# Ordinary Differential Equations

## for Engineers and Scientists

Gregg Waterman
Oregon Institute of Technology

# Contents

# 0 Introduction to This Book

## 0.1 Goals and Essential Questions

Differential equations are perhaps the most central mathematical topic of science and engineering. Our quest in those areas is to understand and predict the behavior of some sort of "system" consisting of a collection of "parts" that could be things like electrical or mechanical components, living organisms, or some part of the natural world. We often wish to construct a **mathematical model** that describes the behavior of the system reasonably well; such a model usually consists of one of three things:

- An equation or a set of equations (an **analytical** model).
- A general but imprecise description or a graph (a **qualitative** model).
- "Snapshots" of the state of the system at discrete points in time and/or space (a **numerical** model).

The problem is that we generally can't construct directly the equation or equations making up an analytical model of a system. What we will usually have at our disposal are pieces of information about how a system is changing and/or how forces are acting on and within the system. Those pieces of information are combined to form an equation containing the changes or forces, in the form of derivatives. Such an equation containing derivatives is called a **differential equation**. Once a differential equation is obtained, we hope we can use some mathematical technique to extract a model without derivatives that describes the behavior of the system.

This book is a fairly straightforward introduction to differential equations, with an applied emphasis. The student should be aware that this is a huge subject, with lifetimes of study possible. Our hope is that this collection of explanations, examples and exercises will create a solid foundation for understanding differential equations when they are encountered in subject-specific courses, and for further study of differential equations themselves.

In the past an introduction to differential equations has usually consisted of learning specific techniques for solving a variety differential equations. It should be no surprise that those techniques are easily forgotten in short order! We will look at techniques for obtaining solutions - that is an essential part of the subject. However, we will also attend to the "bigger picture," in the hopes of giving the student an overall understanding of the subject that will be more lasting than just a bunch of 'recipes" for obtaining solutions. Our study of the subject of differential equations will be guided by some overarching goals, and essential questions related to those goals.

### Goals

Upon completion of his/her study, the student should understand what differential equations, initial value problems, and boundary value problems are, and what their solutions consist of. For ordinary differential equations (ODEs) and associated initial value or boundary value problems, the student should understand

- where such problems come from,
- what their solutions consist of,
- how solutions are obtained,
- how parameters of a system and initial or boundary conditions influence the nature of solutions.

Our pursuit of these goals will take place through the consideration of some related *essential questions*.

**Essential Questions:**

- What are differential equations and why do we need them?
- What is a solution to a differential equation? What do we mean by a family of solutions to a differential equation?
- What are initial value problems, and what are boundary value problems? How are the two alike and how are they different?
- What is meant by an analytical solution? A qualitative solution? A numerical solution?
- How do we go about finding solutions to differential equations?
- How do parameters differ from variables? What is the role of parameters in differential equations?
- What is a mathematical model? How do differential equations and their solutions model systems and their responses?

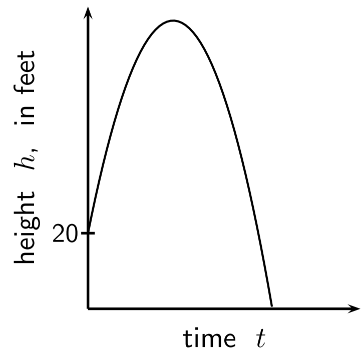
It has been demonstrated experimentally that retaining the things we learn can be enhanced if those things are learned through spaced repetition. To that end, I have attempted to write this book like a novel in which most of the characters are introduced early on, and are then developed and fleshed out as the plot unfolds. A large number of the important concepts of differential equations (including at least a little bit about partial differential equations) are first seen in Chapter 1, then taken up again at later points in the book, where they are reinforced and expanded upon.

Those having prior knowledge of ordinary differential equations (in most cases, the instructor) will notice that the focus of this book is more on the important concepts related to differential equations (both ordinary and partial) rather than techniques for solving a broad range of types of equations. This is based on my conviction that most students will quickly forget the specific procedures for solving differential equations unless those techniques are used in other courses taken shortly after this one. However, a person with a good working understanding of differential equations, initial value problems and boundary value problems should be able to go to any of the many resources available and quickly remind themselves of techniques previously learned, or even techniques not seen in this course!

## 0.2 An Illustrative Example

In this section we will use a simple and perhaps familiar problem to illustrate many of the main ideas of this book. In the process we will begin our quest to answer the essential questions. Let's consider the following situation and question: A rock is fired straight upward with a velocity of 60 feet per second, from a height of 20 feet off the ground. What will the rock be doing after being fired?

We will begin by offering what we'll call a **qualitative solution** to the problem that was posed. Intuitively, we all know that the ascent of the rock will slow as time goes on, until at some point the rock will come to a complete stop at its maximum height. It will then begin to pick up speed downward, falling until it hits the ground. Letting $h$ represent the height (above the ground) of the rock and $t$ represent time, we say the *height is a function of the time*, and we could graph this behavior to obtain the graph shown to the right. Note that the horizontal axis is for the variable of time, so the graph *does not* indicate the the trajectory of the projectile. The trajectory is straight up and then straight down.




This qualitative solution might be fine for our purposes if all we wish to know is the general behavior of the rock after being fired. Suppose, though, that we wanted more. We might wish to know how high the rock was at any time after being fired, or the maximum height it obtains. We can determine those things by finding what we'll call an **analytical solution** to the problem, consisting of an equation that gives the height of the rock at any time. To obtain such a solution, we will need to recall a few things from differential calculus and physics:

- When the position of an object is changing, the rate at which its position is changing is its velocity, given by the first derivative of position with respect to time.
- The rate at which the velocity of an object is changing with respect to time is the acceleration of the object, which is also the second derivative of position with respect to time.
- $F = ma$, the force on an object is its mass times its acceleration.

Going back to our rock, its velocity and acceleration at any time are $\dfrac{dh}{dt}$ and $\dfrac{d^2h}{dt^2}$. We will assume that the only force acting on the rock is the force due to gravity. (Later we will consider the possibility that air resistance applies a force to the rock as well.) The force of gravity on an object of mass $m$ has been experimentally determined to be $mg$, where $g$ is the gravitational constant. For the surface of the earth, that constant has value $g = 9.8$ m/sec$^2$ or $g = 32$ ft/sec$^2$. So we know two things about the force acting on our rock: it is $F = ma = m\dfrac{d^2h}{dt^2}$ and it is also $-mg = -32m$. (Later we will discuss why this is negative.) We set our two expressions for force equal to each other, then divide both sides by the mass $m$:

$$m\frac{d^2h}{dt^2} = -32m \qquad \Longrightarrow \qquad \frac{d^2h}{dt^2} = -32 \tag{1}$$

Both the equations above are what we call **differential equations**, which simply means that they are equations that contain derivatives. (They are equivalent equations, but we'll use the second because it is simpler. Note that the mass of the rock will not be needed from here on, and our result will be independent of the mass.)

Our goal now is to determine an equation for $h$, in terms of $t$. We know that if the second derivative is $-32$, as stated in (1) above, then the first derivative must be

$$\frac{dh}{dt} = -32t + C_1, \tag{2}$$

where $C_1$ is some unknown constant. Thus

$$h = -16t^2 + C_1 t + C_2, \tag{3}$$

where $C_2$ is yet another unknown constant. This gives us an equation for the height as a function of time, but it has the problem that it contains two unknown constants. This function is a **solution** to the differential equation (1).

Because the highest (and only) derivative in (1) is a second derivative, the differential equation is called a **second order differential equation**. What we see in (3) is typical - the solution to a second order differential equation contains two arbitrary (meaning they can have any value) constants. However, the differential equation is not the only information we have. We also know that $h = 20$ when $t = 0$, and $\dfrac{dh}{dt} = 60$ when $t = 0$. These pieces of information are called **initial conditions**, and they will usually be written in the function form

$$h(0) = 20, \qquad h'(0) = 60. \tag{4}$$

We can substitute the second initial condition into (2) to get

$$60 = -32(0) + C_1,$$

resulting in $C_1 = 60$. If we substitute that value into (3) along with the first condition of (4), we get

$$20 = -16(0)^2 + 60(0) + C_2,$$

giving us that $C_2 = 20$. Thus

$$h = -16t^2 + 60t + 20. \tag{5}$$

This is our analytical solution to the problem, which allows us to compute the height of the projectile at any time (until it hits the ground), as well as other things like its maximum height.

The solution (5) was determined from three pieces of information, the differential equation and two initial conditions. Those three things together constitute what we call an **initial value problem** consisting of a differential equation and any conditions we know are placed on our solution. In this case the initial value problem is

$$\frac{d^2h}{dt^2} = -32, \qquad h(0) = 20,\ h'(0) = 60$$

and its solution is $h = -16t^2 + 60t + 20$. Let's now come back to the issue of the signs in our differential equation and its solution, and let's see if we can interpret what the solution is telling us.

When solving any problem that has a spatial component, we need to begin by establishing a coordinate system. In this case the coordinate system is a vertical number line, with its origin (zero) at ground level and the positive direction being up. So the fact that the rock starts 20 feet above the ground means that $h$ is positive 20 at time zero. The signs of velocity and acceleration must be consistent with the coordinate system. When the rock is going upward its velocity is positive, which is why the initial velocity of 60 is positive, and when the rock is going downward its velocity is negative. The acceleration is always negative, because it is caused by gravity. When the rock is travelling upward the acceleration is working against the velocity, causing the rock to slow down. When the rock is travelling downward the acceleration is working with the velocity, causing the rock to speed up.

Now think about this: If we had launched the rock from ground level at time zero, and *if there was no gravity*, the height of the rock at any time $t$ would be $h = 60t$ ("distance equals rate times time"). If there was still no gravity, but we want the rock to start at an initial height of 20 feet, we need to modify our equation to $h = 60t + 20$. Finally, the effect of gravity needs to be added in, which is where the $-16t^2$ term comes in.

**Section 0.2 Exercises**

1. (a) Determine the height of the rock after three seconds. Round your answer to the nearest tenth, and include units.

   (b) Determine when, to the nearest hundredth of a second, the rock is at a height of 50 feet.

2. The solution (5) of the differential equation gives the height of the rock as a function of time. Although we *could* determine $h$ values for all real number values of $t$, that would not make sense in the context of the problem. Determine the values of $t$ for which it really does make sense to find values of $h$. We'll call the set of all those values the **domain** of the solution (as opposed to the domain of the function, which is all real numbers). *Give your answer using interval notation.*

3. Determine the maximum height of the rock and when it occurs, both to the nearest tenth. Give your answer as a complete sentence that includes both pieces of information asked for. (**Hint:** If you are not sure how to approach this, read about the qualitative solution again for a hint.)

4. At some time that we'll call zero you are 380 miles from Klamath Falls, *on your way TO Klamath Falls*. You are driving at a constant speed of 58 mph. Let $x$ represent your distance from Klamath Falls and $t$ the time after time zero.

   (a) Draw a graph with time on the horizontal axis and distance from Klamath Falls on the vertical axis, and draw a graph showing what is happening for you in your car, starting at time zero.

   (b) Write two mathematical statements representing each of the two numerical pieces of information given. (One of those statements should contain a derivative, the other won't.)

   (c) The two pieces of information you gave in (a) constitute an initial value problem. (It is first order, so its solution will only have one constant and only one initial condition is needed in order to determine that constant.) Solve the initial value problem. That is, find an equation for the distance $x$ from Klamath Falls as a function of time $t$.

   (d) Your answer to (c) is your analytical solution to the initial value problem. Explain how you know that this agrees with your qualitative solution from (a). (If it doesn't agree, fix the problem and *then* answer the question.)

   (e) What are the units of $x$ and $t$? Give your answer as a sentence.

# 1 Functions and Derivatives, Variables and Parameters

**Learning Outcomes:**

1. Understand functions and their derivatives, variables and parameters. Understand differential equations, initial and boundary value problems, and the nature of their solutions.

   **Performance Criteria:**

   (a) Determine the independent and dependent variables for functions modeling physical and biological situations. Give the domain(s) of the independent variable(s).

   (b) For a given physical or biological situation, sketch a graph showing the qualitative behavior of the dependent variable over the domain (or part of the domain, in the case of time) of the independent variable.

   (c) Interpret derivatives in physical situations.

   (d) Find functions whose derivatives are given constant multiples of the original functions.

   (e) Identify parameters and variables in functions or differential equations.

   (f) Identify initial value problems and boundary value problems. Determine initial and boundary conditions.

   (g) Determine the independent and dependent variables for a given differential equation.

   (h) Determine whether a function is a solution to an ordinary differential equation (ODE); determine values of constants for which a function is a solution to an ODE.

   (i) Classify differential equations as ordinary or partial; classify ordinary differential equations as linear or non-linear. Give the order of a differential equation.

   (j) Identify the functions $a_0(x)$, $a_1(x)$, ..., $a_n(x)$ and $f(x)$ for a linear ordinary differential equation. Classify linear ordinary differential equations as homogenous or non-homogeneous.

   (k) Write a first order ordinary differential equation in the form $\dfrac{dy}{dx} = F(x,y)$ and identify the function $F$. Classify first-order ordinary differential equations as separable or autonomous.

   (l) Determine whether a function satisfies an initial value problem (IVP) or boundary value problem (BVP); determine values of constants for which a function satisfies an IVP or BVP.

Much of science and engineering is concerned with understanding the relationships between measurable, changing quantities that we call **variables**. Whenever possible we try to make these relationships precise and compact by expressing them as equations relating variables; often such equations define *functions*. In this chapter we begin by looking at ideas you should be familiar with (functions and derivatives), but hopefully you will now see them in a deeper and more illuminating way than you did in your algebra, trigonometry and calculus courses.

We then go on to introduce the idea of a differential equation, and we will see what we mean by a solution to a differential equation, initial value problem, or boundary value problem. We will also learn various classifications of differential equations. This is important in that the method used to solve a differential equation depends on what type of equation it is.

It is valuable to understand these fundamental concepts before moving on to learning techniques for solving differential equations, which are addressed in the remainder of the text.

## 1.1 Functions and Variables

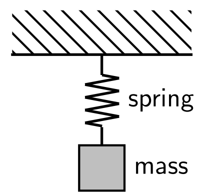
**Performance Criteria:**

1. (a) Determine the independent and dependent variables for functions modeling physical and biological situations. Give the domain of the independent variable(s).

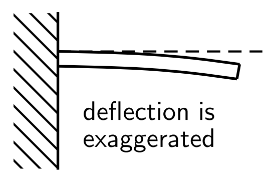
   (b) For a given physical or biological situation, sketch a graph showing the qualitative behavior of the dependent variable over the domain (or part of the domain, in the case of time) of the independent variable.

As scientists and engineers, we are interested in relationships between measurable physical quantities, like position, time, temperature, numbers or amounts of things, etc. The physical quantities of interest are usually changing, so are called **variables**. When one physical quantity (variable) depends on one or more other quantities (variables), the first quantity is said to be a **function** of the other variable(s).

◇ **Example 1.1(a):** Suppose that a mass is hanging on a spring that is attached to a ceiling, as shown to the right. If we lift the mass, or pull it down, and let it go, it will begin to oscillate up and down. Its height (relative to some fixed reference, like its height before we lifted it or pulled it down) varies as time goes on from when we start it in motion. We say that *height is a function of time*.



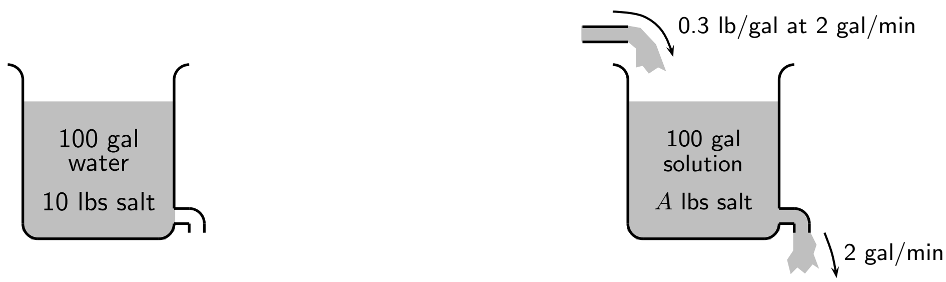

◇ **Example 1.1(b):** Consider a beam that extends horizontally ten feet out from the side of a building, as shown to the right. The beam will deflect (sag) some, with the distance below horizontal being greater the farther out on the beam one looks. *The amount of deflection at a point on the beam is a function of how far the point is from the wall the beam is embedded in.*




◇ **Example 1.1(c):** Consider the equation $y = \dfrac{12}{x^2}$. For any number other than zero that we select for $x$, there is a corresponding value of $y$ that can be determined by substituting the $x$ value and computing the resulting value of $y$. $y$ depends on $x$, or $y$ *is a function of* $x$

◇ **Example 1.1(d):** A drumhead with a radius of 5 inches is struck by a drumstick. The drum head vibrates up and down, with the height of the drumhead at a point determined by the location of that point on the drumhead and how long it has been since the drumhead was struck. *The height of the drumhead is a function of the two-dimensional location on the drumhead and time.*

◇ **Example 1.1(e):** Suppose that we have a tank containing 100 gallons of water with 10 pounds of salt dissolved in the water, as shown to the left below. At some time we begin pumping a 0.3 pounds salt per gallon (of water) solution into the tank at two gallons per minute, mixing it thoroughly with the solution in the tank. At the same time the solution in the tank is being drained out at two gallons per minute as well. See the diagram to the right below.




Because the rates of flow in and out of the tank are the same, the volume in the tank remains constant at 100 gallons. The initial concentration of salt in the tank is 10 pounds/100 gallons = 0.1 pounds per gallon. Because the incoming solution has a different concentration, the amount of salt in the tank will change as time goes on. (The amount will increase, since the concentration of the incoming solution is higher than the concentration of the solution in the tank.) We can say that *the amount of salt in the tank is a function of time*.

---

◇ **Example 1.1(f):** Different points on the surface of a cube of metal one foot on a side are exposed to different temperatures, with the temperature at each surface point held constant. The cube eventually attains a temperature **equilibrium**, where each point on the interior of the cube reaches some constant temperature. *The temperature at any point in the cube is a function of the three-dimensional location of the point.*

---

In each of the above examples, one quantity (variable) is dependent on (is a function of) one or more other quantities (variables). The variable that depends on the other variable(s) is called the **dependent variable**, and the variable(s) that its value depends on is (are) called the **independent variable(s)**.

◇ **Example 1.1(g):** Give the dependent and independent variable(s) for each of Examples 1.1(a) - (f).

**Solution:** Example 1.1(a): The dependent variable is the height of the mass, and the independent variable is time.

Example 1.1(b): The dependent variable is the deflection of the beam at each point, and the independent variable is the distance of each point from the wall in which the beam is embedded.

Example 1.1(c): The dependent variable is $y$, and the independent variable is $x$.

Example 1.1(d): The dependent variable is the height at each point on the drumhead, and the independent variables are the location (in two-dimensional coordinates) of the point on the drum head, and time. Thus there are *three* independent variables.

Example 1.1(e): The dependent variable is the amount of salt in the tank, and the independent variable is time.

Example 1.1(f): The dependent variable is the temperature at each point in the cube, and the independent variables are the three coordinates giving the position of the point, in three dimensions.

---

When studying phenomena like those given in Examples 1.1(a), (b), (d), (e) and (f), the first thing we do after determining the variables is establish coordinate systems for the variables. The purpose for this is to be able to attach a number (or ordered set of numbers) to each point in the domain, and for different positions or states of the dependent variable:

- When position is an independent variable, we must establish a one (for the spring), two (for the drumhead) or three (for the cube of metal) dimensional coordinate system. This coordinate system will have an origin (zero point) at some convenient location, indication of which direction(s) is(are) positive, and a scale on each axis. (The two space variables for the drumhead in Example 1.1(d) would most likely be given using polar coordinates, since the head of the drum is circular.)
- If time is an independent variable, we must establish a "time coordinate system" by determining when time zero is. (Of course all times after that are considered positive.) We must also decide what the time units will be, providing a "scale" for time.
- It may not be clear that there is a coordinate system for the temperature in the cube of metal, or the amount of salt in the tank. For the temperature, the decision whether to measure it in degrees Fahrenheit or degrees Celsius is actually the establishing of a coordinate system, with a zero point and a scale (both of which differ depending on which temperature scale is used).
- The choice of zero for the amount of salt in the tank will be the same regardless of how it is measured, but the scale can change, depending on the units of measurement.

Once we've established the coordinate system(s) for the variable(s), we should determine the **domain** of our function, which means the values of the independent variable(s) for which the dependent variable will have values. The domain is usually given using inequalities or interval notation. Let's look at some examples.

◇ **Example 1.1(h):** For Example 1.1(a), suppose that we pull the mass down and then let it go at a time we call time zero (the origin of our time coordinate system). Time $t$ is the independent variable, and the values of it for which we are considering the height of the mass are $t \geq 0$ or, using interval notation, $[0, \infty)$.

---

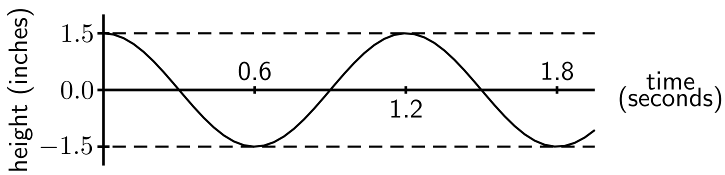
◇ **Example 1.1(i):** For Example 1.1(b), we will use a position coordinate system consisting of a horizontal number line at the top of where the beam emerges from the wall (so along the dashed line in the picture), with origin at the wall and positive values (in feet) in the direction of the beam. Letting $x$ represent the position along the beam, the domain is $[0, 10]$ feet.

---

The four functions described in Examples 1.1(a), (b), (c) and (e) are **functions of a single variable**; the functions in Examples 1.1(d) and (f) are examples of **functions of more than one variable**. The differential equations associated with functions of one variable are called **ordinary differential equations**, and the differential equations associated with functions of more than one variable are (out of necessity) **partial differential equations**. In this class we will study primarily ordinary differential equations.

The function in Example 1.1(c) is a *mathematical* function, whereas all of the other functions from Example 1.1 are not. (We might call them "physical functions.") In your previous courses you have studied a variety of types of mathematical functions, including polynomial, rational, exponential, logarithmic, and trigonometric functions. The main reason that scientists and engineers are interested in mathematics is that many physical situations can be **mathematically modeled** with mathematical functions or equations. This means that we can find a mathematical function that reasonably well describes the relationship between physical quantities. For a mass on a spring (Example 1.1(a)), if we let $y$ represent the height of the mass, then the equation that models the situation is $y = A\cos(bt)$, where $A$ and $b$ are constants that depend on the spring and how far the mass is lifted or pulled down before releasing it. We will see that the deflection of the beam in Example 1.1(b) can be modeled with a fourth degree polynomial function, and the amount of salt in the tank of Example 1.1(c) can be modeled with an exponential function.

Of course one tool we use to better understand a function is its graph. Suppose for Example 1.1(a) we started the mass in motion by lifting it 1.5 inches and releasing it (with no upward or downward force). Then the equation giving the height $y$ at any time $t$ would be of the form $y = 1.5\cos(bt)$, where $b$ depends on the spring and the mass. Suppose that $b = 5.2$ (with appropriate units). Then the graph would look like this:

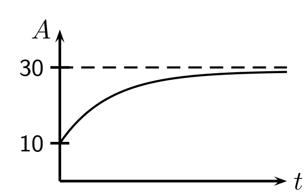



When graphing functions of one variable *we always put the independent variable (often it will be time) on the horizontal axis, and the dependent variable on the vertical axis.* We can see from the graph that the mass starts at a height of 1.5 inches above its equilibrium position $(y = 0)$. It then moves downward for the first 0.6 seconds of its motion, then back upward. It is back at its starting position every 1.2 seconds, the **period** of its motion. This periodic up-and-down motion can be seen from the graph. (Remember that the period $T$ is the time at which $bT = 2\pi$, so $T = \frac{2\pi}{b}$.) Such behavior is called **simple harmonic motion**, and will be examined in detail later in the course because of its importance in science and engineering.

Note that even if we didn't know the value of $b$ in the equation $y = 1.5\cos(bt)$ we could still create the given graph, we just wouldn't be able to put a scale on the time (horizontal) axis. In fact, we could even create the graph without an equation, using our intuition of what we would expect to happen. Let's do that for the situation from Example 1.1(e).

