to zero being the best approximation. This seems to indicate that

$$\sin x \approx x$$

for values of x near zero. Since of course $\sin(5)$ must be a number between -1 and 1 this will certainly not hold in that case!

Now consider the function $f(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$. Here we find that if we round to five places past the decimal we get $f(0.5) = 0.47943$, the value of $\sin(0.5)$ when rounded to the same number of decimal places. Let's try something bigger:

$$f(1.3) = 0.96477, \quad \sin(1.3) = 0.96356 \qquad\qquad f(2.0) = 0.93333, \quad \sin(2.0) = 0.90930$$

It appears that this function f comes pretty close to approximating the sine function, especially for values of x nearer to zero.

The fraction coefficients in the equation for the function f appear to be a bit mysterious, but it turns out that $6 = 3\cdot2\cdot1$ and $120 = 5\cdot4\cdot3\cdot2\cdot1$. We call the quantity $n(n-1)(n-2)\cdots3\cdot2\cdot1$ "n factorial," denoted by $n!$ So $6 = 3!$, $120 = 5!$ and

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

It turns out that (in a sense that you really must take a course in sequences and series to understand)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

This is called the (infinite) **series representation** of the sine function. As you have seen, for values of x near zero, very good approximations of $\sin x$ can be obtained by using just the first term or few terms of this series. For values farther from zero, more terms must be used to get a good approximation.

Cosine and the exponential function have series representations as well:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

and

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

Recall now i, the imaginary unit (j for you electrical engineering students), for which $i^2 = -1$. We can also compute things like

$$i^1 = i, \qquad i^3 = i^2 \cdot i = -i, \qquad i^4 = i^3 \cdot i = -i \cdot i = -i^2 = 1, \qquad i^5 = i^4 \cdot i = 1i = i$$

and so on. Because we can compute these, we can now find something like $e^{i\theta}$ by using the series representation of e^x:

$$\begin{aligned}
e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots \\
&= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} + \cdots\right) \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\
&= \cos\theta + i\sin\theta
\end{aligned}$$

A similar computation can be done for $e^{-i\theta}$. The result of the two of these is

Euler's Relations:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

D Solutions to Exercises

D.6 Solutions for Appendices

Section B.1 Solutions Back to B.1 Exercises

1. (a) $y' = 15x^4 - 28x^3 + 2x - 3$ (b) $h'(t) = -32t + 23.7$
   (c) $f'(x) = \frac{8}{3}x^3 + 15x^2 - \frac{1}{4}x$ (d) $s'(t) = 3t^2 - 4t + 3$
2. (a) $y'' = 60x^3 - 84x^2 + 2$ (b) $h''(t) = -32$
   (c) $f''(x) = 8x^2 + 30x - \frac{1}{4}$ (d) $s''(t) = 6t - 4$
3. (a) $f'(x) = -\dfrac{6}{x^3}$ (b) $g'(t) = \dfrac{1}{3}t + \dfrac{12}{t^3}$
   (c) $y' = \dfrac{1}{10\sqrt{x}}$ (d) $h'(t) = -\dfrac{4}{3\sqrt[3]{x^4}}$
4. (a) $f''(x) = \dfrac{18}{x^4}$ (b) $g''(t) = \dfrac{1}{3} - \dfrac{36}{t^4}$
5. (a) $g(x) = \dfrac{x}{\sqrt{16 - x^2}}$ (b) $y' = -\frac{3\pi}{2}\sin\left(\frac{\pi}{2}t\right)$ (c) $A'(t) = -150e^{-0.3t}$
6. (a) $y' = 10e^{2x}$ (b) $x' = 12\cos 3t$ (c) $g'(x) = -\frac{14(2x-4)}{(x^2-4x)^7}$
   (d) $s'(t) = -\frac{2}{5}\sin\left(\frac{2}{5}t\right)$ (e) $y' = 2xe^{x^2}$
7. (a) $15t^3e^{5t^3}$ (b) $-15\sin(5x - 2)$ (c) $4e^{-2t}\sin(5t + 3)$
8. (a) $-8e^{-2t}\sin(5t + 3)$ (b) $14te^{-t}\cos(3t - 1)$ (c) $12x^3\ln x$
9. (a) $f'(t) = 7t^2e^{7t} + 2te^{7t}$ (b) $y' = -6x\sin 2x + 3\cos 2x$
   (c) $h'(t) = 2\pi e^{-3t}\cos\pi t - 6e^{-3t}\sin\pi t$
10. (a) $f'(x) = \dfrac{6e^{7t}\cos 2t - 21e^{7t}\sin 2t}{e^{14t}}$ (b) $y' = \dfrac{-20t^2e^{-5t} - 8te^{-5t}}{t^4}$
    (c) $g'(x) = \dfrac{-12x^3\sin 6x - 6x^2\cos 6x}{4x^6}$