◇ **Example 1.1(j):** A tank contains 100 gallons of water with 10 pounds of salt dissolved in it. At time zero a 0.3 pounds per gallon solution begins flowing into the tank at 2 gallons per minute and, at the same time, thoroughly mixed solution is pumped out at 2 gallons per minute. (See Example 1.1(e).) Sketch a graph of the amount of salt in the tank as a function of time.

**Solution:** The initial amount of salt in the tank is 10 pounds. We know that as time goes on the *concentration* of salt in the tank will approach that of the incoming solution, 0.3 pounds per gallon. This means that the *amount* of salt in the tank will approach 0.3 lbs/gal × 100 gal = 30 pounds, resulting in the graph shown to the right, where $A$ represents the amount of salt, in pounds, and $t$ represents time, in minutes.



**NOTE:** We have been using the notation $\sin(bt)$ or $\cos(bt)$ to indicate the sine or cosine of the quantity $bt$. It gets to be a bit tiresome writing in the parentheses every time we have such an expression, so we will just write $\sin bt$ or $\cos bt$ instead.

**Section 1.1 Exercises** To Solutions

1. Some material contains a radioactive substance that decays over time, so the amount of the radioactive substance is decreasing. (It doesn't just go away - it turns into another substance that is not radioactive, in a series of steps. For example, uranium eventually turns into lead when it decays.)

   (a) Give the dependent and independent variables.

   (b) Sketch a graph of the amount $A$ of radioactive substance versus time $t$. *Label each axis with its variable - this will be expected for all graphs.*

2. A student holds a one foot plastic ruler flat on the top of a table, with half of the ruler sticking out and the other half pinned to the table by pressure from their hand. They then "tweak" the end of the ruler, causing it to vibrate up and down. (This is roughly a combination of Examples 1.1(a) and (b).)

   (a) Give the dependent and independent variables. (**Hint:** There are *two* independent variables.)

   (b) Give the domains of the independent variables.

3. Consider the drumhead described in Example 1.1(e). Suppose that the position of any point on the drumhead is given in polar coordinates $(r, \theta)$, with $r$ measured in inches and $\theta$ in radians. Suppose also that time is measured in seconds, with time zero being when the head of the drum is struck by a drumstick. Give the domains of each of these three independent variables.

4. Consider the cube of metal described in Example 1.1(f). Suppose that we position the cube in the first octant (where each of $x$, $y$ and $z$ is positive), with one vertex (corner) of the cube at the origin and each edge from that vertex aligned with one of the three coordinate axes. Each point in the cube then has some coordinates $(x, y, z)$. Give the domains of each of these three independent variables.

5. Consider again the scenario from Section 0.2, in which a rock is fired straight upward with a velocity of 60 feet per second, from a height of 20 feet off the ground. In that section we derived the equation

$$h = -16t^2 + 60t + 20$$

   for the height $h$ (in feet) of the rock at any time $t$ (in seconds) after it was fired. Using the equation, determine the domain of the independent variable time.

6. When a solid object with some initial temperature $T_0$ is placed in a medium (like air or water) with a constant temperature $T_m$, the object will get cooler or warmer (depending on whether $T_0$ is greater or less than $T_m$), with its temperature $T$ approaching $T_m$. The rate at which the temperature of the object changes is proportional to the difference between its temperature $T$ and the temperature $T_m$ of the medium, so it cools or warms rapidly while its temperature is far from $T_m$, but then the cooling or warming slows as the temperature of the object approaches $T_m$.

   (a) Suppose that an object with initial temperature $T_0 = 80°$ F is place in a water bath that is held at $T_m = 40°$ F. Sketch a graph of the temperature as a function of time. You should be able to indicate two important values on the vertical axis.

   (b) Repeat (a) for $T_m = 40°$F and $T_0 = 30°$F.

   (c) Repeat (a) for $T_m = 40°$F and $T_0 = 40°$F.

7. (a) Suppose that a mass on a spring hangs motionless in its equilibrium position. At some time zero it is set in motion by giving it a sharp blow downward, and there is no resistance after that. Sketch the graph of the height of the mass as a function of time.

   (b) Suppose now that the mass is set in motion by pulling it downward and simply releasing it, and suppose also that the mass is hanging in an oil bath that resists its motion. Sketch the graph of the height of the mass as a function of time.

8. As you are probably aware, populations (of people, rabbits, bacteria, etc.) tend to grow exponentially *when there are no other factors that might impeded that growth.*

   (a) The variables in such a situation are time and the number of individuals in the population. Which variable is independent, and which is dependent?

   (b) Using $t$ for time and $N$ for the number of individuals, sketch (and label, of course) a graph showing growth of such a population.

   (c) Often there are environmental conditions that lead to a **carrying capacity** for a given population, meaning an upper limit to how many individuals can exist. Suppose that 500 fish are stocked in a sterile lake (no fish in it) that has a carrying capacity of 3000 fish. When a population like this starts at well below the carrying capacity, it experiences "almost exponential" growth for a while, then the growth levels off as the population approaches the carrying capacity. Sketch a graph of the fish population versus time. This sort of growth is called **logistic growth**.

   (d) Sketch a graph showing how you would expect the population of fish to behave if 5000 fish were introduced into the same lake having a carrying capacity of 3000 fish.

In this course we will often be interested in just a few "families" of functions, like

$$y(t) = A\sin\omega t + B\cos\omega t, \qquad y = a + be^{-rt}, \qquad y(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$

where each of $A$, $B$, $\omega$, $a$, $b$, $r$, $a_0$, $a_1$, $a_2$, $a_3$, $a_4$ are constants that we will call **parameters** (more on this in Section 1.3) and $x$, $t$, and $y$ are variables. ($\omega$ is the Greek letter *omega*.) We will often omit showing the dependence of $y$ on $x$ or $t$, as done for the second function above. The behavior of each of the above functions remains roughly the same, but varies somewhat depending on the values of the parameters. The point of the following exercises is to see what the graph of each type of function looks like in general, and how the values of the parameters affect various aspects of the graph.

9. We begin with the graph of $y = A\sin\omega t + B\cos\omega t$.

   (a) Sketch what you expect the graph to look like when $A = 1$, $B = 0$ and $\omega = 1$. Sketch a separate graph for $A = 0$, $B = 1$ and $\omega = 1$. These two graphs should be familiar to you. Do you have any idea what the graph would look like if $A = B = \omega = 1$?

   (b) Enter the function into *Desmos*, using $w$ in place of $\omega$. It will ask you if you want sliders for $A$, $w$, $B$ and $t$. select the first three, but not $t$. Check your answers to part (a) by setting the sliders for the appropriate values.

   (c) Recall that trigonometric graphs have three important characteristics:

   - **Amplitude** - The maximum distance the function ($y$ value) gets from the horizontal axis. (We will assume no vertical shifting of the graph, which we don't need in this course.)
   - **Period** - The distance along the horizontal axis from any point on the graph to where the graph first repeats itself.
   - **Phase** - This refers to the horizontal point on the graph where it crosses the vertical axis. For example does the graph cross the vertical axis at a peak, trough, near the top of an "upslope," etc.?

   Set $B = 0$. Now use the slider to change $A$. Which of the above characteristics does changing $A$ seem to affect? Which characteristics does changing $w$ affect? Which characteristic is not affected by changing $A$ or $w$ when $B = 0$?

   (d) Now set $B = 1$ and change $A$. Which characteristic or characteristics does this affect? What does changing $w$ affect? How about changing $B$?

   **Note:** We'll see later how to take a function of the form $y = A\sin\omega t + B\cos\omega t$ and turn it into just a sine function of the form $y = C\sin(\omega t + \phi)$, which is a little easier to work with.

10. The type of function we looked at in the previous exercise models a mass on a spring, as described in Example 1.1(a), as well as certain electric circuits. We will see that $\omega$ is determined by the mass and the spring, but $A$ and $B$ are determined by how the mass is set in motion. We assume there is nothing resisting the motion of the mass, a situation we refer to as **undamped**. In applications we often have some sort of resisting force called damping. It shows up mathematically as an exponential function times a function of the sort we saw in Exercise 9:

$$y = e^{-rt}(A\sin\omega t + B\cos\omega t), \qquad r > 0.$$

   Speculate, based on either this function equation or the physical situation, what the graph of such a function would look like. Check your conjecture using *Desmos*.

11. Now we consider the function $y = a + be^{-rt}$, where $r > 0$.

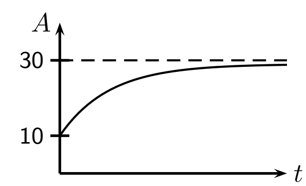
   (a) What do you expect the graph of this function to look like when $a = 0$ and $b = r = 1$? Enter the function in *Desmos*, setting sliders for $a$, $b$ and $r$ and check your conjecture. When working with this kind of function we are really just interested in its behavior for $t \geq 0$. You can restrict the graph in *Desmos* by entering $\{t > 0\}$ after the function - do that now.

   (b) For graphs of this sort of function, we are interested in three things:

   - What the $y$-intercept is.
   - What $y$ value the graph tends toward as time goes on. This is the **horizontal asymptote** of the graph, and it can be expressed in the language of calculus as the limit of $y$ as $t$ goes to infinity: $\lim_{t \to \infty} y(t)$
   - How rapidly the function approaches the aymptote.

   How do the parameters $a$, $b$ and $r$ affect each of the above? Be as specific as possible. Remember that we are only interested in positive values of $r$. For the applications we will be interested in we will usually have only positive values of $a$, but $b$ will sometimes be positive, sometimes negative.

12. To the right is the graph from Example 1.1(j), for a tank containing 100 galloms of water with 10 pounds of salt dissolved in it. The horizontal axis is time, in minutes, and the vertical axis is the amount of salt dissolved in the water, in pounds.




   (a) Determine values of $a$, $b$ and $r$ for which the graph of $A = a + be^{-rt}$ is the graph shown. Use *Desmos* to check your answer.

   (b) In part (a) you should have found that you cannot determine the value of $r$ from the graph given. Graph your answer to (a) with *Desmos*, for $r = 0.1$. Sketch what you see, using a dotted line for the graph. Sketch in, as a dashed line and a solid line, what you think the graph would look like for $r = 0.05$ and $r = 0.5$. Check your answers with *Desmos*.

13. You cook a potato in a microwave oven, and when you take it out, its temperature is $160°$F. It is too hot to eat, so you decide to let it cool. In the meantime you start playing a video game, and completely forget about the potato for several hours. The temperature in your house is $70°$F.

   (a) Sketch a graph of what you expect the temperature $T$ of the potato to be as a function of time, $t$.

   (b) Give values of $a$ and $b$ for which the graph of $T = a + be^{-rt}$ has the appearance of your graph from part (a).

14. There are two facts that are helpful in understanding the appearance of the graph of $y = a+be^{-rt}$, where we emphasize again that we are only interested in $r > 0$. Those two facts are

$$e^0 = 1 \qquad \text{and} \qquad \lim_{t\to\infty} e^{-rt} = 0 \text{ for any fixed } r > 0.$$

(a) What is the value of $t$ at the $y$-intercept of the graph of $y = a + be^{-rt}$? Given that and the above, what is the $y$-intercept of $y = a + be^{-rt}$?

(b) Based on the above, what is the limit of $y$ as $t$ goes to infinity? What does that tell us about the graph of $y = a + be^{-rt}$?

15. Use what you discovered in the previous exercise to sketch the graph of each of the following functions for some value of $r > 0$. Use *Desmos* to check your answers.

(a) $y = 100 + 50e^{-rt}$ (b) $y = 100 - 50e^{-rt}$ (c) $y = 200e^{-rt}$

16. The functions that model how a horizontal beam deflects (a fancy way of saying "sags") under its own weight are always of the form

$$y = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0, \tag{1}$$

where $x$ is the distance along the beam from one end (usually the left end as we look at it from the side), $y$ is the amount of deflection, and $c_4$, $c_3$, $c_2$, $c_1$ and $c_0$ are constants determined by properties of the beam and how it is supported at its ends. We will consider the following ways of supporting a beam at its ends:

- **Embedded** - The end of the beam is held at a fixed angle (which we will always take to be horizontal) coming out of a wall.
- **Pinned** - Sometimes called **simply supported**. The end of the beam is held up on the end by a hinged joint that allows it to pivot at that point.
- **Free** - The end of the beam is not supported at all. In this case the other end must be embedded.

Give the Roman numeral of the form of equation $(1)$ given below that models a ten foot horizontal beam with the given left-right end conditions. Graph each function using *Desmos* to help you do this, and enter $\{0 < x < 10\}$ after each equation to restrict the graph to zero to ten feet.

(a) pinned-pinned (b) embedded-embedded (c) embedded-free

I. $y = -0.001x^4 + 0.02x^3 - 0.1x^2$ II. $y = -0.0001x^4 + 0.004x^3 - 0.06x^2$

III. $y = -0.001x^4 + 0.02x^3 - x$

## 1.2 Derivatives and Differential Equations

**Performance Criteria:**

1. (c) Interpret derivatives in physical situations.
   (d) Find functions whose derivatives are given constant multiples of the original functions.

Scientists and engineers are usually concerned with the behavior of a **system**, which is a collection of physical objects. Some examples of physical systems that we saw in the previous section and will see again later include

- a mass on a spring, hanging from a "ceiling"
- a horizontal beam, supported somehow on one or both ends
- a tank or reservoir of liquid, with liquid added and removed over time

More complex examples of systems are electrical components and devices (including computers), heating and cooling systems, mechanical systems, and constructed things like roads, building structures, bridges and so on. (We will focus on systems of interest in mechanical, civil and electrical engineering, but things like biological and sociological systems can be modeled using differential equations as well!)

In the previous section's exercises you graphed the behaviors of some physical systems. Those graphs are *models* of the systems' behaviors; that is, they are human constructed descriptions of how the systems behave. Such graphical models are good for giving us an overall *qualitative* idea of the behavior of a system, but are generally inadequate if we would like to know precise values of the dependent variable based on a value (or values in the case of more than one) of the independent variable(s). When we desire such *quantitative* information, we attempt to develop an *analytical* model, which usually consists of an equation of a function.

Analytical models for physical situations can often be developed from various principles and laws of physics. The physical principles do not usually lead us directly to the functions that model physical situations, but to equations involving derivatives of those functions. This is because what we usually know is how our variables are changing in relationship with each other, and such change is described with derivatives. Equations containing derivatives are called **differential equations**. In this section we will review the concept of a derivative and see an example of a simple differential equation, along with how it arises.

When you hear the word "derivative," you may think of a process you learned in a first term calculus class. Throughout this course it will be important that you can carry out the process of "finding a derivative"; *if you need review or practice, see Appendix B*. In this section our concern is not the mechanics of finding derivatives, but instead we wish to recall what derivatives are and what they mean.

To reiterate what was said in the previous section, a function is just a quantity that depends on one or more other quantities, one in most cases that we will consider. Again, we refer to the first quantity (the function) as the dependent variable and the second quantity, that it depends on, is the independent variable. If we were to call the independent variable $x$ and the dependent variable $y$, then you should recall the **Leibniz notation** $\dfrac{dy}{dx}$ for the derivative. This notation can be loosely interpreted as *change in $y$ per unit of change in $x$*. Technically speaking, any derivative of a function is really the derivative of the dependent variable (which *IS* the function) *with respect to the the independent variable*. We sometimes use the notation $y'$ instead of $\frac{dy}{dx}$. Obviously it is easier to write $y'$, but that notation does not indicate what the independent variable is and it does not suggest a ratio, or rate.

Let's consider a couple examples of the meaning of the derivative in physical situations.

◇ **Example 1.2(a):** Suppose again that we take a mass hanging from a ceiling on a spring, lift it and let it go, and suppose the equation of motion is $y = 1.5\cos 5.2t$. The derivative of this function is $\frac{dy}{dt} = -7.8\sin 5.2t$, a new function of the independent variable. This function's value at any time $t$ can be interpreted as how fast the the height of the mass is changing with respect to time, *at that particular time*. If the height units are inches and the time units seconds, then the units of the derivative are $\frac{\text{inches}}{\text{seconds}} =$ inches per second, indicating that the derivative of the function $y$ at a given time is the velocity of the mass at that time. For example, the derivative at time 0.5 seconds is

$$\left.\frac{dy}{dt}\right|_{t=0.5} = y'(0.5) = -7.8\sin[(5.2)(0.5)] = -4.02 \text{ in/sec},$$

telling us that the mass is moving downward (indicated by the negative sign) at about four inches per second at one half second after being set in motion. **NOTE:** *Your calculator will need to be set in radians for all trigonometric computations in this course!*

◇ **Example 1.2(b):** Now recall the beam of Example 1.1(b), sticking out from a wall that it is embedded in. If $x$ represents a horizontal position along the beam and $y$ represents the deflection ("sag") of the beam at that horizontal position, then the derivative $\frac{dy}{dx}$ is the change in deflection per unit of horizontal change, which is just the slope of the beam at that particular point.

We'll now take a break from actual physical situations to ask some questions about derivatives, in a mathematical sense. After doing so, we'll see that such questions relate directly to certain "real-life" situations.

◇ **Example 1.2(c):** Find a function whose derivative is seven times the function itself.

**Solution:** Note that the derivative of $y = e^{kt}$, where $k$ is a constant, is $y' = ke^{kt}$. This shows that exponential functions are essentially their own derivatives, with perhaps a constant multiplier. If $k$ was seven, the original function would be $y = e^{7t}$ and the derivative would be $y' = 7e^{7t} = 7y$, seven times the original function $y$.

◇ **Example 1.2(d):** Find a function whose second derivative is sixteen times the function itself.

**Solution:** Here we should again be expecting an exponential function, but it will get multiplied twice because of the chain rule. Note that if $y = e^{4t}$, then $y' = 4e^{4t}$ and $y'' = 16e^{4t} = 16y$, so $y = e^{4t}$ is the function we are looking for. But in fact it is not *the* function, but only one such function. The function $y = e^{-4t}$ is another such function, as is $y = 5e^{-4t}$. (You should verify this last claim for yourself.) In fact, $y = Ce^{-4t}$ is a solution *for any value of* $C$. We will see later why this is, and what we do about it.

◇ **Example 1.2(e):** Find a function whose derivative is $-16$ times the function itself.

**Solution:** The previous example shows that the desired function is not an exponential function, as the only likely candidates were shown to have second derivatives that are *positive* sixteen times the original function. What we want to note here is that if we take the derivative of sine or cosine twice, we end up back at sine or cosine, respectively, but with opposite sign. However, each time we take the derivative of a sine or cosine of $kx$, the chain rule gives us a factor of $k$ on the "outside" of the trig function. Thus we see that

$$y = \sin 4x \qquad \Longrightarrow \qquad y' = 4\cos 4x \qquad \Longrightarrow \qquad y'' = -16\sin 4x = -16y$$

$$y = \cos 4x \qquad \Longrightarrow \qquad y' = -4\sin 4x \qquad \Longrightarrow \qquad y'' = -16\cos 4x = -16y$$

This shows that $y = \sin 4x$ and $y = \cos 4x$ are functions whose derivatives are $-16$ times the original functions themselves.

---

Examples 1.2(c) and (d) show the importance of exponential functions in the study of derivatives and differential equations. Regarding Example 1.2(e), we can see that if $y = e^{4ix}$ where $i^2 = -1$,

$$y' = 4ie^{4ix} \qquad \Longrightarrow \qquad y'' = (4i)^2 e^{4ix} = 16i^2 e^{4ix} = -16e^{4ix} = -16y. \tag{1}$$

The same sort of computation would show that the second derivative of $y = e^{-4ix}$ would also be $-16y$. Later we will see that these two functions are "equivalent" to the sine and cosine, in some sense. The point, for now, is that the functions we are looking for are again exponential functions.

Consider Example 1.2(e) above. The words "the second derivative is $-16$ times the original function" can be written symbolically as

$$\frac{d^2y}{dx^2} = -16y, \tag{2}$$

since *the function is the dependent variable* $y$. This is a **differential equation**, an equation containing a derivative. (Differential equations can contain derivatives of any order. The **order of a differential equation** is the highest order derivative occurring in the differential equation, so this is a second order differential equation.) Any function that makes such an equation true is a **solution to the differential equation**, so Example 1.2(e) shows that both $y = \sin 4x$ and $y = \cos 4x$ are both solutions to the differential equation $(2)$, and $(1)$ shows that $y = e^{4ix}$ is as well. We will find later that every solution to $(2)$ is a function of the form

$$y = C_1 \sin 4x + C_2 \cos 4x,$$

where $C_1$ and $C_2$ are constants that can be any numbers (and that must be able to be complex numbers to account for the fact that $y = e^{4ix}$ is a solution).

We now show that Example 1.2(e) and equation $(2)$ are not just an exercise in understanding derivatives or a mathematical curiosity, but can arise from a physical situation. Suppose one end of a spring is attached to a ceiling, as shown in Figure 1.2(a) at the top of the next page. We then hang an object with mass $m$ (we will refer to both the object itself *and* its mass as the "mass" - one must note from the context which we are talking about) on the spring, extending it by a length $l$ to where the mass hangs in equilibrium. See Figure 1.2(b). There are two forces acting on the mass, a downward force of $mg$, where $g$ is the acceleration due to gravity, and an upward force of $kl$, where $k$ is the **spring constant**, a measure of how hard the spring "pulls back" when stretched. The spring constant is a property of the particular spring. When the mass hangs in equilibrium these two force are equal in magnitude to each other, but in opposite directions. This is expressed by $mg = kl$.

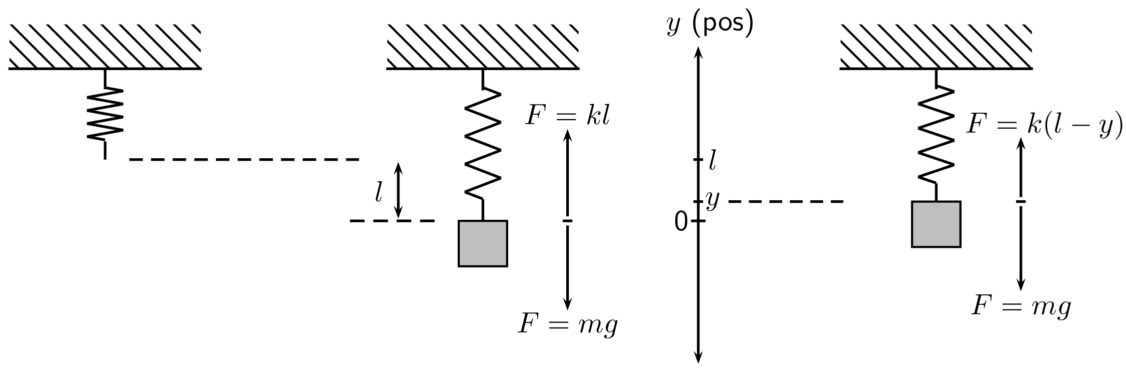



Figure 1.2(a) Figure 1.2(b) Figure 1.2(c)

We will put a coordinate system (with a scale in appropriate length units, like inches) beside the mass, with the zero at the point even with the top of the mass at rest and *with the positive direction being up*. If we then lift the mass up to a position $y_0$, where $y_0 < l$, and release it, it will oscillate up and down. If we assume (for now) that there is no resistance, it will oscillate between $y_0$ and $-y_0$ forever; as noted in the previous section, this is called **simple harmonic motion**. Consider the mass when it is at some position $y$ in this oscillation, as shown in Figure 1.2(c) above. There will be an upward force of $k(l - y)$ due to the spring and a downward force of $-mg$ (the negative indicating downward) due to gravity. Remembering that force is mass times acceleration and that acceleration is the second derivative of position with respect to time, the net force is then

$$F = ma = m\frac{d^2y}{dt^2} = k(l - y) - mg = kl - ky - mg = -ky,$$

since $kl = mg$.

Extracting the equation $m\frac{d^2y}{dt^2} = -ky$ from the above and dividing both sides by $m$ gives $\frac{d^2y}{dt^2} = -\frac{k}{m}y$. If the values of $k$ and $m$ are such that $\frac{k}{m} = 16$, this becomes $\frac{d^2y}{dt^2} = -16y$, the equation describing the situation from Example 1.2(e) (with the variable $x$ replaced with $t$). Based on the discussion from the previous page, the equation that models the motion of the mass is

$$y = C_1 \sin 4t + C_2 \cos 4t. \tag{3}$$

This is to be expected, as we know the mass will oscillate up and down. The values $C_1$ and $C_2$ will depend on how the mass is set in motion (more on that in Section 1.4), but as long as $\frac{k}{m} = 16$ we will have a solution of the form (3). All of this shows that what seems like a whimsical mathematical question about derivatives (posed in Example 1.2(e)) is actually very relevant for a practical application.

**Section 1.2 Exercises** To Solutions

1. Find the derivative of each function without using your calculator. You *MAY* use the course formula sheet. Give your answers using correct derivative notation.

   (a) $y = 2\sin 3x$ (b) $y = 4e^{-0.5t}$ (c) $x = t^2 + 5t - 4$

   (d) $y = 3.4\cos(1.3t - 0.9)$ (e) $y = te^{-3t}$ (f) $x = 4e^{-2t}\sin(3t + 5)$

2. Find the second derivatives of the functions from parts (a) - (c) of Exercise 1. Give your answers using correct derivative notation.

3. The temperature $T$ of an object (in degrees Fahrenheit) depends on time $t$, measured in minutes, and $\frac{dT}{dt} = 2.7$ when $t = 7$. $\left( \text{We sometimes write this as } \left.\frac{dT}{dt}\right|_{t=7} = 2.7 \right)$ Interpret the derivative in a sentence, using either *increasing* or *decreasing*.

4. The amount $A$ of salt in a tank depends on the time $t$. If $A$ is measured in pounds and $t$ is measured in minutes, interpret the fact that $\left.\frac{dA}{dt}\right|_{t=12.5} = -1.3$. Again, use *increasing* or *decreasing*.

5. The height of a mass on a spring at time $t$ is given by $y$, where $t$ is in seconds and $y$ is in inches.

   (a) Interpret the fact that $\frac{dy}{dt} = -5$ when $t = 2$.

   (b) Interpret the fact that $\frac{d^2y}{dt^2} = 3$ when $t = 2$.

   (c) Based on the values of these two derivatives, is the mass speeding up or slowing down at time $t = 2$? Explain.

6. The number of bacteria in a test dish is denoted by $N$, and time $t$ is measured in hours. Write a sentence interpreting the fact that $\frac{dN}{dt}$ is 430 when $t = 5.4$. Include one of the words *increasing* or *decreasing* in your answer.

7. For this exercise, consider the beam of Examples 1.1(b) and 1.2(b). *Note that deflection downward is generally considered positive for this situation!*

   (a) Will the value of the derivative $\frac{dy}{dx}$ be positive, or negative, for points $x$ with $x > 0$?

   (b) Suppose that $0 \le x_1 < x_2 \le 10$. Which is greater, the *absolute value* of the derivative at $x_1$, or the absolute value of the derivative at $x_2$?

8. (a) Find a function $y(x)$ whose derivative is $-3$ times the original function. Is there more than one such function? If so, give another.

   (b) Find a function $y(t)$ whose second derivative is $-9$ times the original function. Is there more than one such function? If so, give another.

   (c) Find a function $x(t)$ whose second derivative is 9 times the original function. Is there more than one such function? If so, give another.

   (d) Find a function $y(x)$ whose second derivative is $-5$ times the original function. Is there more than one such function? If so, give another.

9. For each of the situations in Exercise 8, write a differential equation whose solution is the desired function. (See the second paragraph after Example 1.2(e).) *Use the given independent and dependent variables, and give your answers using Leibniz notation.*

## 1.3 Parameters and Variables

**Performance Criterion:**

1. (e) Identify parameters and variables in functions or differential equations.

If you have not recently read the explanation of the spring-mass system at the end of the last section, you should probably skim over it again before reading this section. Recall that for the spring-mass system, the independent variable is time and the dependent variable is the height of the mass. Assuming no resistance, once the mass is set in motion, it will exhibit periodic oscillation (simple harmonic motion). It should be intuitively clear that changing either the amount of the mass or the stiffness of the spring (expressed by the spring constant $k$) will change the period of oscillation. The mass $m$ and the spring constant $k$ are what we call **parameters**, and they should not be confused with the variables, which are time and the height of the mass. When working with real world mathematical models of physical systems, parameters will show up in three places:

- As characteristics of the physical systems themselves, quantified by numerical values.
- As constants within differential equations.
- As constants in the solutions to differential equations.

Let's illustrate these three manifestations of parameters using our spring-mass system. As mentioned above, the two physical parameters are the mass of the object hanging on the spring, and the stiffness of the spring, given by the spring constant. If the mass was $m = 0.5$ kg and the spring constant was $k = 8$ N/m (Newtons per meter) the differential equation would be

$$0.5\frac{d^2y}{dt^2} = -8y.$$

Here we see the two physical parameters, characteristics of the physical system, showing up in the differential equation. If we multiply both sides by two and subtract the right side from both sides we obtain

$$\frac{d^2y}{dt^2} + 16y = 0,$$

where the $16$ is the new parameter $\frac{k}{m}$, which we often rename as $\omega^2$. In this case $\omega^2 = 16\frac{1}{\sec^2}$. So the physical parameters $k$ and $m$ give us the parameter $\omega^2$ in the differential equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0. \tag{1}$$

As stated in the previous section, the most general solution to this equation is $y = C_1 \sin \omega t + C_2 \cos \omega t$. The variables are $t$ and $y$, and $C_1$, $C_2$ and $\omega$ are parameters. $\omega$ determines the period of oscillation and $C_1$ and $C_2$ determine the amplitude and phase shift. The parameter $\omega$ depends on the mass and spring constant $\left(\omega = \sqrt{\dfrac{k}{m}}\right)$ and the parameters $C_1$ and $C_2$ depend on how the mass is set in motion, by what we will call **initial conditions**. In Sections 1.4 and 1.7 we will see what initial conditions are and how they are used to determine $C_1$ and $C_2$.

We now consider the horizontal beam of Example 1.1(b). One might guess that some parameters that determine the amount of deflection of the beam would be the material the beam is made of, the thickness and cross-sectional shape of the beam (square, "I-beam," etc.), the length of the beam, and perhaps other things.

◇ **Example 1.3(a):** The differential equation, and its solution, for the beam of Example 1.1(b) are

$$EI\frac{d^4y}{dx^4} = w \qquad \text{and} \qquad y = \frac{w}{24EI}x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

where $E$ is Young's modulus of elasticity of the material the beam is made of, $I$ is the cross-sectional moment of inertia of the beam about the "neutral axis," and $w$ is the weight per unit of length. Give the variables and parameters for both the differential equation and the solution.

**Solution:** We can see from the derivative in the differential equation that the independent variable is $x$ and the dependent variable is $y$. The remaining letters all represent parameters: the modulus of elasticity $E$, the cross-sectional moment of inertia $I$, and the weight per unit of length $w$. In the solution we see these parameters again, combined as the single parameter $\frac{w}{24EI}$, along with four others, $c_0$, $c_1$, $c_2$ and $c_3$.

---

The last four parameters $c_0$, $c_1$, $c_2$, and $c_3$ in the solution will depend on the length of the beam and how it is supported, in this case by being embedded in the wall at its left end and having no support at the right end. These things are what are called **boundary conditions**. We'll discuss them more in the next section and Section 1.7, and look at specific applications involving boundary conditions in Chapter 5.

In summary, the physical parameters are variables that can change from situation to situation, but once the situation is determined the values of those parameters are constant. At that point, the only things that change are the variables. The physical parameters then show up alone, or with each other, as parameters in the differential equation modeling the physical situation. Finally, the solution to the differential equation will be some familiar type of function like an exponential function, trigonometric function or polynomial function, with its exact behavior determined by parameters that are dependent on the parameters in the differential equation and the initial or boundary conditions.

**NOTE:** *In this course we will never again refer to parameters as variables, and we will consider them distinct from the variables of interest.*

## Section 1.3 Exercises

To Solutions

1. As mentioned previously, when a solid object with some initial temperature $T_0$ is placed in a medium (like air or water) with a constant temperature $T_m$, the object's temperature $T$ will approach $T_m$ as time goes on. The rate at which the temperature of the object changes is proportional to the difference between its temperature $T$ and the temperature $T_m$ of the medium, giving us the differential equation

$$\frac{dT}{dt} = k(T_m - T),$$

where $k$ is a constant dependent on the material the object is made from.

(a) Keeping in mind that parameters are quantities that vary from situation to situation but do not change once the situation is fixed, give all of the parameters.

(b) Give the independent variable(s).

(c) Give the dependent variable.

2. Suppose that a mass on a spring hangs motionless in its equilibrium position. At some time zero it is set in motion by pulling it downward and simply releasing it, and suppose also that the mass is hanging in an oil bath that resists its motion. The independent variable is time, and the dependent variable is the height of the mass. Give as many physical parameters as you can think of for this situation - there are three or four that occur to me.

3. When dealing with certain electrical circuits we obtain the differential equation and solution

$$L\frac{di}{dt} + Ri = E \qquad \text{and} \qquad i = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right)e^{-\frac{R}{L}t}.$$

Give the independent variable, dependent variable, and all the parameters.

4. At some time a guitar string is plucked, and the dependent variable that we are interested in is the displacement of the string from its initial position.

   (a) What is(are) the independent variable(s)?

   (b) What are some physical parameters of importance?

5. Recall the situation of Example 1.1(d): A tank containing 100 gallons of water with 10 pounds of salt dissolved in the water. At some time we begin pumping a 0.3 pounds salt per gallon (of water) solution into the tank at two gallons per minute, mixing it thoroughly with the solution in the tank. At the same time the solution in the tank is being drained out at two gallons per minute as well. Our interest is the amount, in pounds, of salt in the tank at any time.

   (a) What are the independent and dependent variables, in that order?

   (b) What are the parameters?

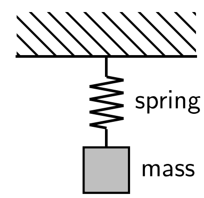
## 1.4 Initial Conditions and Boundary Conditions

**Performance Criterion:**

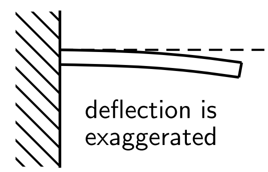
1. (f) Identify initial value problems and boundary value problems. Determine initial or boundary conditions.

Recall Examples 1.1(a) and 1.1(b):

◇ **Example 1.1(a):** Suppose that a mass is hanging on a spring that is attached to a ceiling, as shown to the right. If we lift the mass, or pull it down, and let it go, it will begin to oscillate up and down. The height $y$ of the mass (relative to some fixed reference, like its height before we lifted it or pulled it down) is a function of the time $t$ that has elapsed since we set the mass in motion.




◇ **Example 1.1(b):** Consider a beam that extends horizontally ten feet out from the side of a building, as shown to the right. The beam will deflect (sag) some, with the distance below horizontal being greater the farther out on the beam one looks.




The differential equations modeling these two situations are

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \qquad \text{and} \qquad EI\frac{d^4y}{dx^4} = w,$$

where $k$, $m$, $E$, $I$ and $w$ are physical parameters, as described in the previous section. We've already seen in Example 1.2(d) that a differential equation can have infinitely many different solutions, all of which are obtained by varying one (or more) constants. In this case, the most general solutions to the above two differential equations are

$$y = C_1 \sin \omega t + C_2 \cos \omega t \qquad \text{and} \qquad y = \frac{w}{24EI}x^4 + c_3x^3 + c_2x^2 + c_1x + c_0,$$

where $C_1$, $C_2$, $c_3$, $c_2$, $c_1$ and $c_0$ are arbitrary (meaning they can have any values) constants *differing from, and not depending on, the parameters* $k$, $m$, $E$, $I$ *and* $w$. (Remember that we are case sensitive in mathematics, science and engineering, so $C_1$ and $c_1$ are not necessarily the same value.) *Note that the number of such arbitrary constants in the solution of a differential equation is equal to the order of the differential equation.* (Again, the **order of a differential equation** is the highest order derivative in the differential equation - more on this in Section 1.6.)

The height of the mass at any time $t$ depends on the amount of the mass and the "stiffness" of the spring (given quantitatively by the spring constant $k$), but it also depends on how we set the mass in motion. It can be lifted or pulled down and let go, or given a blow, or some combination of these things. The combination that sets it in motion are what are called **initial conditions**. Suppose that we set the mass in motion by simply pulling it down by two units and then letting it go. Calling the moment we let it go time zero, we would have that

$$y = -2 \qquad \text{and} \qquad \frac{dy}{dt} = 0 \qquad \text{when} \qquad t = 0.$$

(Remember that the derivative is the velocity, so the second statement says that the mass has zero velocity at the moment we let it go.) Using function notation and the fact that the derivative is the function $y'$, this is usually expressed by

$$y(0) = -2, \quad y'(0) = 0.$$

The two numbers $-2$ and $0$ are called **initial values**, a term we will use interchangeably with initial conditions, even though the concepts are slightly different. We will see later how these two pieces of information can be used to determine the values of the constants $C_1$ and $C_2$ in the solution $y = C_1 \sin \omega t + C_2 \cos \omega t$.

Let's now think about the horizontal beam. The independent variable is $x$, the horizontal distance along the beam, measured from the wall. Time is not a variable at all; the beam deflects immediately when put into place, then retains its displacement from then on. The deflection, though, is dependent on what is going on at the two ends of the beam. At the left end the beam is what we call **embedded**. The effect of this is two things: the displacement of the beam is zero at that point, and the slope of the beam is zero right where it leaves the wall. We can express these two things by

$$y(0) = 0 \qquad \text{and} \qquad y'(0) = 0,$$

which are **boundary conditions**. The right end of the beam is "free," which is described mathematically by the boundary conditions

$$y''(10) = 0 \qquad \text{and} \qquad y'''(10) = 0.$$

We'll discuss the origin of these two conditions a bit more in Chapter 5. Altogether we have *four* boundary conditions

$$y(0) = 0, \quad y'(0) = 0, \quad y''(10) = 0, \quad y'''(10) = 0$$

which allow us to determine the four constants $c_3$, $c_2$, $c_1$ and $c_0$ in the solution

$$y = \frac{w}{24EI}x^4 + c_3x^3 + c_2x^2 + c_1x + c_0.$$

The numerical values of zero for all these derivatives at $x = 0$ and $x = 10$ are **boundary values**. As with initial values/initial conditions, we will blur the distinction between boundary values and boundary conditions. Let's now look at some more examples of initial and boundary conditions.

◇ **Example 1.4(a):** Consider the mass on the spring, set in motion by lifting it one inch and letting it go. Give the height and velocity of the mass at the time it is let go, using function notation.

**Solution:** Taking up to be positive, at time zero (the moment we set the mass in motion) the height of the mass is one inch, so we write $y(0) = 1$. Since we simply release the mass at time zero, the velocity at time zero is zero. Recalling that velocity is the first derivative of position, we can describe this by $y'(0) = 0$. The initial conditions are then $y(0) = 1$, $y'(0) = 0$.

◇ **Example 1.4(b):** Consider the mass on the spring, this time setting it in motion by hitting it downward at three inches per second from its position at rest. Give the initial conditions for the height function $y$.

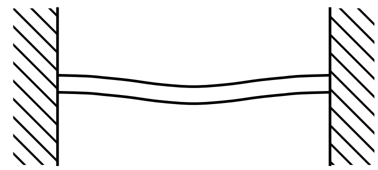
**Solution:** Because we are forcing the mass from its position at rest, its initial height is zero. This is given using function notation by $y(0) = 0$. The fact that it has downward velocity of three inches per second at time zero gives us the initial condition $y'(0) = -3$.

◇ **Example 1.4(c):** Suppose that the mass is set in motion by pulling it down two inches, then giving it an upward velocity of five inches per second to begin. Give the initial conditions for the height function $y$.

**Solution:** The initial conditions are $y(0) = -2$ and $y'(0) = 5$.

---

◇ **Example 1.4(d):** Consider a twenty foot beam that is embedded in walls at both ends, as shown to the right. The beam will deflect downward some in the middle; the deflection is exaggerated in the picture. Give the boundary conditions for the beam.



**Solution:** The boundary conditions are $y(0) = 0$, $y'(0) = 0$, $y(20) = 0$ and $y'(20) = 0$.

---

This last example warrants a bit more thought. The independent variable is the distance along the beam (most likely from the left wall) and the dependent variable is the amount of deflection downward. (Again, standard convention for this sort of problem is that down is positive.) The shape the beam takes, given by the deflections at all points, depends on the material the beam is made of, the design (cross section) of the beam, and the way that the beam is supported at the ends. One might then think that the type of support would be a parameter, but it is not. The type of support can be expressed as values of the dependent variable and some of its derivatives, and that is what distinguishes the support (boundary conditions or values) from the parameters. Similarly, the behavior of a mass on a spring is dictated in part by how it is set in motion, but that can be described by values of the dependent variable and its first derivative. Thus the way the mass is set in motion is given by initial conditions/values rather than parameters.

We conclude this section with the following remarks:

- Situations in which time is the independent variable will have initial conditions.
- Situations in which position along a line is the independent variable will have boundary conditions.
- Situations where a function depends on both position and time will have both initial conditions *and* boundary conditions. We will not see these, because they are described by **partial differential equations**.
- Partial differential equations are also required when working with boundary conditions only, when the function of interest is a function of more than one space variable. Such functions would arise when dealing with sheets or solids, rather than beams, which can be thought of as one-dimensional lines.

We will work primarily with initial conditions, but you will see boundary conditions later in the course (Section 1.7 and Chapter 5).

**Section 1.4 Exercises** To Solutions

1. In each of the following, the independent variable is given for a situation (the dependent variable should be clear), along with initial or boundary conditions, in function form. For each, give every initial or boundary condition in the form "$variable = number$ when $variable = number$."

   (a) independent variable $x$, $y(0) = 7$, $y'(0) = -3$

   (b) independent variable $t$, $x(0) = 1$, $x'(0) = 5$

   (c) independent variable $x$, $y(0) = 0$, $y''(0) = 0$, $y(15) = 0$, $y'(0) = 0$.

2. For each of the following, give the initial conditions for a mass on a spring that is set in motion in the way described. *Give the conditions using function notation, as done in Examples 1.4(a), (b) and (c).* Let the dependent variable in each case be $y$.

   (a) The mass is pulled down five units and let go with no initial velocity.

   (b) The mass is not displaced, but it is given an downward velocity of two units per second.

   (c) The mass is lifted by one unit and given an upward velocity of two units per second.

   (d) The mass is pulled down by three units and given an upward velocity of one unit per second.

3. If a mass on a spring is set in motion are there is no resistance to its vibration, it will oscillate in the same manner forever. (Resistance to its motion we will call **damping**, and we'll study its effect in Chapters 3 and 4.) Assuming such conditions, sketch the graph of the displacement of the mass at any time $t$ for each of the sets of initial conditions listed in Exercise 2. Extend your graph far enough to show at least two full periods. You will not be able to label a scale on the horizontal axis, but *for three of the cases you should be able to label the vertical axis with a scale. Take care to make sure that the graph has the correct slope where it leaves the vertical axis.*

4. There is one other condition (besides embedded or free) we'll see at the end of a beam, called **simply supported** or **pinned**. This means that the end is supported but allowed to pivot freely, as shown in the diagram below and to the right. In that case the deflection at the end is zero, and the *second* derivative of deflection is zero there as well. For each of the following scenarios, give the boundary conditions for the beam, assuming a dependent variable of $y$.

   (a) A 20 foot beam that is simply supported at its left end and embedded at its right end.

   (b) A 12 foot beam that is simply supported at both ends.


5. Suppose that we have a 70 centimeter metal rod that is perfectly insulated along the length of the rod, so that no heat can enter or leave along its length, but heat *CAN* enter or leave at its ends. We then put the rod horizontally in front of us and consider a coordinate system that puts zero at the left end of the rod and 70 cm at the right end, and we let $u(x)$ represent the temperature at any point $x$ along the length of the rod, using our coordinate system.

   Suppose also that we hold an ice cube (temperature 32° Fahrenheit at the left end and a hair dryer blowing 115° F air on the right end. Because the independent variable is a space variable $x$, this situation has boundary conditions. Give them, using function notation.

## 1.5 Differential Equations and Their Solutions

**Performance Criteria:**

1. (g) Determine the independent and dependent variables for a given differential equation.
   (h) Determine whether a function is a solution to an ordinary differential equation (ODE); determine values of constants for which a function is a solution to an ODE.

An equation that contains one or more derivatives is called a **differential equation**. Here are some examples that we will be considering:

**Equation 1:** $\dfrac{dy}{dx} + 3y = 0$ **Equation 2:** $y'' + 3y' + 2y = 0$

**Equation 3:** $y'' + 9y = 26e^{-2t}$ **Equation 4:** $15.3\dfrac{d^4y}{dx^4} = 1.4$

**Equation 5:** $\dfrac{dy}{dx} = \dfrac{x}{y}$ **Equation 6:** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial u}{\partial t}$

Note that equations 1, 2, 3 and 5 contain not only derivatives of the function $y$, but the function itself as well. (We can really think of the function as the "zeroth" derivative.)

The first five of these equations are all **ordinary differential equations**, meaning that they contain "ordinary" derivatives, which are appropriate when there is only one independent variable. The last one contains partial derivatives (which are written with the symbol $\partial$ instead of $d$) and is called a **partial differential equation**. (Some of you may have not yet taken a course in which you learn about partial derivatives.) We often use the abbreviations ODE for ordinary differential equation and PDE for partial differential equation. Video Discussion, 0:00 to 2:00

The **order** of a differential equation is the order of the highest derivative in the equation. Equations 1 and 5 above are first order, Equations 2, 3 and 6 are second order, and Equation 4 is fourth order. In this course we will focus almost entirely on ordinary differential equations, and most of the equations we will work with will be first or second order. Video Discussion, 2:00 to 3:40

When looking at a differential equation, it is often possible to determine the independent and dependent variables of interest. Derivatives are always of the dependent variable, and with respect to the independent variable (or one of the independent variables in the case of a function of more than one variable). So for Equation 1, the dependent variable is $y$ and the independent variable is $x$.

◇ **Example 1.5(a):** Give the dependent and independent variables for the rest of the equations.

**Solution:** For Equations 4 and 5 the dependent variable is $y$ and the independent variable is $x$. For equation 3 the dependent variable is $y$, and since the derivative is an ordinary derivative there must be only one independent variable, and it has to be $t$, the only other variable visible in the equation. The dependent variable in Equation 2 is $y$, and it is not possible to determine the independent variable in that case. Lastly, $u$ is the dependent variable in Equation 6. There are three independent variables, $x$, $y$ and $t$, which is why partial derivatives are required. *Any situation with more than one independent variable will result in a partial differential equation.*

Is $x = 5$ a solution to $4x - 2 = 10$? One way to answer this question is to substitute five for $x$ in the left hand side of the equation and see if it simplifies to become the right hand side. If it does, then five is a solution to the equation:

$$4(5) - 2 = 20 - 2 = 18 \neq 10, \text{ so } x = 5 \text{ is not a solution}$$

On the other hand, $x = 3$ is a solution to $4x - 2 = 10$:

$$4(3) - 2 = 12 - 2 = 10, \text{ so } x = 3 \text{ is a solution}$$

What the above shows us is that a solution to an *algebraic* equation is a number that, when substituted for the unknown value, makes the equation true. We should recall that some equations have more than one solution. For example, both $3$ and $-3$ are solutions to the equation $x^2 - 9 = 0$.

In the case of a differential equation, a solution to the equation is *NOT* a number, it is a function.

**Solution to a Differential Equation**

A **solution to a differential equation** is a function for which the function and its relevant derivatives can be substituted into the equation to obtain a true statement.

There are some differential equations whose solutions are *relations* rather than functions; we'll solve a few of those, but for all of the applications we will consider, the solutions to the ODEs modeling the situations will be functions.

When asked to verify that, or determine whether, a function is a solution to an ODE, you need to show some work supporting whatever your conclusion is. The following example shows one way to do this.

◇ **Example 1.5(b):** Show that $y = 5\cos 4t$ is a solution to $\dfrac{d^2y}{dt^2} = -16y$.