Section B.2 Solutions Back to B.2 Exercises

1. (a) $\frac{5}{2}\ln\left|2x + 3\right| + C$ (b) $-\frac{2}{5}\ln\left|3 - 5x\right| + C$ (c) $-12.5\ln\left|1.6 - 0.08A\right| + C$
2. (a) $\frac{1}{3}x^3 - \frac{7}{2}x^2 + 3x + C$ (b) $-\frac{7}{3}\cos 3t + C$
   (c) $-\frac{3}{2}te^{-2t} - \frac{3}{4}e^{-2t} + C$ (d) $-\dfrac{3}{x} + C$
   (e) $-10\ln\left|2.0 - 0.1A\right| + C$ (f) $-e^{-t}\left(\frac{3}{26}\sin 5t + \frac{15}{26}\cos 5t\right) + C$
   (g) $\frac{3}{5}\ln\left|5x - 1\right| + C$ (h) $-\frac{5}{3}t^2e^{-3t} - \frac{10}{9}te^{-3t} - \frac{10}{27}e^{-3t} + C$
   (i) $e^{-4t}\left(\frac{3}{25}\sin 3t - \frac{4}{25}\cos 3t\right) + C$ (j) $\frac{10}{\pi}\sin\frac{\pi}{2}t + C$

Section B.3 Solutions

Back to B.3 Exercises

1. (a) $(-15, -7)$ (b) $(3, 4)$ (c) $(13, -10)$
2. (a) $(1, -1)$ (b) $(-1, 4)$ (c) $(-2, \frac{1}{3})$
3. $(\frac{29}{22}, -\frac{5}{11})$
4. $(8, 6)$
5. (a) $C_1 = \frac{10}{3},\ C_2 = \frac{2}{3}$ (b) $A = \frac{28}{53},\ B = -\frac{8}{53}$
   (c) $A = \frac{1}{2},\ B = -1,\ C = \frac{13}{8}$ (d) $C_1 = -\frac{1}{3},\ C_2 = \frac{1}{3}$
   (e) $A = -\frac{14}{73},\ B = \frac{13}{73}$ (f) $C_1 = -19,\ C_2 = \frac{185}{3}$

Section B.4 Solutions

Back to B.4 Exercises

1. (a) $\dfrac{4x+7}{x^2+5x+6} = \dfrac{5}{x+3} + \dfrac{-1}{x+2} = \dfrac{5}{x+3} - \dfrac{1}{x+2}$
   (b) $\dfrac{-14}{x^2-3x-10} = \dfrac{-2}{x-5} + \dfrac{2}{x+2}$
   (c) $\dfrac{11-x}{x^2-x-2} = \dfrac{-4}{x+1} + \dfrac{3}{x-2}$
   (d) $\dfrac{4x-10}{x^2-1} = \dfrac{7}{x+1} + \dfrac{-3}{x-1} = \dfrac{7}{x+1} - \dfrac{3}{x-1}$

Appendix C Solutions

Back to Appendix C Exercises

1.

n	t_n	y_n	$y(t_n)$	% error
0	0.0	2	2	0
1	0.2	2.4	2.4214	0.88
2	0.4	2.84	2.8918	1.79
3	0.6	3.328	3.4221	2.75
4	0.8	3.8736	4.0255	3.77
5	1.0	4.48832	4.7183	4.87

2. (a) $y_{n+1} = 1.05y_n - 0.05t_n$
   (b)

n	t_n	y_n	$y(t_n)$	% error
0	0.00	1.4	1.4	0.00
1	0.05	1.47	1.4705	0.03
2	0.10	1.541	1.5421	0.07
3	0.15	1.6131	1.6147	0.10
4	0.20	1.6863	1.6886	0.14

3. (a)

n	t_n	y_n	$y(t_n)$	% error
0	0.0	2	2	0.00
1	0.1	2	2.01	0.50
2	0.2	2.02	2.0404	1.00
3	0.3	2.0604	2.0921	1.52

   (b) $y = 2e^{-\frac{1}{2}t^2}$

4. (a) $1.1y_n - 0.1t_n$ (b) $y_{n+1} = 1.22y_n - 0.22t_n - 0.02$

(c)

n	t_n	y_n	$y(t_n)$	% error
0	0.0	2	2	0
1	0.2	2.42	2.4214	0.06
2	0.4	2.8884	2.8918	0.12
3	0.6	3.4158	3.4221	0.18
4	0.8	4.0153	4.0255	0.25
5	1.0	4.7027	4.7183	0.33

(d) The error using the midpoint method are much smaller than those obtained using Euler's method with the same step size.

5. (a) $k_1 = .2y_n - .2t_n$, $k_2 = 0.22y_n - 0.22t_n - 0.02$, $k_3 = 0.222y_n - 0.222t_n - 0.022$, $k_4 = 0.2444y_n - 0.0444t_n - 0.044 - 0.2t_{n+1}$
Note the appearance of both t_n *and* t_{n+1} *in* k_4.

(b)

n	t_n	y_n	k_1	k_2	k_3	k_4
0	0.0	2	0.4	0.42	0.422	0.4444
1	0.2	2.4214	0.4443	0.4687	0.4712	0.4985
2	0.4	2.8918	0.4984	0.5282	0.5312	0.5646
3	0.6	3.4221	0.5644	0.6009	0.6045	0.6453
4	0.8	4.0255	0.6451	0.6896	0.6941	0.7439
5	1.0	4.7182				

(c) The approximations obtained using this method are very close to the exact values.