**Solution:** We compute the left hand side (LHS) and right hand side (RHS) separately:

$$\frac{dy}{dt} = -20\sin 4t \quad \Longrightarrow \quad \text{LHS} = \frac{d^2y}{dt^2} = -80\cos 4t$$

$$\text{RHS} = -16(5\cos 4t) = -80\cos 4t$$

Because LHS = RHS, $y = 5\cos 4t$ is a solution to $\dfrac{d^2y}{dt^2} = -16y$.

For the above example the left hand side was just one derivative. When the left hand side is more complicated, a standard method of verifying a solution is to first calculate any derivatives that appear on the left hand side of the equation, then substitute them into the left hand side. If the right hand side is fairly simple, we might be able to simplify the left side directly to the right hand side, as done in the next example.

◇ **Example 1.5(c):** Determine whether $y = Ce^{-2t}$, where $C$ is any constant, is a solution to the differential equation $y'' + 3y' + 2y = 0$. Another Example

**Solution:** First we see that $y' = Ce^{-2t}(-2) = -2Ce^{-2t}$ and $y'' = -2Ce^{-2t}(-2) = 4Ce^{-2t}$, so

$$\text{LHS} = 4Ce^{-2t} + 3(-2Ce^{-2t}) + 2(Ce^{-2t}) = 4Ce^{-2t} - 6Ce^{-2t} + 2Ce^{-2t} = 0 = \text{RHS}.$$

Therefore $y = Ce^{-2t}$ is a solution to $y'' + 3y' + 2y = 0$.

---

This last example shows that a differential equation can have an infinite number of solutions (since $C$ can be any real number), and we'll see the same thing in the next example as well.

◇ **Example 1.5(d):** Verify that $y = C_1 \sin 3t + C_2 \cos 3t$, where $C_1$ and $C_2$ are any constants, is a solution to $y'' + 9y = 0$.

**Solution:** First we see that $y' = 3C_1 \cos 3t - 3C_2 \sin 3t$ and $y'' = -9C_1 \sin 3t - 9C_2 \cos 3t$. Therefore

$$\text{LHS} = (-9C_1 \sin 3t - 9C_2 \cos 3t) + 9(C_1 \sin 3t + C_2 \cos 3t) = 0 = \text{RHS},$$

so $y = C_1 \sin 3t + C_2 \cos 3t$ is a solution to $y'' + 9y = 0$.

---

In this last example the function $y = C_1 \sin 3t + C_2 \cos 3t$ is a solution regardless of the values of the parameters $C_1$ and $C_2$. Because $C_1$ and $C_2$ can take any values, we say they are *arbitrary* constants. We will often use the lower case $c$ and upper case $C$ for arbitrary constants, sometimes with subscripts like above. We call all the functions obtained by letting the constants take different values a **family of solutions** for the differential equation. The solution to every first order equation will contain a constant that can take on infinitely many values, and solutions to second order equations contain two arbitrary constants, as in the above example. This may seem to contradict the result of Example 1.5(c), but the most general solution in that case is $y = C_1e^{-2t} + C_2e^{-t}$; the solution verified in that example is for the case in which $C_2 = 0$. The fact that $C_1$ and $C_2$ are subscripted differently means that they are probably, but not necessarily, different constants. Other letters will occasionally be used as constants.

In this next example you will see a situation where a function is a solution only when the parameter takes a certain value; in this case the constant (parameter) is *NOT* arbitrary.

◇ **Example 1.5(e):** Determine any values of $C$ for which $y = Ce^{-2t}$ is a solution to the differential equation $y'' + 9y = 26e^{-2t}$.

**Solution:** The derivatives of the given function are $y' = -2Ce^{-2t}$ and $y'' = 4Ce^{-2t}$. Substituting the second derivative into the left hand side of the ODE gives

$$\text{LHS} = 4Ce^{-2t} + 9Ce^{-2t} = 13Ce^{-2t}.$$

$y = Ce^{-2t}$ is a solution only if LHS = RHS, which requires that $13C = 26$. Therefore $y = Ce^{-2t}$ is a solution to the differential equation $y'' + 9y = 26e^{-2t}$ only when $C = 2$.

---

The equation $y'' + 9y = 0$ is what we will call the **homogenous equation** associated with the equation $y'' + 9y = 26e^{-2t}$. (More on this later.) $y'' + 9y = 0$ is a second order homogenous equation, and Example 1.5(d) shows that the solution to the second order homogenous equation has not one, but two, arbitrary constants. The function $y = 2e^{-3t}$ is what we call a **particular solution** to the **non-homogenous equation** $y'' + 9y = 26e^{-2t}$. A particular solution is one for which the values of constants *are not* arbitrary: The constant in this case *must* be two.

A family of solutions that has one arbitrary constant, like the family from Example 1.5(c), is often referred to as a **one-parameter family of solutions**. The parameter is the constant $C$. The family $y = C_1 \sin 3t + C_2 \cos 3t$ from Example 1.5(d) is a **two-parameter family of solutions**, with the parameters being $C_1$ and $C_2$. Solutions containing all possible arbitrary constants will be called **general solutions**.

This section has contained a lot of information! Let's summarize the important points:

- A **solution to a differential equation** is a function for which the function and its relevant derivatives can be substituted into the equation to obtain a true statement.
- Solutions to first order differential equations contain one arbitrary constant, and solutions to second order differential equations contain two arbitrary constants. All the solutions obtained by letting constants take all possible values are called **families of solutions**.
- A solution to a differential equation that contains constants that are not arbitrary is called a **particular solution** to the differential equation.
- A family that encompasses all possible solutions of a differential equation is called a **general solution** to the differential equation.
- General solutions to first order differential equations contain one arbitrary constant, and general solutions to second order differential equations contain two arbitrary constants. All the solutions obtained by letting constants take all possible values are called **families of solutions**.

In Section 1.7 we will see that if we have initial or boundary conditions along with a differential equation, the values of the arbitrary constants can be determined.

**Section 1.5 Exercises** To Solutions

1. For each of the following differential equations, determine the independent and dependent variables *when possible*. (You should always be able to identify the dependent variable.)

   (a) $\dfrac{dy}{dx} - 2y = 0$ (b) $y'' - y = 0$ (c) $\dfrac{\partial^2 u}{\partial t^2} = 3\left(\dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \dfrac{\partial^2 u}{\partial x_3^2}\right)$

   (d) $y'' + 9y = 26e^{-2t}$ (e) $\dfrac{\partial u}{\partial t} = 0.5\dfrac{\partial^2 u}{\partial x^2}$ (f) $\dfrac{d^2x}{dt^2} - 5\dfrac{dx}{dt} + 6x = 10\sin t$

   (g) $L\dfrac{d^2u}{dx^2} + g\sin u = 0$ (h) $EI\dfrac{d^4y}{dx^4} = w$ (i) $\dfrac{\partial^2 u}{\partial t^2} - c^2\left(\dfrac{\partial^2 u}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial u}{\partial r}\right) = 0$

2. (a) For part (b) of the previous exercise you should not have been able to identify the independent variable. Given that the solution is $y = 3e^{x} - 5e^{-x}$, what is the independent variable?

   (b) The differential equation $y'' - 6y' + 9y = 0$ has solution $y = C_1e^{3t} + C_2te^{3t}$. What are the independent and dependent variables?

3. Is $y = \sin 2t$ a solution to $\dfrac{dy}{dt} + 2y = 0$?

4. Is $y = 3e^{x} - 5e^{-x}$ a solution to $y'' - y = 0$?

5. (a) Verify that $y = -5e^{-3x}$ is a solution to $\dfrac{dy}{dx} + 3y = 0$.

   (b) Verify that $y = Ce^{-3x}$, where $C$ is any constant, is a solution to the same differential equation.

6. (a) Verify that $y = 2e^{-2t}$ is a solution to the differential equation $y'' + 9y = 26e^{-2t}$ and show that $y = 3e^{-2t}$ is *NOT* a solution to the same differential equation.

   (b) Verify that $y = C_1 \sin(3t) + C_2 \cos(3t) + 2e^{-2t}$ is a solution to the differential equation $y'' + 9y = 26e^{-2t}$.

7. Determine values of constants $A$ and $B$ for which $x = A\sin t + B\cos t$ is a solution to the differential equation $\dfrac{d^2x}{dt^2} - 7\dfrac{dx}{dt} + 10x = 8\sin t$.

8. Consider the differential equation $y'' - 6y' + 9y = 0$.

   (a) Verify that $y = ce^{3t}$ is a solution to the differential equation.

   (b) Verify that $y = cte^{3t}$ is a solution to the differential equation.

9. Consider the differential equation $\dfrac{dy}{dx} - y = 4e^{3x}$.

   (a) Is there any value of $c$ for which $y = ce^{x}$ a solution to the equation? If so, what is the value?

   (b) Is there any value of $c$ for which $y = ce^{3x}$ a solution to the equation? If so, what is the value?

   (c) Recall that a solution to a differential equation that cannot have an arbitrary constant in it is called a **particular solution** to the equation. Give a particular solution to the differential equation $\dfrac{dy}{dx} - y = 4e^{3x}$.

   (d) Is there any value of $c$ for which $y = ce^{x}$ a solution to the equation $\dfrac{dy}{dx} - y = 0$? If so, what is the value?

10. Consider the ODE $y'' + 3y' + 2y = 0$. For this exercise the independent variable is $t$, not $x$!

   (a) Because exponential functions are their own derivatives, it is conceivable that $y = e^{rt}$ is a solution for some *constant* value of $r$. Substitute it into the ODE.

   (b) You should be able to factor $e^{rt}$ out of the left side of your result from (a). Now $e^{u}$ is not zero for *any* value of $u$. What does this imply about our situation? (The answer to this is an equation!)

   (c) Your answer to (b) should be an equation. Solve it to determine what value(s) $r$ must have in order for $y = e^{rt}$ to be a solution. Write the solution(s) *to the ODE.*

(d) You now should have at least one solution to the ODE. Write down all the solutions you have found and check each.

11. Repeat Exercise 10 for the ODE $2y'' + 3y' + y = 0$.

12. (a) Determine whether $x = e^{-3t} \sin 2t$ is a solution to the ODE $x'' + 4x' + 13x = 0$. *Note that you will need the product rule when taking derivatives.*

    (b) Determine whether $x = e^{-2t} \cos 3t$ is a solution to $x'' + 4x' + 13x = 0$.

13. An equation of the form $ax^2y'' + bxy' + cy = 0$ is called an Euler equation. (Euler is pronounced "oiler.")

    (a) Determine whether any of $y = x$, $y = x^2$, $y = x^3$ is a solution to the Euler equation $x^2y'' - 3xy' + 4y = 0$.

    (b) Another Euler equation is $4x^2y'' + 4xy' - y = 0$. Show that $y = C_1x^{\frac{1}{2}} + C_2x^{-\frac{1}{2}}$ is a solution to this equation.

    (c) Assume that a solution to the Euler equation $x^2y'' + 4xy' + 2y = 0$ has the form $y = x^r$, for some constant $r$. Substitute into the equation and do a bit of algebra to determine any values of $r$ for which $y = x^r$ is in fact a solution.

## 1.6 Classification of Differential Equations

**Performance Criteria:**

1. (i) Classify differential equations as ordinary or partial; classify ordinary differential equations as linear or non-linear. Give the order of a differential equation.

   (j) Identify the functions $a_0(x)$, $a_1(x)$, ..., $a_n(x)$ and $f(x)$ for a linear ordinary differential equation. Classify linear ordinary differential equations as homogenous or non-homogeneous.

   (k) Write a first order ordinary differential equation in the form $\frac{dy}{dx} = F(x, y)$ and identify the function $F$. Classify first-order ordinary differential equations as separable or autonomous.

There are many different classifications and types of differential equations; we will focus on just a few classifications here. Let's consider the following examples, most of which we saw in the previous section.

**Equation 1:** $\frac{dy}{dx} + 3y = 0$ **Equation 2:** $x^2y'' + xy' + x^2y = 0$

**Equation 3:** $y'' + 9y = 26e^{-2t}$ **Equation 4:** $15.3\frac{d^4y}{dx^4} = 1.4$

**Equation 5:** $\frac{dy}{dx} = \frac{x}{y}$ **Equation 6:** $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

Here are the classifications we'll be interested in: Video Discussion

- **Ordinary differential equations** (ODEs) versus **partial differential equations** (PDEs). We have already discussed this; Equations 1 - 5 are ODEs and Equation 6 is a PDE. It is worth mentioning here that a solution to an ODE is a function of just one variable, whereas a solution to a PDE is a function of more than one variable. The solution to Equation 6 is a function $u$ of the three variables $x$, $y$ and $t$.

- Differential equations are classified by **order**, which is the highest derivative occurring in the equation. Equations 1 and 5 are first order, Equations 2, 3 and 6 are second order (PDEs are classified by order the same way that ODEs are), and Equation 4 is fourth order.

- An ODE that can be written in the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \qquad (1)$$

  where $a_0(x), ..., a_n(x)$ are functions of $x$ (possibly constants), is called a **linear** ordinary differential equation. Equations 1 through 4 are linear ODEs: Another Video Discussion

  - Equation 1 is first order linear, with $a_1(x) = 1$, $a_0(x) = 3$ and $f(x) = 0$.
  - Equation 2 is second order linear, with $a_2(x) = x^2$, $a_1(x) = x$, $a_0(x) = x^2$ and $f(x) = 0$. This particular equation is known as a Bessel equation of order zero (where "order" does not refer to the order of the ODE - how confusing!). It is obtained when working with a PDE called the **wave equation**, used for things like modeling the vibration of a drumhead.

- Equation 3 is second order linear with $a_2(t) = 1$, $a_1(t) = 0$, $a_0(t) = 9$ and $f(t) = 26e^{-2t}$. Note the variable is $t$, rather than $x$, because the independent variable in this case is $t$.
- Equation 4 is fourth order linear with $a_4(x) = 15.3$, $a_3(x) = a_2(x) = a_1(x) = a_0(x) = 0$ and $f(x) = 1.4$.

- A linear equation, so an ODE of the form (1) above, is called **homogeneous** if $f(x) = 0$. Equations 1 and 2 are homogeneous, Equations 3 and 4 are **non-homogeneous**. *One must be a bit careful, because there is another meaning of homogenous associated with ODEs!* The difference between the two uses must be determined by the context in which they are used. The definition just given is the only one we'll be using.

- If we multiply both sides of Equation 5 by $y$ we get $y\dfrac{dy}{dx} = x$, which is not of the form (1). Any effort to get the coefficient of $\dfrac{dy}{dx}$ to be a function of $x$ will fail, so Equation 5 is **non-linear**. Note that if the original equation had instead been $\dfrac{dy}{dx} = \dfrac{y}{x}$, we could multiply both sides by $x$ and subtract $y$ to get $x\dfrac{dy}{dx} - y = 0$. This equation *IS* linear, with $a_1(x) = x$, $a_0(x) = -1$ and $f(x) = 0$.

Suppose that we have a *first* order ODE with independent variable $x$ and dependent variable $y$. Such an equation can always be written in the form $\dfrac{dy}{dx} = F(x, y)$, where $F$ is simply a function of the two variables $x$ and $y$. Consider for example the Equation B below; it can be written as $\dfrac{dy}{dx} = x + 2xy$, so $F(x, y) = x + 2xy$ for that equation.

A. $\dfrac{dy}{dx} - \dfrac{x}{y} = 0$ B. $y' - 2xy = x$ C. $y' + 2y = y^2$ D. $5\dfrac{dy}{dx} - 3y = \sin x$

◇ **Example 1.6(a):** Determine the functions $F(x, y)$ for Equations A, C and D above.

**Solution:** Each of the equations can be solved for $\dfrac{dy}{dx}$ to get

A. $\dfrac{dy}{dx} = \dfrac{x}{y}$ C. $\dfrac{dy}{dx} = y^2 - 2y$ D. $\dfrac{dy}{dx} = \dfrac{3}{5}y + \dfrac{1}{5}\sin x$

We can see now that the functions $F$ for the three equations are

A. $F(x, y) = \dfrac{x}{y}$ C. $F(x, y) = y^2 - 2y$ D. $F(x, y) = \dfrac{3}{5}y + \dfrac{1}{5}\sin x$

We can now define two other categories of first order differential equations.

- When $F$ is the product of a function of $x$ and a function of $y$, written compactly as $F(x, y) = g(x)h(y)$, the ODE is called **separable**.
- When $F$ is really just a function of $y$ (so $F(x, y) = f(y)$) the ODE is called **autonomous**. Note that by letting $g(x) = 1$, any autonomous equation is also separable (but not vice-versa!).

◇ **Example 1.6(b):** Determine whether any of the the equations

A. $\dfrac{dy}{dx} - \dfrac{x}{y} = 0$ B. $y' - 2xy = x$ C. $y' + 2y = y^2$ D. $5\dfrac{dy}{dx} - 3y = \sin x$

are separable or autonomous.

**Solution:** The only one of the equations that can be written in the form $\dfrac{dy}{dx} = f(y)$ is C, so it is autonomous (and therefore separable as well). For Equation A, $F(x, y) = \dfrac{x}{y} = x \cdot \dfrac{1}{y}$, so it is separable, with $g(x) = x$ and $h(y) = \dfrac{1}{y}$. For Equation B, $F(x, y) = x + 2xy = x(1 + 2y)$, so it is also separable, with $g(x) = x$ again and $h(y) = 1 + 2y$. In the case of Equation D $F(x, y) = \frac{3}{5}y + \frac{1}{5}\sin x$, which is clearly not just a function of $y$, so it is not autonomous. We also see that it is not possible to write $F(x, y)$ in the form $g(x)h(y)$ either, so it is also not separable.

---

We will see the significance of separable and autonomous equations later. For now we should note a bit of algebra that can be performed with a separable equation. To begin with, we need to think of the derivative $\dfrac{dy}{dx}$ as being the quotient of the two **differentials** $dy$ and $dx$. Treating each like we would a variable, when we are working with a separable equation we can get all the $x$ "stuff" on one side of the equation and the $y$ "stuff" on the other side:

$$\begin{aligned} \frac{dy}{dx} - \frac{x}{y} &= 0 \\ \frac{dy}{dx} &= \frac{x}{y} \\ dy &= \frac{x}{y}\,dx \\ y\,dy &= x\,dx \end{aligned}$$

Separable equations are often easy to find solutions for, as we'll do in Chapter 2, and computations like the above will be part of the process.

**Section 1.6 Exercises** To Solutions

1. List the letters of all the following that are ordinary differential equations. Assume that any letters not used in derivatives represent constants except $w(x)$ is a function of $x$.

(a) $\dfrac{dy}{dx} - 2y = 0$ (b) $y'' - y = 0$ (c) $\dfrac{\partial^2 u}{\partial t^2} = 3\left(\dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \dfrac{\partial^2 u}{\partial x_3^2}\right)$

(d) $y'' + 9y = 26e^{-2t}$ (e) $\dfrac{\partial u}{\partial t} = 0.5\dfrac{\partial^2 u}{\partial x^2}$ (f) $\dfrac{d^2x}{dt^2} - 5\dfrac{dx}{dt} + 6x = 10\sin t$

(g) $L\dfrac{d^2u}{dx^2} + g\sin u = 0$ (h) $EI\dfrac{d^4y}{dx^4} = w(x)w($ (i) $u_{tt} - c^2(u_{rr} + \frac{2}{r}u_r) = 0$

2. Give the order of each of the following ordinary differential equations. Assume that any letters not used in derivatives represent constants.

(a) $\dfrac{dy}{dt} - 2y = 0$

(b) $y'' - y = 0$

(c) $\dfrac{1}{y}\dfrac{dy}{dx} + y = 1$

(d) $y'' + 9y = 26e^{-2t}$

(e) $\dfrac{1}{x}\dfrac{dy}{dx} + y = 1$

(f) $\dfrac{d^2x}{dt^2} - 5\dfrac{dx}{dt} + 6x = 10\sin t$

(g) $L\dfrac{d^2u}{dx^2} + g\sin u = 0$

(h) $EI\dfrac{d^4y}{dx^4} = w$

(i) $\dfrac{dy}{dx} + xy = 1$

3. For each of the first order equations from Exercise 2, give the function $F$ if the the equation was to be written in the form $\dfrac{dy}{dx} = F(x, y)$. (Use the appropriate variables for the equation.)

4. For each of the ODEs from Exercise 2 that are linear, give the values of the functions $f$, $a_0$, $a_1$, $a_2$, ... (Include the independent variable, like $a_1(x) = x^2$, for example.) If the independent variable cannot be determined, use $x$.

5. For each of the first order equations from Exercise 2 that are separable, give the functions $g$ and $h$, using the appropriate independent variable.

6. Which of the first order equations from Exercise 2 are autonomous?

## 1.7 Initial Value Problems and Boundary Value Problems

**Performance Criterion:**

1. (l) Determine whether a function satisfies an initial value problem (IVP) or boundary value problem (BVP); determine values of constants for which a function satisfies an IVP or BVP.

### Initial Value Problems

Consider again the ODE $\frac{dy}{dx} + 3y = 0$, for which any function of the form $y = Ce^{-3x}$ is a solution. Suppose we impose the additional condition that $y = 4$ when $x = 0$. This is called an **initial condition** and we often write such a condition in the form $y(0) = 4$, where the number four is called an **initial value**. (As discussed in Section 1.4, we will often blur the distinction between initial conditions and initial values.) Substituting these values into $y = Ce^{-3x}$ gives $4 = Ce^{-3(0)}$, leading to $C = 4$.

When we combine a differential equation with one or more initial values, we have what is called an **initial value problem** (IVP). The solution to an initial value problem is a function or equation that satisfies *both* the differential equation and the initial value(s). Thus $y = 4e^{-3x}$ is a solution to the IVP

$$\frac{dy}{dx} + 3y = 0\,, \qquad y(0) = 4$$

The term "initial" implies "starting," so the independent variable for initial value problems is often time. To be a solution to an initial value problem means the following:

**Solution to an Initial Value Problem**

A **solution to an initial value problem** is a function that is a solution to the differential equation and that satisfies all of the initial conditions.

◇ **Example 1.7(a):** Verify that $y = \frac{7}{2}e^{-5t} + \frac{5}{2}\sin t - \frac{1}{2}\cos t$ is the solution to the initial value problem

$$\frac{dy}{dt} + 5y = 13\sin t, \qquad y(0) = 3$$

**Solution:** $\frac{dy}{dt} = -\frac{35}{2}e^{-5t} + \frac{5}{2}\cos t + \frac{1}{2}\sin t$, so

$$\begin{aligned}
\text{LHS} &= -\tfrac{35}{2}e^{-5t} + \tfrac{5}{2}\cos t + \tfrac{1}{2}\sin t + 5\left(\tfrac{7}{2}e^{-5t} + \tfrac{5}{2}\sin t - \tfrac{1}{2}\cos t\right) \\
&= -\tfrac{35}{2}e^{-5t} + \tfrac{5}{2}\cos t + \tfrac{1}{2}\sin t + \tfrac{35}{2}e^{-5t} + \tfrac{25}{2}\sin t - \tfrac{5}{2}\cos t \\
&= \tfrac{26}{2}\sin t \\
&= 13\sin t \\
&= \text{RHS}
\end{aligned}$$

This shows that the function satisfies the differential equation. We must now show that the function satisfies the initial condition. When $t = 0$,

$$y = \tfrac{7}{2}e^{-5(0)} + \tfrac{5}{2}\sin 0 - \tfrac{1}{2}\cos 0 = \tfrac{7}{2} + 0 - \tfrac{1}{2} = \tfrac{6}{2} = 3,$$

so the function satisfies the initial condition also. Therefore it is a solution to the IVP.

---

We should observe in the above example that the process of checking the initial condition is easier than checking the differential equation; this is often the case.

◇ **Example 1.7(b):** Determine whether $y = 5e^{-3t} - 2e^{3t}$ is a solution to the initial value problem

$$y'' - 9y = 0\,, \qquad y(0) = 3\,, \qquad y'(0) = -8$$

**Solution:** This time let's check the initial conditions first. We see that $y(0) = 5 - 2 = 3$, so the first initial condition is met. We next compute $y'(t) = -15e^{-3t} - 6e^{3t}$, so $y'(0) = -15 - 6 \neq -8$, So the second initial condition is not met. Therefore the function is NOT a solution to the IVP.

---

Let's reiterate that in order to be a solution to an IVP, the function must satisfy *BOTH* the ODE and the initial conditions. Since the function in this last example failed to satisfy one of the initial conditions, it doesn't matter whether it satisfies the ODE or not (it does in this case), it still fails to satisfy the IVP.

We will now see how initial values or boundary values are used to determine the values of arbitrary constants for solutions to ODEs for which we also know initial or boundary conditions.

◇ **Example 1.7(c):** It can be shown that Another Example

$$x = C_1e^{-t} + C_2e^{-3t} + 2\sin t \tag{1}$$

is the general solution to the differential equation $x'' + 4x' + 3x = 4\sin t + 8\cos t$. Find the values of $C_1$ and $C_2$ for which the function also satisfies the initial conditions

$$x(0) = -2, \qquad x'(0) = 5.$$

**Solution:** First we can substitute $t = 0$ and $x = -2$ into $(1)$ to get

$$-2 = C_1 + C_2. \tag{2}$$

Next we compute the derivative of $(1)$ to get $x' = -C_1e^{-t} - 3C_2e^{-3t} + 2\cos t$. Substituting $t = 0$ and $x' = 5$ into this gives us

$$5 = -C_1 - 3C_2 + 2. \tag{3}$$

We can simply add $(2)$ and $(3)$ at this point to get

$$3 = -2C_2 + 2,$$

which can then be solved to obtain $C_2 = -\frac{1}{2}$. We then substitute this value into $(2)$ and solve to obtain $C_1 = -\frac{3}{2}$.

---

Note that the above shows that

$$x = -\tfrac{3}{2}e^{-t} - \tfrac{1}{2}e^{-3t} + 2\sin t$$

is a solution to the initial value problem

$$x'' + 4x' + 3x = 4\sin t + 8\cos t, \qquad x(0) = -2,\ x'(0) = 5.$$

**Boundary Value Problems**

When the independent variable we are working with is distance along a line, rather than time, we have **boundary conditions** rather than initial conditions. An example of such a situation occurs when we model the deflection of a horizontal beam. Often when we have boundary conditions they are given at two different values of the independent variable. The numerical values of the dependent variable that describe the boundary conditions are called **boundary values**. A solution to a boundary value problem is a function that satisfies both the differential equation and the boundary conditions.

◇ **Example 1.7(d):** Determine whether $y = C\cos\frac{2}{3}x$ is a a solution to the boundary value problem

$$y'' + \tfrac{4}{9}y = 0, \qquad y'(0) = 0,\ y'(3\pi) = 0.$$

**Solution:** First we see that

$$y = C\cos\tfrac{2}{3}x \qquad \Longrightarrow \qquad y' = -\tfrac{2}{3}C\sin\tfrac{2}{3}x \qquad \Longrightarrow \qquad y'' = -\tfrac{4}{9}C\cos\tfrac{2}{3}x,$$

from which we get

$$y'' + \tfrac{4}{9}y = -\tfrac{4}{9}C\cos\tfrac{2}{3}x + \tfrac{4}{9}C\cos\tfrac{2}{3}x = 0,$$

so $y = C\cos\frac{2}{3}x$ is a solution to the differential equation. Noting that we found $y'$ above, we have

$$y'(0) = -\tfrac{2}{3}C\sin\tfrac{2}{3}(0) = 0 \quad \text{and} \quad y'(3\pi) = \tfrac{2}{3}C\sin\tfrac{2}{3}(3\pi) = \tfrac{2}{3}C\sin 2\pi = 0.$$

These show that $y = C\cos\frac{2}{3}x$ also satisfies the boundary condtions, so it is indeed a solution to the boundary value problem.

---

◇ **Example 1.7(e):** Consider the boundary value problem

$$y'' + \tfrac{1}{4}y = 0, \qquad y(0) = 3,\ y(\pi) = -4. \tag{2}$$

Show that

$$y = C_1\sin\tfrac{1}{2}x + C_2\cos\tfrac{1}{2}x \tag{3}$$

is a solution to the ODE $y'' + \frac{1}{4}y = 0$. Then find values of $C_1$ and $C_2$ for which the function (3) satisfies the boundary conditions $y(0) = 3,\ y(\pi) = -4$.

**Solution:** It is easy to calculate

$$y' = \tfrac{1}{2}C_1\cos\tfrac{1}{2}x - \tfrac{1}{2}C_2\sin\tfrac{1}{2}x \quad \text{and} \quad y'' = -\tfrac{1}{4}C_1\sin\tfrac{1}{2}x - \tfrac{1}{4}C_2\cos\tfrac{1}{2}x,$$

leading to

$$y'' + \tfrac{1}{4}y = (-\tfrac{1}{4}C_1\sin\tfrac{1}{2}x - \tfrac{1}{4}C_2\cos\frac{1}{2}x) + \tfrac{1}{4}(C_1\sin\tfrac{1}{2}x + C_2\cos\tfrac{1}{2}x) = 0,$$

so $y = C_1\sin\frac{1}{2}x + C_2\cos\frac{1}{2}x$ is a solution to $y'' + \frac{1}{4}y = 0$.

For the boundary condition $y(0) = 3$ we substitute $x = 0$ and $y = 3$ into (3) to get

$$3 = C_1\sin\tfrac{1}{2}(0) + C_2\cos\tfrac{1}{2}(0).$$

This gives us $C_2 = 3$. Substituting $x = \pi$ and $y = -4$ into (3) gives us $C_1 = -4$.

We now know that $y = -4\sin\frac{1}{2}x + 3\cos\frac{1}{2}x$ is a solution to the boundary value problem (2).

---

**Section 1.7 Exercises** To Solutions

1. Verify that $y = -\dfrac{1}{x} + 3$ is a solution to the initial value problem

$$x^2\frac{dy}{dx} = 1, \qquad y(1) = 2$$

2. Determine whether $y = 2\sin(3t) + e^{-2t}$ is a solution to the initial value problem

$$y'' + 9y = 13e^{-2t}, \qquad y(0) = 1, \qquad y'(0) = 4$$

3. For each of the following, determine whether the given function is a solution to the initial value problem that is given after it. *If it is not, tell why not.*

   (a) $y = \frac{1}{4}e^{-x} + \frac{1}{2}e^{2x} + \frac{3}{4}e^{3x}$ IVP: $y'' - y' - 2y = e^{3x}, \quad y(0) = \frac{3}{2}, \quad y'(0) = 1$

   (b) $y = 3e^{-x} + \frac{1}{2}\sin x$ IVP: $y' + y = \sin x, \quad y(0) = 3$

   (c) $y = \frac{5}{2}e^{x^2} - \frac{1}{2}$ IVP: $\dfrac{dy}{dx} - 2xy = x, \quad y(0) = 2$

   (d) $x = 2\sin 2t + 3\cos 2t$ IVP: $\dfrac{d^2x}{dt^2} + 4x = 0, \quad x(0) = 3, \quad x'(0) = 4$

4. (a) $y = Ce^{-2t} + 3\cos 2t$ is the general solution to the ODE $y' + 2y = 6\cos 2t - 6\sin 2t$. Determine the solution to the initial value problem

$$y' + 2y = 6\cos 2t - 6\sin 2t, \qquad y(0) = 5.$$

   (b) $x = C_1e^{-t} + C_2e^{-4t} + 3t + 1$ is the general solution to the ODE $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + 5x = 12t + 19$. Find the solution to the IVP

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 5x = 12t + 19, \qquad x(0) = -2, \; x'(0) = 1.$$

   (c) $y = A\sin\sqrt{5}\,t + B\cos\sqrt{5}\,t$ is the general solution to the ODE $y'' + 5y = 0$. Find the solution to the initial value problem

$$y'' + 5y = 0\ , \qquad y(0) = -3, \; y'(0) = \tfrac{2}{3}.$$

   (d) $y = C_1\sin 2t + C_2\cos 2t + e^{-3t}$ is the general solution to the ODE $y'' + 4y = 13e^{-3t}$. Find the solution to the initial value problem

$$y'' + 5y = 13e^{-3t}\ , \qquad y(0) = 7, \; y'(0) = -4.$$

5. We showed in Example 1.7(e) that $y = C_1 \sin \frac{1}{2}x + C_2 \cos \frac{1}{2}x$ is a solution to the differential equation $y'' + \frac{1}{4}y = 0$.

   (a) Determine values of $C_1$ and $C_2$ for which $y = C_1 \sin \frac{1}{2}x + C_2 \cos \frac{1}{2}x$ is a solution to the boundary value problem

   $$y'' + \tfrac{1}{4}y = 0, \qquad y'(0) = 1,\ y'(\pi) = 2.$$

   Note that both boundary conditions are on the first derivative!

   (b) Determine values of $C_1$ and $C_2$ for which $y = C_1 \sin \frac{1}{2}x + C_2 \cos \frac{1}{2}x$ is a solution to the boundary value problem

   $$y'' + \tfrac{1}{4}y = 0, \qquad y(0) = 5,\ y'(2\pi) = -3.$$

6. In each of the following, a boundary value problem and function are givne. In each case, determine whether the function is a solution to the boundary value problem (see Example 1.7(d)). If it is not a solution, tell why not.

   (a) **BVP:** $y'' + \frac{\pi^2}{25}y = 0, \;\; y(0) = 0,\ y'(5) = 0$ **Function:** $y = C \sin \frac{\pi}{5}x$

   (b) **BVP:** $y'' + \frac{\pi^2}{25}y = 0, \;\; y(0) = 0,\ y(5) = 0$ **Function:** $y = C \sin \frac{\pi}{5}x$

   (c) **BVP:** $y'' + \frac{25}{4}y = 0, \;\; y(0) = 0,\ y'(\pi) = 0$ **Function:** $y = C \sin \frac{5}{2}x$

   (d) **BVP:** $y'' + \frac{\pi}{14}y = 0, \;\; y'(0) = 0,\ y(7) = 0$ **Function:** $y = C \cos \frac{\pi}{14}x$

   (e) **BVP:** $y'' + \frac{\pi^2}{25}y = 0, \;\; y'(0) = 0,\ y'(10) = 0$ **Function:** $y = C \cos \frac{\pi}{5}x$

   (f) **BVP:** $y'' + \frac{9}{25}y = 0, \;\; y(0) = 0,\ y'(5\pi) = 0$ **Function:** $y = C \cos \frac{3}{5}x$

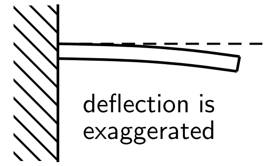
## 1.8 Chapter 1 Summary

- For our purposes, a function is a dependent variable that depends on one or more independent variables. (For most of this course we are concerned only with functions of one independent variable.)
- The values of the independent variable for which values of the dependent variable are obtained are called the domain of the function.
- The graph of a function gives us a quick way to determine the general behavior (sometimes called the qualitative behavior) of the function.
- The derivative of a function gives the rate of change of the dependent variable with respect to the independent variable.
- Exponential functions are essentially their own derivatives (of any order). Sine and cosine are essentially their own second derivatives. "Essentially" means that the function and the derivative differ only by a factor of (multiplication by) a constant.
- Differential equations are equations containing derivatives of a function. When the function is a function of one variable, the differential equation is an ordinary differential equation; when the function is a function of more than one variable, the differential equation is a partial differential equation.
- Parameters are values that change from situation to situation, but that do not change once the situation is set. Variables are values that change once the situation is set. Another way of looking at this is that parameters are characteristics of the physical system, variables are quantities that vary within the physical system.
- When examining a situation in which time is the independent variables, we generally have initial conditions, which describe the state of the dependent variable at time zero (or some other point in time).
- When considering a situation in which position in space (or along a rod, on a surface) is(are) the independent variable(s), we have values of the dependent variable on the boundary of our domain. These are called boundary values.
- Even though they are slightly different, we use initial values and initial conditions synonymously, and the same for boundary values and boundary conditions. Technically, initial conditions are physical states of the system at time zero and initial values are numerical values describing those states. A similar distinction holds for boundary conditions and boundary values.
- A solution to a differential equation is a function (or relation) for which the function(relation) and its relevant derivatives can be substituted into the equation to obtain a true statement.
- When a solution contains arbitrary constants we call it a family of solutions. A family that includes all possible solutions to a differential equation is called a general solution; a solution that contains no arbitrary constants is called a particular solution.
- General solutions to first order differential equations contain one arbitrary constant, and general solutions to second order differential equations contain two arbitrary constants.
- To verify that a function is a solution to a differential equation, we substitute the function and its derivative into the left side and see if the result is the right side, *OR* we substitute the function and its derivatives into both sides and see if the results are equal.

- We classify ordinary (and partial) differential equations:
  - The order of a differential equation is the order of the highest derivative in the equation.
  - In addition to classifying ordinary differential equations by order, we also classify them as linear or non-linear.
  - Linear ODEs are classified as homogeneous or non-homogeneous.
  - First order ordinary differential equations can also be classified as separable (or not), and autonomous (or not).
- Values of arbitrary constants are determined by initial (or boundary) conditions. For a first order equation, one initial condition is needed to determine the one constant. For a second order equation, two initial (or boundary) conditions are needed to determine the two constants.
- A differential equation together with either initial values or boundary values is called an initial value problem or a boundary value problem.
- Recognizing the classification(s) of an ordinary differential equation is important for knowing how to solve the equation.

## 1.9 Chapter 1 Exercises

1. For the ten foot beam in Example 1.1(b), shown to the right, the boundary value problem modeling the situation is given below. In this exercise you will solve the boundary value problem.




$$\frac{d^4y}{dx^4} = 2, \qquad y(0) = 0,\ y'(0) = 0,\ y''(10) = 0,\ y'''(10) = 0$$

(a) The original equation can be written $y^{(4)} = 2$. Given that $y^{(4)}$ is the derivative of $y^{(3)} = y'''$, what must $y^{(3)}$ look like? *Your answer should include a constant.*

(b) Use the boundary condition $y'''(10) = 0$ to determine the constant from your answer to (a). Write the function $y'''$ with that value substituted into your answer to (a).

(c) Now that you know $y'''$, you can use the fact that it is the derivative of $y''$ to find what $y''$ looks like. It contains a constant - use the boundary condition $y''(10) = 0$ to find the value of that constant. Then give $y''$.

(d) Repeat to find $y'$, and then $y$.

2. In this exercise you will solve what is perhaps the easiest boundary value problem there is. Remember the situation from Exercise 5 of Section 1.4: A 70 centimeter metal rod is perfectly insulated along the length of the rod, so that no heat can enter or leave along its length, but heat *CAN* enter or leave at its ends. We impose a one-dimensional coordinate system that puts zero at the left end of the rod and 70 cm at the right end, and we let $u(x)$ represent the temperature at any point $x$ along the length of the rod, using our coordinate system. The temperature at the left end of the rod is held at a constant temperature of 32° Fahrenheit and the right end is held at 115° F.

We now leave the rod in this state for a very long (infinite) period of time. The temperature at each point in the rod will eventually reach a constant value, called its **equilibrium** temperature. Let $u(x)$, for $0 \le x \le 70$, represent the function giving the equilibrium temperature at every point in the rod. Physical principles of heat flow dictate that the function $u$ must satisfy the following BVP:

$$\frac{d^2u}{dx^2} = 0, \qquad u(0) = 32,\ u(70) = 115.$$

The equation $\dfrac{d^2u}{dx^2} = 0$ is called the one-dimensional **Laplace's equation** or the **steady-state heat equation**, and $u(0) = 32$ and $u(70) = 115$ are associated boundary values. Let's solve the boundary value problem!

(a) Draw a graph of what you think the equilibrium temperatures will look like. As we will always do, put the independent variable $x$ on the horizontal axis and the dependent variable $u$ (the temperature) on the vertical axis. Label each axis with its variable *and the units for that variable*.

(b) The ODE can be written as $u'' = 0$. Remembering that $u''$ is the derivative of $u'$, what sort of function must $u'$ be if it has a derivative of zero? Write an equation for $u'$.

(c) What must $u$ look like in order to have the derivative found in (b)? Write an equation for $u$ - it should contain two arbitrary constants.

(d) Substitute the first boundary condition into your answer to (b) to get an equation containing both arbitrary constants but neither of the variables $x$ or $u$. Repeat for the second boundary condition.

(e) You now have a system of two equations containing the two unknown constants. Solve the system to find their values.

(f) Give the function $u$ that is the solution to the BVP. Would the graph of this function look like what you drew for part (a)?

**NOTE:** When we have a similar problem, but for temperatures in a two-dimensional plate of metal rather than a rod, the differential equation will be the two-dimensional **Laplace's equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

This is a partial differential equation - we can tell by the facts that the derivatives are partial derivatives and that there are two independent variables, $x$ and $y$. You should be able to guess what the three-dimensional Laplace's equation would look like.

If we were to, instead of waiting for the temperatures to reach equilibrium in our rod, watch the temperature as time progressed from time zero, then time itself would be an independent variable as well. The temperature $u(x,t)$ at any point and time in the rod is then a function of two variables, and it obeys the one-dimensional **heat equation**

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}.$$

Here the constant $\kappa$ is a parameter that depends on various properties of the material the rod is made of. If our object was three dimensional, the corresponding heat equation would be

$$\frac{\partial u}{\partial t} = \kappa \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right).$$

When we work with Laplace's equation or the heat equation in more than one dimension, the boundary value situation becomes more complicated, as there are infinitely many boundary points. When working with the heat equation (in any number of dimensions) there will also be infinitely many initial values. In all of these cases the boundary values and initial values are given by functions rather than constants (unless the functions happen to be constant functions!).

There are three important classes of partial differential equations, called **elliptic**, **parabolic** and **hyperbolic**. Laplace's equation and the heat equation are the standard examples of elliptic and parabolic equations, respectively. (Mathematicians love to call such examples "canonical" examples.) The canonical example of a hyperbolic equation is the **wave equation**. In three dimensions the wave equation is usually written as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

where $c$ is a constant.

# 2 First Order Equations

**Learning Outcome:**

2. Solve first order differential equations and initial value problems; set up and solve first order differential equations modeling physical problems.

   **Performance Criteria:**

   (a) Solve first order ODEs and IVPs by separation of variables.

   (b) Demonstrate the algebra involved in solving a relation in $x$ and $y$ for $y$; in particular, change $\ln|y| = f(x)$ to $y = g(x)$, showing all steps clearly.

   (c) Sketch solution curves to an ODE for different initial values. Given a set of solution curves for a first order ODE, identify the one having a given initial value.

   (d) Sketch a small portion of the direction field for a first order ODE.

   (e) Given the direction field and an initial value for a first order IVP, sketch the solution curve.

   (f) Use an integrating factor to solve a first order linear ODE or IVP.

   (g) Determine whether an ODE is autonomous.

   (h) Create a one-dimensional phase portrait for an autonomous ODE.

   (i) Determine critical points/equilibrium solutions of an autonomous ODE, and identify each as stable, unstable or semi-stable.

   (j) Sketch solution curves of an autonomous ODE for various initial values.

   (k) Solve an applied problem modeled by a first order ODE using separation of variables or an integrating factor.

   (l) Give an ODE or IVP that models a given physical situation involving growth or decay, mixing, Newton's Law of Cooling or an RL circuit.

   (m) Sketch the graph of the solution to a mixing or Newton's Law of Cooling problem, indicating the initial value and the steady-state asymptote.

   (n) Identify the transient and steady-state parts of the solution to a first order ODE.

In the first chapter we found out what what ordinary differential equations (ODEs), initial value problems (IVPs) and boundary value problems (BVPs) are, and what it means for a function to be a solution to an ODE, IVP or BVP. We then saw how to determine whether a function is a solution to an ODE, IVP or BVP, and we looked at a few "real world" situations where ODEs, BVPs and IVPs arise from physical principles.

Our goal for the rest of the course is to solve ODEs, IVPs and BVPs and to see how the ODEs, IVPs, BVPs and their solutions apply to real situations. We can "solve" ODEs (and PDEs) in three ways:

- Analytically, which means "paper and pencil" methods that give exact solutions in the form of algebraic equations.

- Qualitatively, which means determining the general behavior of solutions without actually finding function values. Results of qualitative methods are often expressed graphically.
- Numerical methods which result in values of solutions only at discrete points in time or space. Results of numerical methods are often expressed graphically or as tables of values.

In this chapter you will learn how to find solutions qualitatively and analytically for first order ODEs and IVPs. (Numercial methods are discussed in Appendix C.) You will see two analytical methods, **separation of variables** and the **integrating factor** method.

- Separation of variables is the simpler of the two methods, but it only works for separable ODEs, which you learned about in Section 1.6. It is a useful method to look at because when it works it is fairly simple to execute, and it provides a good opportunity to review integration, which we will need for the other method as well.
- Solving with integrating factors is a method that can be used to solve any *linear* first order ODE, whether it is separable or not, as long as certain integrals can be found. The method of solution is more complicated than separation of variables, but not necessarily any more difficult to execute once you learn it.

After learning these two methods we will again look at applications, but only for first order ODEs and IVPs at this time.

## 2.1 Solving By Separation of Variables

**Performance Criteria:**

2. (a) Solve first order ODEs and IVPs by separation of variables.
   (b) Demonstrate the algebra involved in solving a relation in $x$ and $y$ for $y$; in particular, change $\ln|y| = f(x)$ to $y = g(x)$, showing all steps clearly.

So far you have learned how to determine whether a function is a solution to a differential equation, initial value problem or boundary value problem. But the question remains, *"How do we find solutions to differential equations?"* We will spend much of the course learning some analytical methods for finding solutions. If the ODE is separable, we can apply the simplest method for solving differential equations, called **separation of variables**. The bad news is that separation of variables only "works" for separable (so necessarily also first order) equations; the good news is that those sorts of equations actually occur in some "real life" situations. Let's look at an example of how we solve a separable equation. Video Discussion

◇ **Example 2.1(a):** Solve the differential equation $y' - \dfrac{6\sin 3x}{y} = 0$. Another Example

**Solution:** Note that we can write the ODE as $\dfrac{dy}{dx} = 6\sin 3x \cdot \dfrac{1}{y} = g(x)h(y)$, where $g(x) = 6\sin 3x$ and $h(y) = \dfrac{1}{y}$. (It doesn't really matter where the $6$ is, it can be included in either $g$ or $h$.) Therefore the ODE is separable; let's separate the variables and solve:

| | |
|---|---|
| $y' - \dfrac{6\sin 3x}{y} = 0$ | The original equation. |
| $\dfrac{dy}{dx} = \dfrac{6\sin 3x}{y}$ | Change to $\frac{dy}{dx}$ notation and get the term with the derivative alone on one side. |
| $dy = \dfrac{6\sin 3x}{y}\,dx$ | Multiply both sides by $dx$. |
| $y\,dy = 6\sin 3x\,dx$ | Do some algebra to get all the "$x$ stuff" on one side and the "$y$ stuff" on the other. At this point the variables have been separated. |
| $\displaystyle\int y\,dy = \int 6\sin 3x\,dx$ | Integrate both sides. |
| $\frac{1}{2}y^2 + C_1 = -2\cos 3x + C_2$ | Compute the integrals. |
| $\frac{1}{2}y^2 = -2\cos 3x + C$ | Subtract $C_1$ from both sides and let $C = C_2 - C_1$. DO NOT solve for $y$ unless asked to. |

The resulting solution for the above example is not a function, but is instead a *relation*. In some cases we will wish to solve for $y$ as a function of $x$ (or whatever other variables we might be using), but you should only do so when asked to.

In the next example you will see a simple, but a very useful, type of differential equations.

◇ **Example 2.1(b):** Solve the differential equation $\frac{dy}{dt} + 0.5y = 0$ by separation of variables, and solve the result for $y$.

**Solution:** First let's solve the ODE by separation of variables:

$$\begin{aligned} \frac{dy}{dt} + 0.5y &= 0 \\ \frac{dy}{dt} &= -0.5y \\ dy &= -0.5y\,dt \\ \frac{dy}{y} &= -0.5\,dt \\ \int \frac{dy}{y} &= \int -0.5\,dt \\ \ln|y| + C_1 &= -0.5t + C_2 \\ \ln|y| &= -0.5t + C_3 \end{aligned}$$

where $C_3 = C_2 - C_1$. We now solve for $|y|$, using the facts that the inverse of the natural logarithm is the exponential function with base $e$ and if $|x| = u$, then $x = \pm u$ (the definition of absolute value):

$$\begin{aligned} \ln|y| &= -0.5t + C_3 && \\ e^{\ln|y|} &= e^{-0.5t+C_3} && \text{take } e \text{ to the power of each side} \\ |y| &= e^{-0.5t}\,e^{C_3} && \text{inverse of natural log and } x^a x^b = x^{a+b} \\ |y| &= C_4 e^{-0.5t} && e^{C_3} \text{ is just another constant, which we call } C_4 \\ y &= \pm C_4 e^{-0.5t} && \text{the definition of absolute value} \\ y &= Ce^{-0.5t} && \text{“absorb” the } \pm \text{ into } C_4, \text{ calling the result } C \end{aligned}$$

---

The last step above might seem a bit “fishy,” but it is valid. In most cases we have initial values, which then determine the constant $C$, including its sign:

◇ **Example 2.1(c):** Solve the initial value problem $\frac{dy}{dt} + 0.5y = 0, \quad y(0) = 7.3.$

**Solution:** We already solved the differential equation in the previous example, so we just need to find the value of the constant by substituting the initial values into the solution $y = Ce^{-0.5t}$:

$$\begin{aligned} 7.3 &= Ce^{-0.5(0)} \\ 7.3 &= C \end{aligned}$$

The solution to the IVP is $y = 7.3e^{-0.5t}$.

---

Don't assume that the the constant is always the initial value!

◇ **Example 2.1(d):** Solve the initial value problem $y' - \dfrac{6\sin 3x}{y} = 0, \quad y(0) = 4.$

**Solution:** We already solved the differential equation in Example 2.1(a), so we just need to find the value of the constant. Substituting $x = 0$ and $y = 4$ into the solution $\frac{1}{2}y^2 = -2\cos 3x + C$:

$$\begin{aligned} \tfrac{1}{2}(4)^2 &= -2\cos 3(0) + C \\ 8 &= -2 + C \\ C &= 10 \end{aligned}$$

The solution to the IVP is $\frac{1}{2}y^2 = -2\cos 3x + 10$.

---

The next, and last, example in this section illustrates something we will see again soon.

◇ **Example 2.1(e):** Solve the ODE $(x^2 + 4x - 5)y' = x + 17$.

**Solution:** The derivative $y'$ is $\dfrac{dy}{dx}$. When we separate the variables we get

$$dy = \frac{x + 17}{x^2 + 4x - 5}\, dx.$$

If we do the partial fraction decomposition of the fraction on the right side (see Example A.4(a)) we can proceed as follows:

$$\begin{aligned} dy &= \left(\frac{3}{x-1} - \frac{2}{x+5}\right) dx \\ \int dy &= \int \left(\frac{3}{x-1} - \frac{2}{x+5}\right) dx \\ \int dy &= \int \frac{3}{x-1}\, dx - \int \frac{2}{x+5}\, dx \\ y + C_1 &= 3\int \frac{dx}{x-1} - 2\int \frac{dx}{x+5} \\ y + C_1 &= 3\ln|x-1| + C_2 - 2\ln|x+5| + C_3 \end{aligned}$$

From here we can combine the constants and apply properties of logarithms to obtain

$$\begin{aligned} y &= \ln|x-1|^3 - \ln|x+5|^2 + C \\ y &= \ln\frac{|x-1|^3}{|x+5|^2} + C, \end{aligned}$$

which can also be written as

$$y = \ln\left|\frac{(x-1)^3}{(x+5)^2}\right| + C.$$

---

**Section 2.1 Exercises** To Solutions

1. Use separation of variables to solve each of the following ODEs. **Don't solve for** $y$.

   (a) $\dfrac{dy}{dx} = -x\sec y$

   (b) $dx + x^3 y\, dy = 0$

   (c) $x^2 + y^4\,\dfrac{dy}{dx} = 0$

   (d) $y' = \dfrac{y}{2x+3}$

   (e) $x^2\, dy = e^y\, dx$

   (f) $y' = \dfrac{5x+3}{y}$

2. Solve each of the following initial value problems. **DO NOT solve for** $y$**, and give constants in exact form.**

   (a) $y' = xy, \qquad y(1) = 3$

   (b) $x\dfrac{dx}{dt} + 5t = 3, \qquad x(2) = 4$

   (c) $\dfrac{dy}{dx}\dfrac{e^y}{x} = 3, \qquad y(0) = 2$

   (d) $y' = y^4 \cos t, \qquad y(0) = 2$

3. Some of the following initial value problems can be solved by separation of variables. Solve the ones that CAN be solved by that method. **DO solve for** $y$ **and give constants in exact form again.**

   (a) $\dfrac{dy}{dx} - 3y = 0\ , \quad y(0) = 4$

   (b) $x\dfrac{dy}{dx} - y = x\ , \quad y(1) = 2$

   (c) $y' - 4xy = 0\ , \quad y(0) = 2$

   (d) $y' - 2x = xy\ , \quad y(2) = 5$

   (e) $\dfrac{dy}{dx} - y = e^{3x}\ , \quad y(0) = 4$

   (f) $\dfrac{dy}{dx} = \dfrac{y-1}{x+3}\ , \quad y(1) = 3$

4. (a) Solve the initial value problem $y' - 2xe^{-y} = e^{-y}\ , \quad y(0) = 0$.

   (b) Solve the initial value problem $y' - 2xe^{-y} = e^{-y}\ , \quad y(1) = 3$. **Give the exact form for the unknown constant**.

5. (a) Solve the differential equation $y' + 2ty = 0$. You should get $\ln y = -t^2 + C$.

   (b) We now want to get $y$ as a function of $t$. "$e$" both sides of the equation and use the fact that $e^{\ln u} = u$. Use also the facts that $x^{a+b} = x^a x^b$ and $e$ raised to a constant power is yet another constant. You should now have a family of solutions to the differential equation.

   (c) Use the initial condition $y(0) = 7$ to determine the value of the arbitrary constant. You now have a solution to the initial value problem.

6. Later we will solve certain second order linear ODEs using a method called **reduction of order**. At one point in the process we will need to solve first order ODEs that will be expressed with independent variable $x$ and dependent variable $v$. An example of such an equation is the rather harmless looking equation

$$xv' + v = 0.$$

Solving this requires a bit of delicate handling - you will be led through the process in this exercise.

(a) Separate the variables, noting that $x$ cannot be zero.

(b) When integrating the right side, note that there is a negative sign that can be taken out of the integral.

(c) The result of integrating the right side is $-\ln|x|$. Apply the property of logarithms stating that $\log(u^c) = c\log u$, and combine constants as usual.

(d) "$e$" both sides and apply the fact that $|u|^r = |u^r|$ when $u^r$ is defined.

(e) Apply the fact that if $|u| = C|v|$, then $u = \pm Cv$, absorb the $\pm$ into the constant, get rid of the negative exponent and you are done!

7. Solve each of the following ODEs. You will use separation of variables and partial fraction decomposition for each.

(a) $(x^2+3x)\dfrac{dy}{dx} = 2x+3$

(b) $(x^2+3x)\dfrac{dy}{dx} = 3$

(c) $(x^2-3x-10)\dfrac{dy}{dx} = -14$ (see Exercise 2(b) from Appendix A.4 to check your partial fraction decomposition)

8. In this exercise you will solve the differential equation $\dfrac{dy}{dx} = -\frac{1}{3}y^2 + y$ with various initial values. This will lead into Sections 2.2 and 2.4, and will illustrate the sort of calculations that we must perform to solve certain applied problems related to something called the **logistic equation**. Some of these calculations are not really needed but make the expressions involved a bit simpler.

(a) Solve this system by separation of variables and partial fraction decomposition. (Be sure to begin by multiplying both sides by $-3$ to clear the fraction.) *This situation is a bit different from the other ones you've encountered, in that you will be doing the partial fraction decomposition with the dependent variable this time.*

(b) Check that your answer to (a) agrees with the solution given in the back of the book. Now "$e$" both sides, putting the right side in the form demonstrated in Example 2.1(b). As in that example, the absolute value can be removed.

(c) Now here comes a bit of algebra: Get rid of the fraction on the left by multiplying both sides by its denominator. Multiply both sides by $e^{-x}$ and then solve for $y$. This is the solution to the ODE.

(d) Determine the values of the constant, and then the solution to the corresponding initial value, for each of the following initial conditions:

$$y(0) = -\tfrac{1}{2}, \qquad y(0) = 0, \qquad y(0) = 1, \qquad y(0) = 4$$

*DO NOT give your answers as complex fractions*: Multiply the numerator and denominator both by the same value in order to eliminate smaller fractions within them.

(e) Use your calculator or a graphing utility like *www.desmos.com* to graph your solutions. (For $y(0) = -\frac{1}{2}$ we want only the part of the solution that goes through that initial value.) Sketch a single grid with all four solutions on it. We call these **solution curves** for the ODE, each corresponding to a different initial value. In Section 2.4 we will see how to obtain these curves without even solving the ODE!

(f) Remembering that $e^{-x} \to 0$ as $x \to \infty$, give the limit of each of your solutions from part (d). This should agree with what you see in the graph from (e).

(g) Attempt to determine the value of the constant for the initial condition $y(0) = 3$. What happens/what do you get?

## 2.2 Solution Curves and Direction Fields

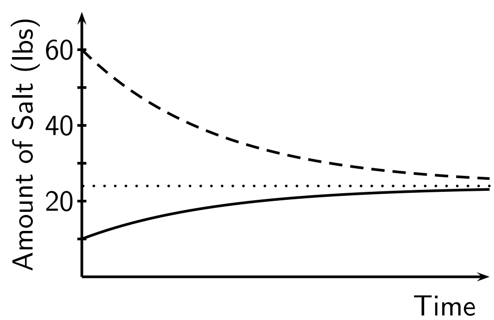
**Performance Criteria:**

2. (c) Sketch solution curves to an ODE for different initial values. Given a set of solution curves for a first order ODE, identify the one having a given initial value.
   (d) Sketch a small portion of the direction field for a first order ODE.
   (e) Given the direction field and an initial value for a first order IVP, sketch the solution curve.

Suppose that a tank contains 80 gallons of water with 10 pounds of salt dissolved in it. Fluid with a 0.3 pounds per gallon salt concentration is being pumped into the tank at a rate of 7 gallons per minute. The fluid is continually mixed and, at the same time, the fluid is being drained from the tank at a rate of 7 gallons per minute. (This is similar to the situation from Example 1.1(c).)

A quick computation reveals that the initial concentration of the solution in the tank is 0.125 pounds per gallon, less than the concentration of the fluid that is replacing it. Therefore the concentration of the fluid in the tank will increase, but it can never exceed the concentration of the incoming fluid. If all of the fluid in the tank had the concentration of the incoming fluid, there would be $(0.3)(80) = 24$ pounds of salt. If we were then to graph the amount of salt in the tank as a function of time we would get the solid curve graphed below and to the right. The limit of the amount of salt in the tank is 24 pounds, indicated by the dotted line.

Now suppose that the tank had $A_0 = 60$ pounds of salt initially, giving an initial concentration of 0.5 pounds per gallon, higher than the concentration of the incoming fluid. In this case the amount of salt in the tank will decrease, with a limit of 24 pounds again. The dashed line on the graph to the right shows the amount of salt as a function of time, for the initial amount of 60 pounds. The dotted curve is the solution curve for the initial amount of 24 pounds, and it is also the asymptote for the other solutions.




As we saw before, the situation with the tank can be modeled with a differential equation, and the general solution to that differential equation is a family of functions. The graphs of the functions in the family are called **solution curves** for the ODE. Each curve is associated with a particular initial value. The graph above shows the graphs of the solutions for the initial values 10, 24 and 60 pounds of salt in the tank. Notice that none of the solution curves cross each other; this is not always the case, but will be for most of the ODEs that we'll look at. For an initial salt amount of 15 pounds, the solution curve will lie midway between the curves for initial amounts of 10 and 24 pounds, without crossing either.

In the exercises you will use your calculator or a graphing utility to plot solution curves for various ODEs.

### Direction Fields

To obtain a graph like the one above we need to either find actual solutions to the differential equation for various initial values, or we have to have a good intuitive idea of what is happening. What if we don't have either of those two things? Well, for first order equations it is usually fairly easy to determine what solution curves look like from just the differential equation itself, as we will now see.

Consider the first order linear differential equation $\frac{dy}{dx} + y = x$. We can solve for $\frac{dy}{dx}$ to get $\frac{dy}{dx} = x - y$. Now remember that by "solving" the differential equation we mean finding a function $y = y(x)$ that makes the equation true; there are infinitely many such functions, with the graph of each representing a particular solution curve. Recall also that when considering the graph of a function, the derivative of the function at some point is the slope of the tangent line to the graph of the function at that point. So the equation $\frac{dy}{dx} = x - y$ gives us a formula for finding the slope of the tangent line to the unknown function $y(x)$ at any point $(x, y)$.

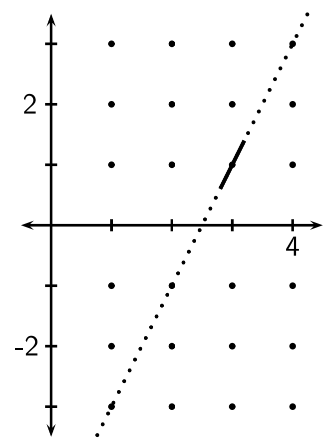
To be more specific, consider the point $(3, 1)$. The equation

$$\frac{dy}{dx} = x - y$$

tells us that at that point the slope of the tangent line to the solution curve will be

$$\left.\frac{dy}{dx}\right|_{(3,1)} = 3 - 1 = 2$$

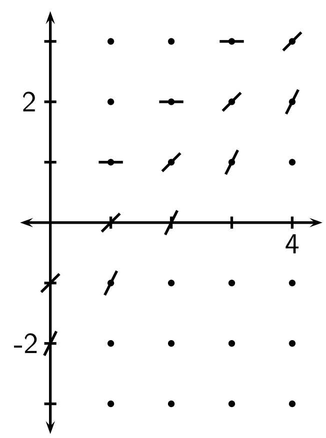
So the tangent line of the solution curve passing through the point $(3, 1)$ has slope 2 at that point. The dotted line to the right has slope two and passes through the point $(3, 1)$. We will just keep the small part of it that actually goes through the point. In the following example we continue on to find slopes at other points with integer coordinates.



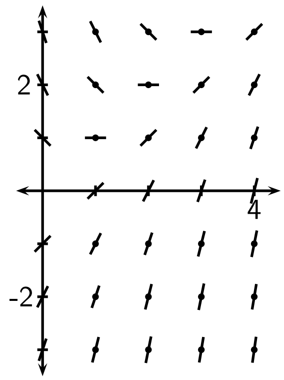

◇ **Example 2.2(a):** Find slopes for the remaining grid points, for $\frac{dy}{dx} = x - y$.

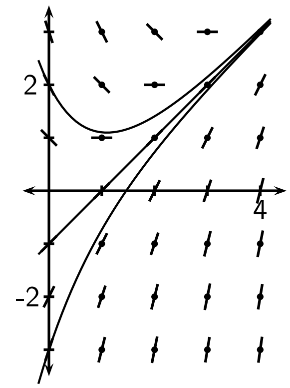
Another Example

**Solution:** It is often easiest to determine slopes not by going point to point, but to find all points where the slope is the same. For this equation, the slope will be zero at every point where $x = y$, so at $(0.0, (1, 1), (2, 2)$, and so on. Similarly, the slope will be one at all the points where $x$ is one unit larger than $y$; for the above grid those are the points $(1, 0)$, $(2, 1)$, $(3, 2)$ and $(4, 3)$. Similarly, the slope will be two at the points where $x$ is two greater than $y$: $(0, -2)$, $(1, -1)$, $(2, 0)$, $(3, 1)$ and $(4, 2)$. The slope lines for slopes zero, one and two are plotted on the grid shown to the right. The remaining slopes can be seen on the left graph at the top of the next page.




The graph of the result of what we have been doing is something called a **direction field** or **slope field**; the completed direction field can be seen to the left at the top of the next page. Direction fields are a way of studying the behavior of solutions to first order differential equations without actually solving the equations analytically. The slope lines that we have drawn in on the direction field are not all that are possible - *such a slope line exists for every point in the plane where the derivative exists.* Given a direction field and an initial value, we can sketch a solution curve by drawing a curve that starts at the initial value point and that is tangent to the "imagined" slope lines at all points that a curve goes through. To the right below you can see the solution curves corresponding to the initial values $y(0) = 2$, $y(0) = -1$ and $y(0) = -3$. Each curve is began by sketching a curve that is tangent to the slope line at the initial value, then continues to be tangent to other slope lines it passes through or near as the curve is constructed. Video Example




**Section 2.2 Exercises** To Solutions

1. The general solution to the ODE $\dfrac{dy}{dx} + y = x$ from Example 2.2(a) is $y = x - 1 + Ce^{-x}$. Find the values of the constant $C$ and graph the solution curves for each of the following initial values. Sketch each of the curves on the same grid as each other, for $-1 \le x \le 4$. Use a graphing tool if you wish.

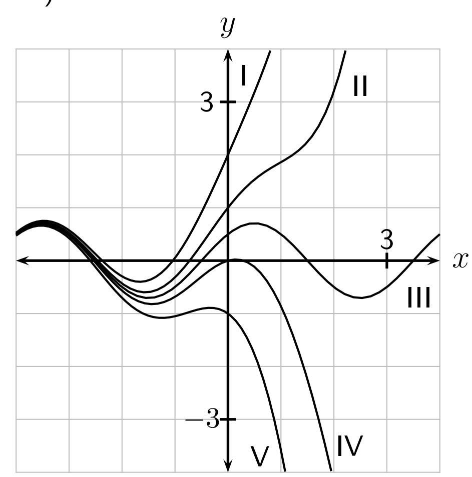
   (a) $y(0) = 2$ (b) $y(0) = 0$ (c) $y(0) = -1$ (d) $y(0) = -3$

2. On your graph from Exercise 1, sketch what you think the graphs for initial conditions $y(0) = -2$, $y(0) = 1$ and $y(0) = 3$ would look like. Then graph them with a graphing tool to check yourself. (You will need to find the values of $C$ for each to do this.)

3. The graph of some solution curves for a differential equation are shown to the right. Give the Roman numeral that corresponds to each given initial condition.

   (a) $y(0) = 1$ (b) $y(-\frac{1}{2}) = 1$

   (c) $y(1) = \frac{1}{2}$ (c) $y(0) = 0$



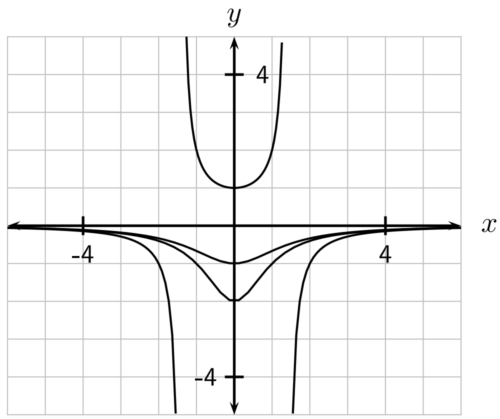

4. (a) On the grid for Exercise 3, sketch in what you think the solution curve for the initial value $y(0) = \frac{3}{4}$ would look like.

   (b) The general solution for the ODE for which some solution curves are shown in Exercise 3 is $y = \frac{1}{2}(\sin x + \cos x) + Ce^{x}$. Determine the value of $C$ for the initial value $y(0) = \frac{3}{4}$ and plot the solution curve using technology. Compare with the curve you sketched for part (a).

5. (a) The graph below shows some solutions to $\dfrac{dy}{dx} = xy^2$. Label each that you can with its initial value $y(0) =$ ____.

   (b) The solution to the ODE is $y = \dfrac{-2}{x^2 + C}$. Find a point on one of the curves for which you couldn't find an initial value and substitute it into the solution to determine the value of $C$.

   (c) Use technology to graph the solution, for the value of $C$ that you found in (b). Explain what is going on. What are the asymptotes for the parts of the graph that go out the top and bottom edges of the grid?




6. Suppose that a group of $N_0$ individuals is put in an environment that can only support $K$ individuals, and suppose that the growth rate of the population without any restrictions would be $r$ percent (in decimal form!) per year. Then the population $N$ at any time $t$ years is given by

$$N = \frac{K}{1 - (1 - K/N_0)e^{-rt}}$$

The value $N_0$ is called the **initial population**, $K$ is called the **carrying capacity**. Suppose that for some population the carrying capacity is 100 and the growth rate is 20%. Graph the functions $N$ for the initial populations below all on the same grid, for zero to forty years, using technology. Your graph will need to go up to at least 150 individuals. Sketch the graph.

(a) $N_0 = 20$ (b) $N_0 = 150$ (c) $N_0 = 100$ (d) $N_0 = 0$

7. Think about the graph you got in the previous exercise, and make sure that you understand (from a population growth point of view) why each curve looks the way it does.

8. For each ODE given, plot the direction field at integer coordinates over the values given for each variable.

   (a) $\dfrac{dx}{dt} = \dfrac{1}{2}xt, \quad -1 \le t \le 2, \ -2 \le x \le 2$

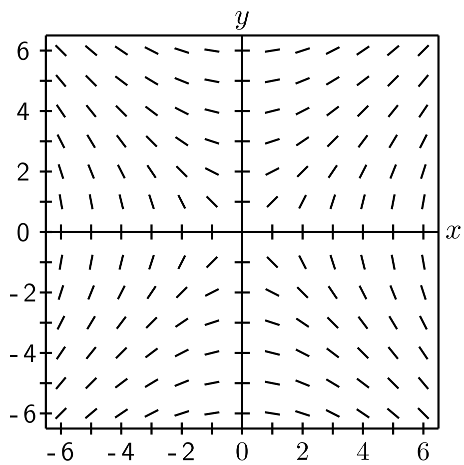
   (b) $\dfrac{dy}{dx} = x^2 - 2x, \quad 0 \le x \le 4, \ -2 \le y \le 4$

   (c) $\dfrac{dy}{dx} = y^2 - 2y, \quad 0 \le x \le 4, \ -2 \le y \le 4$

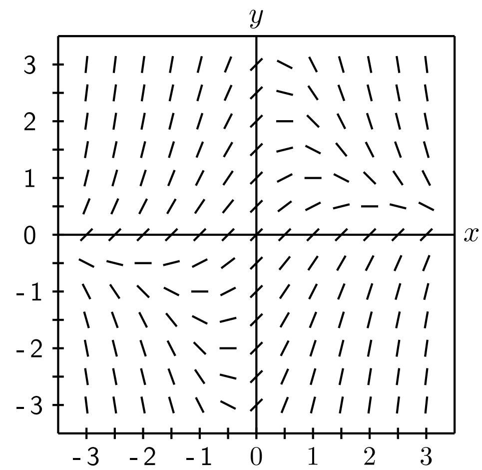
   (d) $\dfrac{dy}{dt} = y + t. \quad -2 \le t \le 2, \ -2 \le y \le 2$

9. On the direction field below and to the left, sketch the solution curves going through the given points.

(a) $(-4, 5)$ (b) $(-2, -2)$ (c) $(6, -4)$



Exercise 9



Exercise 10

10. On the direction field above and to the right, sketch the solution curves going through the given points.

(a) $(-2, 0)$ (b) $(-1.5, -1)$ (c) $(3, 0)$

## 2.3 Solving With Integrating Factors

**Performance Criterion:**

2. (f) Use an integrating factor to solve a first order linear ODE or IVP.

Let's begin with an example that demonstrates the limitation of separation of variables.

◇ **Example 2.3(a):** Solve $\dfrac{dy}{dx} - 3y = e^{5x}$.

**Solution:** Note that if we try to separate the variables we get

$$\begin{aligned} \frac{dy}{dx} - 3y &= e^{5x} \\ \frac{dy}{dx} &= 3y + e^{5x} \\ dy &= (3y + e^{5x})\, dx \end{aligned}$$

Here we see that there is no way to get the $3y$ term back over to the left side with $dy$. (This is because $3y + e^{5x}$ cannot be written in the form $g(x)h(y)$.) Therefore this equation cannot be solved by separation of variables.

---

The following derivative computation provides the key for solving equations like the one above.

◇ **Example 2.3(b):** Suppose that $y = y(x)$ is some function of $x$. Find the derivative of $ye^{-3x}$ (with respect to $x$).

**Solution:** Because both $y$ and $e^{-3x}$ are functions of $x$, we must use the product rule:

$$\frac{d}{dx}(ye^{-3x}) = y\frac{d}{dx}(e^{-3x}) + e^{-3x}\frac{d}{dx}(y) = -3ye^{-3x} + e^{-3x}\frac{dy}{dx} = e^{-3x}\left(\frac{dy}{dx} - 3y\right)$$

---

Notice that multiplying the left side of the ODE of Example 2.3(a) by $e^{-3x}$ gives the result of Example 2.3(b). This indicates an idea for solving the ODE: Video Example

$$\frac{dy}{dx} - 3y = e^{5x} \qquad \text{Original equation}$$

$$e^{-3x}\left(\frac{dy}{dx} - 3y\right) = e^{-3x}e^{5x} \qquad \text{Multiply both sides by } e^{-3x}$$

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{2x} \qquad \text{Distribute } e^{-3x} \text{ and apply } x^a x^b = x^{a+b}$$

$$\frac{d(ye^{-3x})}{dx} = e^{2x} \qquad \text{From Example 2.3(b)}$$

$$d(ye^{-3x}) = e^{2x}\, dx \qquad \text{Multiply both sides by } dx$$

$$\int d(ye^{-3x}) = \int e^{2x}\,dx \qquad \text{Integrate both sides}$$

$$ye^{-3x} + C_1 = \tfrac{1}{2}e^{2x} + C_2 \qquad \text{Carry out the integrations}$$

$$ye^{-3x} = \tfrac{1}{2}e^{2x} + C \qquad \text{Combine constants}$$

$$ye^{-3x}e^{3x} = \tfrac{1}{2}e^{2x}e^{3x} + Ce^{3x} \qquad \text{Multiply both sides by } e^{3x}$$

$$y = \tfrac{1}{2}e^{5x} + Ce^{3x} \qquad \text{Apply properties of exponents}$$

Thus the solution to $\dfrac{dy}{dx} - 3y = e^{5x}$ is $y = \tfrac{1}{2}e^{5x} + Ce^{3x}$. *The reason for multiplying both sides by* $e^{3x}$ *was to get* $y$ *alone on the left side.*

The method just shown for finding the solution to $\dfrac{dy}{dx} - 3y = e^{5x}$ probably seems a bit mysterious, to say the least! This is called the **integrating factor** method, which we now summarize. Note that it only applies to *linear* first order ODEs, which can always be put into the form $\dfrac{dy}{dx} + p(x)y = q(x)$.

**Solving a 1st Order Linear ODE Using An Integrating Factor**

To solve a first order ODE of the form $\dfrac{dy}{dx} + p(x)y = q(x)$,

1) Compute $u = \int p(x)\,dx$. The integrating factor is $e^u$ (not just $u$).
2) Multiply both sides of the equation by the integrating factor $e^u$. The left side of the differential equation then becomes $\dfrac{d(ye^u)}{dx}$.
3) Multiply both sides of the equation by $dx$ and integrate both sides. The left side will become $ye^u$.
4) Solve for $y$ by multiplying both sides by $e^{-u}$.

Note that after integrating both sides of the equation there will be a constant added to the right side. *This constant will be multiplied by* $e^{-u}$ *in the solution.* For the equation $\dfrac{dy}{dx} - 3y = e^{5x}$, $p(x) = -3$ so $u = \int p(x)\,dx = -3\int dx = -3x$ and $e^u = e^{-3x}$.

*Any* first order linear ODE can be solved using the integrating factor method, as long as $p(x)$ and $e^u q(x)$ can be integrated; sometimes you can use either this method or separation of variables and they both will work. Now let's take a look at executing the above steps with another example.

◇ **Example 2.3(c):** Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$ for $x > 0$ by the integrating factor method.

**Solution:** First we note that $p(x) = \dfrac{1}{x}$ and $q(x) = x^2$. Because $x > 0$, $|x| = x$ and $u = \int \frac{1}{x}\,dx = \ln x$. Therefore $e^u = e^{\ln x} = x$. We now carry out steps (2) through (4) above, as shown at the top of the next page.

$$\begin{aligned}
\frac{dy}{dx} + \frac{y}{x} &= x^2 && \text{original equation} \\
x\frac{dy}{dx} + y &= x^3 && \text{multipy both sides by } e^u, \text{ which in this case is } x \\
\frac{d(xy)}{dx} &= x^3 && \text{use the product rule "in reverse" to "collapse" the left side} \\
\int d(xy) &= \int x^3\,dx && \text{multiply both sides by } dx \text{ and integrate} \\
xy &= \tfrac{1}{4}x^4 + C && \text{include a single constant of integration on the right side} \\
y &= \tfrac{1}{4}x^3 + \frac{C}{x} && \text{multiply both sides by } e^{-u} = \frac{1}{e^u} = \frac{1}{x}
\end{aligned}$$

---

Note that in the next to last step we simply put the constant on the right that results from combining the constants from both sides. From here on we will simply put a constant on one side (usually the right side) when we integrate both sides of an equation.

**Section 2.3 Exercises** To Solutions

1. Solve the IVP $\dfrac{dy}{dx} - 3y = e^{5x}$, $y(0) = -1$.

2. (a) Use an integrating factor to solve $y' + 2y = 0.4e^{-2t}$. (Note that $y$ is now a function of $t$.) *Solve for* $y$*.*

   (b) Solve the IVP $y' + 2y = 0.4e^{-2t}$, $y(0) = 3$.

3. (a) Use an integrating factor to solve $\dfrac{dy}{dx} - \dfrac{1}{2}y = 0$.

   (b) Solve the same ODE by separation of variables. Solve for $y$ and compare with your answer to (a) (and take any action that might be suggested by this comparison!).

   (c) Solve the IVP $\dfrac{dy}{dx} - \dfrac{1}{2}y = 0$, $y(0) = \frac{3}{2}$.

4. (a) Solve the ODE $y' - 5y = 3\cos 2t$. *Use your formula sheet to avoid some very messy integration.*

   (b) Solve the IVP $y' - 5y = 3\cos 2t$, $y(0) = -4$.

5. (a) Solve the ODE $\dfrac{dy}{dt} + 3y = t^2 + 5t - 1$.

   (a) Solve the IVP $\dfrac{dy}{dt} + 3y = t^2 + 5t - 1$, $y(0) = 2$.

6. The IVPs from Exercises 3(b) and 3(e) of Section 2.1 couldn't be solved by separation of variables, but they can be done with integrating factors. You will do them here.

   (a) Solve the IVP $\dfrac{dy}{dx} - y = e^{3x}$ , $y(0) = 4$.

   (b) Solve the IVP $x\dfrac{dy}{dx} - y = x$ , $y(1) = 2$. Begin by multiplying through by $\dfrac{1}{x}$.

7. The IVP $y' - 2x = xy$ , $y(2) = 5$ from Exercise 3(d) of Section 3.1 can be solved by both separation *and* using an integrating factor. Solve it using an integrating factor. *Be sure to get it in the right form before multiplying by the integrating factor!*

8. In this exercise you will see another method for solving the ODE $y' - 5y = 3\cos 2t$ from Exercise 4. This method will be used later when we solve second order ODEs.

   (a) The equation $y' - 5y = 0$ is the **homogeneous equation** associated with $y' - 5y = 3\cos 2t$. Substitute $y = Ce^{rt}$ into the homogenous equation to determine what value $r$ must have in order for $y = Ce^{rt}$ to be a solution. For that value of $r$, $y = Ce^{rt}$ is called the **homogeneous solution** to $y' - 5y = 3\cos 2t$.

   (b) Find the values of $A$ and $B$ for which $y = A\sin 2t + B\cos 2t$ is a solution to $y' - 5y = 3\cos 2t$. Do this as follows:

   - Find $y'$ and substitute it and $y$ into the differential equations to get an equation involving sines and cosines of $2t$.
   - Combine the like terms on the left side of the equation to get only one sine term and one cosine term.
   - You will need to note that on the right side of your equation $3\cos 2t$ is the same as $3\cos 2t + 0\sin 2t$. Equate the coefficient of $\cos 2t$ on the left side with the coefficient of $\cos 2t$ on the right side to get an equation involving the unknowns $A$ and $B$. Then repeat for sine to get another equation with $A$ and $B$.
   - Solve two equations for the two unknowns $A$ and $B$. The resulting $y = A\sin 2t + B\cos 2t$ is called the **particular solution** to $y' - 5y = 3\cos 2t$.

   (c) Write the sum of the homogeneous and particular solutions. This is known as the **general solution**, and should match what you found in Exercise 4(a).

9. Use the method of Exercise 8 to solve the ODE $\dfrac{dy}{dx} - 3y = e^{5x}$ from Exercise 1 with this difference: For part (b), find the value of $A$ for which $y = Ae^{5x}$ is a solution to the ODE.

10. Use the method of Exercise 8 to solve the ODE $\dfrac{dy}{dt} + 3y = t^2 + 5t - 1$ from Exercise 5, but for part (b), find the values of $A$, $B$ and $C$ for which $y = At^2 + Bt + C$ is a solution to the ODE.

## 2.4 Phase Portraits and Stability

**Performance Criteria:**

2. (g) Determine whether an ODE is autonomous.
   (h) Create a one-dimensional phase portrait for an autonomous ODE.
   (i) Determine critical points/equilibrium solutions of an autonomous ODE, and identify each as stable, unstable or semi-stable.
   (j) Sketch solution curves of an autonomous ODE for various initial values.

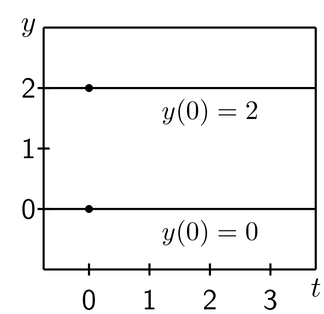
Recall that a first order ODE is called **autonomous** if it can be written in the form $\frac{dy}{dx} = f(y)$. That is, when we get the derivative alone on the left side of the equation, the right hand side is a function of only the dependent variable. $\frac{dy}{dt} = y^2 - 2y$ is an example of an autonomous ODE. Note that $\frac{dy}{dt} = 0$ whenever $y = 0$, so if we had an initial condition of $y(0) = 0$, then the value of $y$ would never change because the rate of change with respect to time is zero. Therefore the solution to the IVP

$$\frac{dy}{dt} = y^2 - 2y, \qquad y(0) = 0$$

is the constant function $y = 0$. We have solved the IVP without doing any calculations! Now suppose that, for the same ODE, $y(0) = 2$. We see that when $y = 2$ we again have $\frac{dy}{dt} = 0$, so the value of $y$ will again not change. This means that $y = 2$ is the solution to the IVP

$$\frac{dy}{dt} = y^2 - 2y, \qquad y(0) = 2$$

The graph to the right shows the two solution curves that we have obtained so far for the ODE $\frac{dy}{dt} = y^2 - 2y$, for the initial values $y(0) = 0$ and $y(0) = 2$. (The dots represent the initial values themselves - note the position of zero on the horizontal axis.) We will call constant solutions like those two **equilibrium solutions**; the word equilibrium essentially meaning unchanging as time goes on. The question that should occur to you is "What happens for other initial values of $y$?" With a little thought we should be able to figure that out. There are three key observations we can make that will help answer the question:

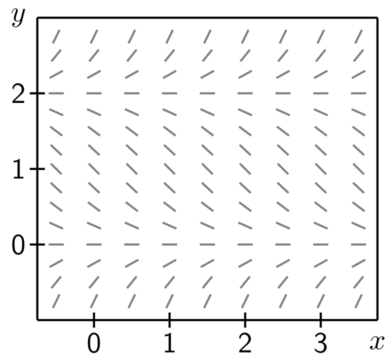


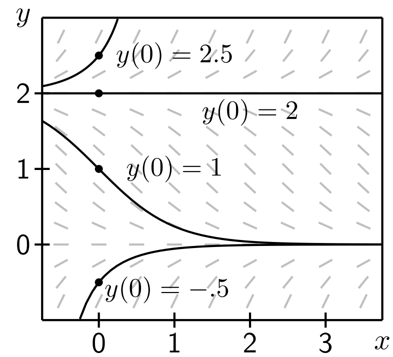

Video Discussion/Example

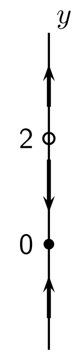
- The direction field depends only on $y$ alone, so for any given value of $y$ the slope remains constant.
- The right hand side of the ODE can be factored to $y(y-2)$. From that we see that if $y < 0$ both $y$ and $y - 2$ will be negative, so $\frac{dy}{dt} = y(y-2)$ will be positive. Therefore any solution with an initial value less than zero will be increasing. When $0 < y < 2$, $y$ is positive and $y - 2$ is negative, so $\frac{dy}{dt}$ is negative and any solution with an initial value between zero and two will be decreasing. Finally, when $y > 2$ we have $\frac{dy}{dt} > 0$ because both $y$ and $y - 2$ are positive when $y > 2$. Any solution with an initial value greater than two will be increasing.

- Whether positive or negative, the value of $\dfrac{dy}{dt}$ approaches zero as $y$ gets nearer to either zero or two. Therefore the direction field lines become "flatter" (closer to horizontal) for values of $y$ close to zero and two.

From these observations we can deduce that the direction field for $\dfrac{dy}{dt} = y^2 - 2y$ has the appearance shown to the left below. The direction field with solutions corresponding to four different initial conditions is shown in the center.




We will summarize the information in the three bullets above with something called a **phase portrait**, shown above and to the right. (Technically it is a *one-dimensional phase portrait*. Those of you taking the second term of this course may see two-dimensional phase portraits.) The vertical line indicates $y$ values, with the two **critical points** zero and two indicated. The critical points divide the line into three intervals, and the arrow in each interval indicates whether $y$ is increasing or decreasing, in the particular interval, as time increases.

Look again at the direction field in the middle above, with the four solutions curves drawn in. Note that solutions with an initial $y$ value less than two (including less than zero) all tend toward the constant solution $y = 0$, as the phase portrait tells us they will. When this occurs we say that the solution $y = 0$ is a **stable equilibrium solution**. You can sort of think that if we have an initial condition of zero the solution will be zero, and if we "bump off" from zero a bit with our initial condition, the new solution obtained will tend back toward zero as time goes on. This is indicated by putting a solid dot at $y = 0$ on the phase portrait.

On the other hand you can see that if $y$ starts with the value two it will remain two, but if it starts at any value close to, but not equal to, two the solution will diverge away from two. For this reason we call the solution $y = 2$ an **unstable equilibrium solution** and we indicate it on the phase portrait with an open circle at $y = 2$. Other language you might hear is that $y = 0$ is a **stable critical point** on the phase diagram, and $y = 2$ is an **unstable critical point**. As just stated, on our phase portraits we will indicate stable critical points with solid dots, and unstable critical points with open circles. (We will also use open circles for semi-stable equilibria, which you will see later.)

Again, the entire analysis you have just seen is based on the fact that the ODE is autonomous. Two points of interest here are

- Autonomous first order ODEs are not just a curiosity - they occur naturally in many applications.
- Autonomous ODEs can be difficult or impossible to solve. However, an analysis like we just did can make it very easy to determine how solutions to such an equation behave, in a qualitative sense.

It is important to be able to recognize autonomous differential equations.

◇ **Example 2.4(a):** Determine which of the following first order ODEs are autonomous:

$$\frac{dy}{dx} = y^2 - x \qquad y' = 2y - 1 \qquad \frac{dy}{dt} = t^2 - 5t + 1 \qquad x\frac{dx}{dt} = x + 1$$

**Solution:** Clearly the first equation is not autonomous and the second is. The third and fourth equations might be a bit confusing, as the variables are no longer $x$ and $y$. In the third equation the dependent variable is $y$ and the independent variable is $t$. Because the right hand side is a function of only the independent variable $t$, the equation is not autonomous. In the fourth equation the dependent variable is $x$, the independent variable is $t$, and the equation can be rewritten as $\dfrac{dx}{dt} = \dfrac{x+1}{x}$. The right hand side is then a function of $x$ alone, so the equation is autonomous.

---

If we can determine the phase portrait for an autonomous ODE, we then have a pretty good idea what all solutions to the ODE look like, without having to go to the trouble of creating a direction field. The next example shows how this is done.

◇ **Example 2.4(b):** Sketch the phase portrait for $\dfrac{dy}{dt} = -y^3 + 6y^2 - 9y$ and use it to sketch solution curves for the initial conditions $y(0) = 4$, $y(0) = 2$ and $y(0) = -1$. Identify each equilibrium solution as stable or unstable.

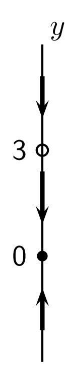
**Solution:** We factor the right hand side of the ODE, starting by factoring $-y$ out:

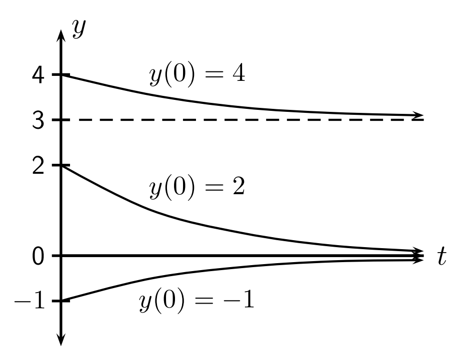
$$\frac{dy}{dt} = -y^3 + 6y^2 - 9y = -y(y^2 - 6y + 9) = -y(y-3)^2$$

From this we can see that the equilibrium solutions are $y = 0$ and $y = 3$. Testing values of $y$ in each of the three intervals $(-\infty, 0)$, $(0, 3)$ and $(3, \infty)$ gives us the following:

- When $y < 0$, $\dfrac{dy}{dt} > 0$
- When $0 < y < 3$, $\dfrac{dy}{dt} < 0$
- When $y > 3$, $\dfrac{dy}{dt} < 0$

This gives us the phase portrait shown to the left below, which indicates that there are equilibrium solutions of $y = 0$ and $y = 3$. If $y(0) < 0$ the solution is increasing, but will approach the equilibrium solution $y = 0$; if $0 < y(0) < 3$ the solution decreases toward $y = 0$. If $y > 3$ the solution also decreases, but toward $y = 3$. These behaviors are shown to the right below, for the three solutions with the given initial values. The solution $y = 0$ is a stable equilibrium solution, and the solution $y = 3$ is neither stable nor unstable, but is what we call a **semi-stable equilibrium** solution.




---

(Long) Video on This Section

**Section 2.4 Exercises** To Solutions

1. Determine which of the following first order ODEs are autonomous.

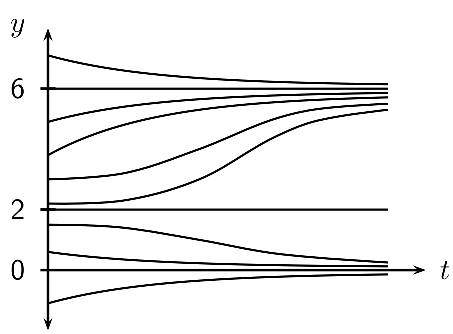
   (a) $\dfrac{dy}{dt} - 2y = 0$ (b) $\dfrac{dy}{dx} + xy = 1$ (c) $\dfrac{1}{y}\dfrac{dy}{dx} + y = 3$

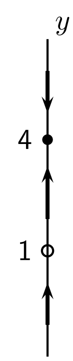
   (d) $y' = y^2 - 7y + 10$ (e) $\dfrac{1}{x}\dfrac{dy}{dx} + y = 1$ (f) $\dfrac{dx}{dt} + 6x^2 = x^3 + 9x$

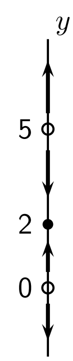
2. For each of the ODEs in Exercise 1 that are autonomous,

   - draw a phase portrait
   - *on a single separate graph*, sketch in all equilibrium solution curves, and one solution curve with an initial value in each interval of the real line created by the values of the equilibrium solutions
   - give all equilibrium solutions and, for each, tell what kind of equilibrium it is

3. In (i) below, the graph of some solution curves to an ODE are given. (ii) and (iii) are phase portraits for two other ODEs.

   (i)



   (ii)



   (iii)



   (a) For the solution curves in (i) above, identify each equilibrium solution and tell whether it is a stable equilibrium, unstable equilibrium, or semi-stable equilibrium.

   (b) Repeat (a) for the phase portrait (ii). (c) Repeat (a) for the phase portrait (iii).

4. (a) Draw the phase portrait for the ODE with solution curves shown in (i) of the previous exercise.

   (b) Draw some solution curves for the ODE whose phase portrait is shown in (ii) of the previous exercise. Be sure to include curves for the equilibrium solutions and at least one solution with initial value in each of the intervals created by the equilibrium points.

   (c) Draw some solution curves for the ODE whose phase portrait is shown in (iii) of the previous exercise. Be sure to include curves for the equilibrium solutions and at least one solution with initial value in each of the intervals created by the equilibrium points.

5. (a) Give an ODE of the form $\frac{dy}{dx} = f(y)$, with $f(y)$ in factored form, that could have the solution curve graph shown in Exercise 3(a).

   (b) Repeat (a) for the phase portrait shown in Exercise 3(b).

   (c) Repeat (a) for the phase portrait shown in Exercise 3(c).

6. In Exercise 7 of Section 2.1 you solved the ODE $\frac{dy}{dx} = -\frac{1}{3}y^2 + y$. When working with it in this exercise, you will find it useful to factor $-\frac{1}{3}$ out of the right hand side first.

   (a) When we tried to solve the ODE with the initial value $y(0) = 3$ we were not able to obtain a solution. What do we now know happens for that initial condition?

   (b) Give the equilibrium solutions, and what kind each is.

   (c) Sketch the phase portrait and some solution curves for the ODE.

7. In the next section the ODE

$$\frac{dA}{dt} = 2.1 - 0.0875A$$

will arise in a mixing problem. Give all equilibrium points and identify each as stable, semi-stable or unstable. The sketch a phase portrait and a graph of solution curves for the initial values $A = 10$, $A = 40$ and $A = 80$.

8. Consider the ODE $\frac{dT}{dt} = -k(T - 50)$, where $k > 0$. Sketch a graph of the equilibrium solution and several other solution curves with initial values different from that of the equilibrium solution.

9. Sketch several solution curves for the ODE $\frac{1}{4}\frac{di}{dt} + 15i = 12$, including any equilibrium solutions.

## 2.5 Applications of First Order ODEs

**Performance Criteria:**

2. (k) Solve an applied problem modeled by a first order ODE using separation of variables or an integrating factor.

   (l) Give an ODE or IVP that models a given physical situation involving growth or decay, mixing, Newton's Law of Cooling or an RL circuit.

   (m) Sketch the graph of the solution to a mixing or Newton's Law of Cooling problem, indicating the initial value and the steady-state asymptote.

   (n) Identify the transient and steady-state parts of the solution to a first order ODE.

### Radioactive Decay and Population Growth

In general, we can assume that the rate at which a quantity of radioactive material decays is proportional to the amount present. For example, 20% of the material might decay in any 600 year period. If there were 1000 pounds initially, 200 pounds would decay over 600 years, but if there were only 100 pounds initially, only 20 pounds would decay over 600 years. If we let $A$ represent the amount of material at any time $t$, then the rate at which the material decays is given by the derivative $\frac{dA}{dt}$. The above discussion tells us that there is some constant of proportionality $r$ for which

$$\frac{dA}{dt} = rA$$

We will find that $r$ is negative because the amount $A$ (which is positive) is *decreasing*.

Similarly, suppose that $N$ represents the number of individuals (which could be people or any other animals) in a region, and assume the that the population is growing. If there were no constraints like famine, disease and such, the population should grow continuously. The derivative $\frac{dN}{dt}$ would represent the rate of change of population with respect to time. When the population is small we would expect a small change in population over a fixed time period, but when the population is large we'd expect a greater increase in population over the same time period, because there is a larger population having offspring. We'd again expect the rate of change to be proportional to the population itself, resulting in the differential equation $\frac{dN}{dt} = rN$, but in this case the constant $r$ would be positive because here there is growth rather than decay.

Clearly the differential equations for both radioactive decay and population growth are the same, and both can easily be solved by separation of variables, or even just by guessing, as long as we remember that the solution must contain an arbitrary constant! The arbitrary constant is in addition to the constant $r$; the additional constant is introduced by the fact that we must essentially integrate once to determine the function $N = N(t)$.

◇ **Example 2.5(a):** Five hundred rainbow trout are introduced into a previously barren (no fish in it) lake. Three years later, biologists estimate that there are 1730 rainbow trout in the lake. Assuming that the population satisfies the ODE $\frac{dN}{dt} = rN$, determine the function $N = N(t)$ that gives the number $N$ of rainbow trout in the lake at time $t$.

**Solution:** The ODE $\frac{dN}{dt} = rN$ says that we are looking for a function $N(t)$ whose derivative is $r$ times the function itself, and $N = e^{rt}$ clearly is such a function. This function contains no constant of integration, but we recall that $N = Ce^{rt}$ is also a solution, for any value of $C$. The general solution is then $N = Ce^{rt}$. The fact that there are 500 fish in the lake at time zero gives us $500 = Ce^0$, so $C = 500$ and $N = 500e^{rt}$.

To find $r$ we substitute $N = 1730$ when $t = 3$ into our solution to get $1730 = 500e^{3r}$, and solve to find $r = 0.414$. The equation for the number of rainbow trout at any time $t$ is then $N = 500e^{0.414t}$.

Here's how one would solve the differential equation from the last example by separation of variables, with some of the steps combined:

$$
\begin{aligned}
\frac{dN}{dt} &= rN \\
\frac{dN}{N} &= r\,dt && \text{multiply by } dt \text{ and divide by } N \\
\ln N &= rt + C && \text{integrate both sides - absolute value is not needed because } N \text{ must be greater than zero} \\
N &= Ce^{rt} && \text{exponentiate both sides and apply } x^a x^b = x^{a+b}
\end{aligned}
$$

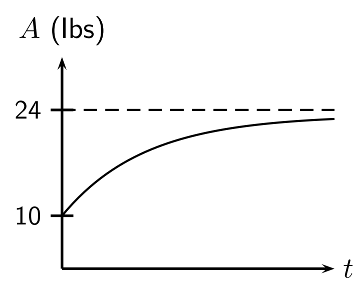
**Mixing Problems**

We've discussed this sort of situation previously, so let's go straight to an example:

◇ **Example 2.5(b):** A tank contains 80 gallons of water with 10 pounds of salt dissolved in it. Fluid with a 0.3 pounds per gallon salt concentration is being pumped into the tank at a rate of 7 gallons per minute. The fluid is continually mixed and, at the same time, the fluid is being drained from the tank at a rate of 7 gallons per minute. Letting $A$ represent the amount of salt in the tank, in pounds, sketch a graph of $A$ as a function of time $t$. Label any values you can on the $A$ axis.

**Solution:** In the next two examples we will set up and solve the initial value problem for this situation analytically, arriving at an equation that will give us the amount of salt at any time $t$. Before doing that it would be good to have some idea of what the behavior of the solution would be. We did this previously, in Example 1.1(j), but let's repeat the reasoning here.

The initial amount of salt in the tank is 10 pounds. We know that as time goes on the concentration of salt in the tank will approach that of the incoming solution, 0.3 pounds per gallon. This means that the amount of salt in the tank will approach 0.3 lbs/gal × 80 gal = 24 pounds, resulting in the graph shown to the right, where $A$ represents the amount of salt, in pounds, and $t$ represents the time, in minutes.




Now let's set up an IVP and solve it:

◇ **Example 2.5(c):** Letting $A$ represent the amount of salt in the tank, in pounds, give an initial value problem describing this situation.

**Solution:** Taking the concentration of salt in the incoming fluid times the rate at which the fluid is coming in, we get that salt is entering the tank at a rate of

$$(0.3\tfrac{\text{lbs}}{\text{gal}})(7\tfrac{\text{gal}}{\text{min}}) = 2.1\tfrac{\text{lbs}}{\text{min}}.$$

Now let $A = A(t)$ be the amount (in pounds) of salt in the tank at any time $t$ minutes. Then the concentration of salt in the tank is $\frac{A}{80}$, so by the same sort of calculation that we just did the rate at which salt is *leaving* the tank is $\frac{7A}{80} = 0.0875A\frac{\text{lbs}}{\text{min}}$. The net rate of change of salt in the tank is the amount coming in minus the amount going out, or $(2.1 - 0.0875A)\frac{\text{lbs}}{\text{min}}$. But this quantity, being a rate of change, is also a derivative. Namely it is $\dfrac{dA}{dt}$, giving us the differential equation

$$\frac{dA}{dt} = 2.1 - 0.0875A$$

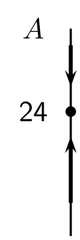
We also have the initial value $A(0) = 10$ pounds, so we have the initial value problem

$$\frac{dA}{dt} = 2.1 - 0.0875A\,, \qquad A(0) = 10 \tag{1}$$

Note that the ODE is autonomous, and the graph obtained in the previous exercise can be obtained from the ODE by the methods of the previous section.

◇ **Example 2.5(d):** The ODE in (1) above is autonomous. Determine the equilibrium solution and whether it is stable, unstable, or semi-stable. Sketch a phase portrait for the situation.

**Solution:** The equilibrium solution occurs when




$$\frac{dA}{dt} = 2.1 - 0.0875A = 0.$$

Solving $2.1 - 0.0875A = 0$ for $A$ gives us an equilibrium solution of $A = 24$ pounds. When $A < 24$ we find that $\dfrac{dA}{dt} > 0$, and when $A > 24$, $\dfrac{dA}{dt} < 0$. Therefore $A = 24$ is a stable equilibrium. The phase portrait is shown to the right.

Note that the phase portrait agrees with the solution curve obtained in Example 2.5(b).

We will see differential equations like the one in (1) above in several contexts, and it can always be solved using separation of variables or an integrating factor (and you should be able to do it either way). When we separate variables we get

$$\frac{dA}{2.1 - 0.0875A} = dt.$$

The left side can be integrated by $u$-substitution, but such integrals come up often enough in practice that we should use the following formula instead, obtained by $u$-substitution:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|\,ax+b\,| + C \tag{2}$$

We can now use this result to solve the IVP (1).

◇ **Example 2.5(e):** Solve the IVP $\dfrac{dA}{dt} = 2.1 - 0.0875A\,,\; A(0) = 10.$ Another Example

**Solution:** Multiplying both sides of the ODE by $dt$ and dividing by the quantity $2.1 - 0.0875A$ gives us

$$\frac{dA}{2.1 - 0.0875A} = dt$$

To use equation (2) from the previous page we note that our left side is like the left side of (2), but with $x = A$, $a = -0.0875$ and $b = 2.1$. The formula then tells us that the left side integrates to $-\frac{1}{0.0875}\ln(2.1 - 0.0875A) + C$. (We don't need absolute value because the quantity $2.1 - 0.0875A$ is the rate at which the amount of salt is changing, and this is positive due to the concentration of the incoming solution being higher than the initial concentration in the tank.) Thus when we integrate both sides and combine the constants we have

$$\begin{aligned} -\tfrac{1}{0.0875}\ln(2.1 - 0.0875A) &= t + C \\ \ln(2.1 - 0.0875A) &= -0.0875t + C \\ 2.1 - 0.0875A &= e^{-0.0875t + C} \\ 2.1 - 0.0857A &= Ce^{-0.0875t} \\ -0.0875A &= -2.1 + Ce^{-0.0875t} \\ A &= 24 + Ce^{-0.0875t} \end{aligned}$$

Applying the initial value $A(0) = 10$ we get $10 = 24 + C$, so $C = -14$ and the solution to the IVP is $A = 24 - 14e^{-0.0875t}$.

Note that when $t = 0$ the solution $A = 24 - 14e^{-0.0875t}$ gives us $A = 10$, as it should. Also, as $t \to \infty$, $A$ goes to $24$ as expected.

### Newton's Law of Cooling

**Newton's Law of Cooling**

Suppose that a solid object with initial temperature $T_0$ is placed in a medium with a constant temperature $T_m$, and let $T = T(t)$ be the temperature of the object at any time $t$ after it is placed in the medium. The rate of change of the temperature $T$ with respect to time is proportional to the difference between the temperatures of the medium and that of the object. That is,

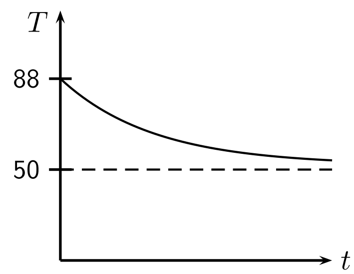
$$\frac{dT}{dt} = -k(T - T_m) \tag{3}$$

for some constant $k > 0$. Together with $T(0) = T_0$, this gives us an initial value problem for the temperature of the object.

The medium that the object is placed in might be something like air, water, etc., and $T_m$ stands for "temperature of the medium," sometimes called **ambient temperature**. We will always consider situations for which this temperature is constant. Note that if the temperature of the medium is greater than the temperature of the object, the rate of change of temperature must be positive, which is why $k$ must be positive. A little thought will tell you that $k$ must be positive if the temperature of the medium is less than the temperature of the object as well.

◇ **Example 2.5(f):** Suppose that a solid object with initial temperature 88°F is placed in a medium with ambient temperature 50°F, and after one hour the object has a temperature of 65°F. Determine the equation for the temperature $T$ as a function of the time $t$.

**Solution:** Because the object's initial temperature of 88°F is higher than the ambient temperature of 50°F, it will cool after being placed in the medium. Newton's Law of Cooling tells us that it cools more rapidly at first, when the temperature difference between the object and the medium is large. As the object cools to temperatures closer to the ambient temperature, the rate of cooling decreases. This is shown by the graph to the right.




Now the initial value problem for this situation is

$$\frac{dT}{dt} = -k(T-50)\,, \qquad T(0) = 88$$

The differential equation becomes $\dfrac{dT}{T-50} = -k\,dt$, and the method of Example 2.4(e) gives us $T = 50 + Ce^{-kt}$ before applying the initial condition. Using the initial condition, we obtain the solution $T = 50 + 38e^{-kt}$. To determine $k$ we apply $T(1) = 65$ to get $k = 0.930$, so the final solution is $T = 50 + 38e^{-0.930t}$.

## Electric Circuits

We will work with some basic electrical circuits in this class. The first sort of circuit we'll look at consists of a **voltage source**, a **resistor** and an **inductor**. The voltage source can be constant (so-called "direct current," or "DC"), or it can be variable, usually in an oscillating manner ("alternating current," or "AC"). The voltage source causes electrons to move in the circuit, and the flow of electrons is called **current**. (Somewhat confusingly, the current flows in the direction *opposite* the flow of the electrons.) The voltage source provides an **electromotive force** which we can think of as sort of "pushing" current through the circuit, analogous to a pump pushing fluid through a network of pipes. The units of the electromotive force are **volts**. We will use the symbol $i$ for current, and it is measured in units called **amperes**. ("Amps," for short.)

The resistor has a characteristic called **resistance**, which is measured in units called **ohms**. The inductor's characteristic is called **inductance**, which is measured in **henries**. Although resistance and inductance could be variable, they will always be constants in our considerations. We will use $E = E(t)$ for the voltage, $R$ for the resistance and $L$ for the inductance. To the right is a *schematic diagram* of such a circuit. We will usually think of our circuits as having a switch as well, which is "open" (off) until time zero, when it is "closed" (turned on). From that point on the current is (usually) changing, and is a function of time: $i = i(t)$.

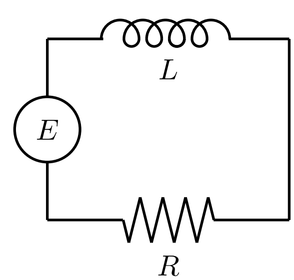



Figure 2.5(a)

**RL Circuit**

Consider an electric circuit as described above, with an applied voltage $E(t)$ volts (possibly a function of time) and constant resistance of $R$ ohms and constant inductance of $L$ henries. The current $i = i(t)$, in amperes, satisfies the first order linear differential equation

$$L\frac{di}{dt} + Ri = E(t) \tag{4}$$

Because the ODE $(4)$ is first order linear, it can be solved using an integrating factor. If the voltage $E(t)$ is constant, the equation can be solved by separation of variables as well. Let's examine the case where the voltage source $E(t)$ is a constant function, to observe why mathematics is so powerful in science and engineering. Suppose that $L = \frac{1}{4}$ henry, $R = 15$ ohms, and $E = 12$ volts; in this case the ODE is $\frac{1}{4}\frac{di}{dt} + 15i = 12$. If we separate the variables we obtain $\dfrac{di}{12 - 15i} = 4\,dt$ or $\dfrac{di}{48 - 60i} = dt$. Let's look at the first of these along with the separated equations arising in Examples 2.5(e) and 2.5(f): Video Example

$$\frac{di}{12 - 15i} = 4\,dt \qquad\qquad \frac{dA}{2.1 - 0.0875A} = dt \qquad\qquad \frac{dT}{T - 50} = -k\,dt$$

Note that these are all of the form $\dfrac{dx}{ax + b} = c\,dt$, where $a$, $b$ and $c$ are constants. This illustrates the fact that

*physical situations that seem to have nothing in common lead to the same differential equation.*

We will see this principle in action again when we study second order ODEs.

**Response of a System**

Suppose that we have a circuit like that show in Figure 2.5(a), but without a voltage source $E$ (but with the circuit closed). If there is no current in the circuit initially, then there will not be any current at any future time. However, if there is some current in the circuit initially (which can be made to happen by including a voltage source, then removing it and completing the circuit in its absence), then there will be current in the future as well. Similarly, if we set a mass on a spring (like shown in Example 1.1(a)) in motion it will continue to oscillate for some time.

We will refer to the circuit without a voltage and the mass on a spring as **systems**. In the case of the circuit the variable of interest is the current in the circuit at any time, and for the mass we are interested in the vertical position at any time. With some initial "stimulus" (non-zero initial conditions) in each case the current or vertical position will vary with time. The current or vertical position will be referred to as the **response** of the system to the initial conditions.

Now suppose that we have either an $RL$ circuit with a voltage source or a spring-mass system with some outside force pushing or pulling on the mass. The outside force or voltage (which is also sometimes referred to as an **electromotive force**) we will refer to as a **forcing function** for the system. In the presence of a voltage source, the circuit will have current at all future times. Similarly, a mass on a spring with a forcing function will continue to oscillate

We will revisit the spring-mass system, along with slightly more complex electrical circuits, in Chapters 3 and 4. For the time being, let's focus on the circuit shown in Figure 2.5(a) and the governing differential equation

$$L\frac{di}{dt} + Ri = E(t). \tag{5}$$

Here the left side $L\dfrac{di}{dt}+Ri$ of the equation represents the system, and the right side $E(t)$ represents the forcing function. Our goal is usually to solve an associated initial value problem for the current $i = i(t)$. That current, the solution to the IVP, is the response of the system to the forcing function $E(t)$ and initial current. Another way of thinking about this is that the forcing function and initial condition(s) are "inputs" to the system, and the response is the "output" of the system for that input.

Let us consider for example an $RL$ circuit where again $L = \frac{1}{4}$ henry and $R = 15$ ohms, and for which $E(t) = \sin 3t$. In this case we would not be able to separate the variables, so we'd solve the equation using an integrating factor. In doing so, we would obtain the solution

$$i = \tfrac{4}{63}\sin 3t + Ce^{-60t}$$

where $C$ is a constant that would be determined by an initial condition. Note that the solution has two parts:

- The part $\frac{4}{63}\sin 3t$, which is periodic and "goes on forever in the same way." This part of the solution is called the **steady-state solution** or **steady-state response** of the system.
- The part $Ce^{-60t}$, approaches zero as time goes on, so it "dies out." Such a solution or part of a solution is called the **transient solution** or **transient response** of the system.

To clarify a little, *we will define a steady-state solution to be any solution that is either constant (and, to avoid triviality, not zero) or periodic.*

In practice, when a system like a machine with moving parts or an electrical circuit is "turned on," it often exhibits a certain behavior as it starts up, which is the transient response of the system. Then it will "settle in" to a steady state behavior or response. (Note that *the ideas of transient and steady state solutions only make sense when the independent variable is time.*) In general only the steady-state response is important in terms of what we want the system to do from a practical viewpoint, but the transient response might be of interest because it might cause some sort of stress on the system that could cause a problem.

For the scenario described above but with $E$ having the constant value of 12 volts, the solution to the differential equation is

$$i = \tfrac{12}{15} + Ce^{-60t},$$

where $C$ is again determined by the initial current in the circuit. (Ordinarily we would be expected to reduce $\frac{12}{15}$, but we'll leave it as is to see the voltage and resistance.) We can see that here the steady-state solution is $i = \frac{12}{15}$, where 12 is the voltage and 15 is the resistance, saying that "in the long run" the circuit will exhibit Ohm's Law $V = IR$ $\left(\text{solved for } I = \dfrac{V}{R}\right)$. This is because the current approaches a constant value, and the inductor only affects the circuit when there is change in the current, causing flux in the coil of the inductor.

◇ **Example 2.5(g):** In Example 2.5(f) we found that when a solid object with initial temperature 88°F is placed in a medium with ambient temperature 50°F, and after one hour the object has a temperature of 65°F, the temperature $T$ at any time $t$ is given by $T = 50 + 38e^{-kt}$. Give the transient and steady-state parts of the solution.

**Solution:** The transient part of the solution is $38e^{-kt}$ and the steady-state part is $50$.

## Section 2.5 Exercises

To Solutions

1. When a person takes a medication, the amount in their body decreases exponentially, in the same way that a radioactive element decays. Suppose that a person takes 80 grams of some medication, and that we somehow know (???) that 12 hours later they still have 23 grams in their system.

   (a) Give the initial value problem for this situation, using $A$ for amount in grams and $t$ for time in hours.

   (b) The ODE is autonomous - what is the equilibrium solution? Is it stable?

   (c) Solve the IVP. Your answer should still contain an unknown constant $k$.

   (d) Determine $k$ and give the amount function $A$.

2. An underground storage tank contains 1000 gallons of water with 87 pounds of contaminant in it. At some time we will call time zero, clean water is pumped into the tank at a rate of 300 gallons per hour and the thoroughly mixed solution is pumped out at the same rate.

   (a) Set up an IVP for this situation, using $A$ for the amount of contaminant, in pounds, and $t$ for time, in minutes.

   (b) Solve the IVP.

   (c) Determine when the amount of contaminant has decreased to five pounds.

   (d) Give the transient and steady-state solutions.

3. (a) A solid object is placed in a medium with ambient temperature 70°F. Solve the differential equation (2) for this situation. The constant $k$ will be unknown for now, and there will be another constant that arises as well.

   (b) Suppose that the initial temperature of the object is 32°F. Solve for the constant that arose in solving the ODE.

   (c) After one hour the object has a temperature of 58°F. Use this information to determine the constant $k$. Give units with your answer.

   (d) What is the steady-state solution? What is the transient solution?

4. (a) Suppose that the voltage in resistor-inductor series circuit is supplied by a 12 volt battery, so $E(t) = 12$. The inductance of the circuit is $\frac{1}{2}$ henry, and the resistance is 10 ohms. At time zero the current in the circuit is zero. Find the current function $i(t)$ by solving the initial value problem just described.

   (b) Now suppose that the voltage is variable, with equation $E(t) = 10\sin 2t$, and the initial current is zero. Solve the IVP.

   (c) Give the transient and steady-state parts of your solution to part (b). (Make it clear of course which is which!)

5. In general, the solution to the differential equation for Newton's Law of Cooling is

$$T(t) = T_m + (T_0 - T_m)e^{-kt} \; , \qquad (5)$$

   where $T_0$ is the initial temperature.

(a) What happens if $T_0 = T_m$?

(b) What is the steady-state solution to (5)?

(c) What is the transient solution to (5)?

6. The ODE $\dfrac{dA}{dt} = 2.1 - 0.0875A$ from Example 2.5(c) is autonomous.

   (a) Determine all equilibrium solutions, and tell whether each is stable, unstable or semi-stable.

   (b) Sketch some solution curves for $A(0) > 0$, including some with $A(0)$ greater than the largest equilibrium solution.

7. When an owner arrives home in their car, it is at 29°F from being outside all day. The owner parks it in a heated garage, which is at a temperature of 73°F.

   (a) We wish to determine the temperature $T$ of the car as a function of time $t$, assuming it follows Newton's Law of Cooling. Write a differential equation to be solved to find that function, the solution we are looking for. Also, give any initial condition(s).

   (b) Will there be a steady-state solution? If so, what will it be?

   (c) Solve the initial value problem. Your answer will still contain a constant, but you should be able to determine the value of another.

   (d) Suppose that two hours later the owner is ready to go back out in the car, and by that time it has warmed up to 47°F. Determine the function modeling the temperature of the car in the two hours that it was in the garage.

8. An RL circuit contains an inductor with an inductance of $\frac{3}{4}$ henry and a 15 ohm resistor. It is driven by a variable voltage $E(t) = 6\cos 2t$, and the initial current in the circuit is 2 amperes.

   (a) Give the initial value problem to be solved, and solve it. *Determine exact values for all constants.*

   (b) Give the steady-state and transient solutions.

9. A 150 gallon tank contains a 3 pounds (lbs) per gallon (gal) salt solution. At time zero, solution will begin being pumped out of the tank at a rate of 7 gallons per minute and a 1 pounds per gallon solution will begin being pumped into the tank at the same rate. Assume that there is constant mixing in the tank, so that it has the same concentration all over in the tank at any given time. Let $A$ represent the amount of salt in the tank (in pounds) and let $t$ represent time (in minutes).

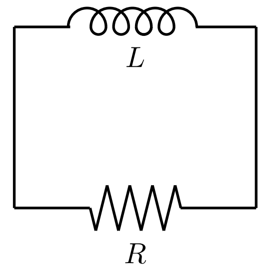
   (a) Sketch the graph of the amount $A$ of salt in the tank as a function of time, from just the given information.

   (b) Give the initial value problem to be solved for $A$, and solve it.

   (c) Graph your solution using some technology, and compare with your answer to (a).

   (d) Give the steady-state and transient solutions.

10. Suppose that we have an RL circuit with no voltage, as shown to the right. The resistor has a resistance of 8 ohms, and the inductor has an inductance of $\frac{1}{3}$ henry, and there is an initial current in the circuit of 5 amperes.




(a) Solve the initial value problem.

(b) Give the transient and steady-state parts of the solution.

## 2.6 Chapter 2 Summary

- We can "solve" ODEs (and PDEs) in three ways:
  - Analytically, which means "paper and pencil" methods that give exact algebraic solutions.
  - Qualitatively, which means determining the general behavior of solutions without actually finding function values. Results of qualitative methods are often expressed graphically.
  - Numerically, which result in values of solutions only at discrete points in time or space. Results of numerical methods are often expressed graphically or as tables of values.

- The two most commonly applicable methods for solving first order ODEs analytically are separation of variables and the integrating factor method.

- Separation of variables only works for equations that can be written $\dfrac{dy}{dx} = g(x)h(y)$, and for which the antiderivatives $\displaystyle\int g(x)\,dx$ and $\displaystyle\int \frac{dy}{h(y)}$ can be determined.

- The integrating factor method only applies to linear first order ODEs. Such ODEs can be put in the form $\dfrac{dy}{dx} + p(x)y = q(x)$, and to carry out the integrating factor method the antiderivatives $u = \displaystyle\int p(x)\,dx$ and $\displaystyle\int e^u\, q(x)\,dx$ must exist.

- Some applications of first order ODEs are population growth and radioactive decay, mixing problems, Newton's Law of Cooling problems, and $RL$ electric circuits.

- Very different physical situations often result in the same differential equation.

- Suppose that the independent variable for an ODE is time, so the solution is a function of time. Any part of the solution that goes to zero as time goes to infinity is called the transient part of the solution, and any part of the solution that is a nonzero constant or periodic is called the steady state part of the solution.

- It is not necessary that all parts of solutions exhibit transient or steady state behavior, but it is often the case that they do.

## 2.7 Chapter 2 Exercises

1. In Example 2.1(b) the ODE $\dfrac{dy}{dt} + 0.5y = 0$ is solved by separation of variables, and it can also easily be solved using an integrating factor.

   (a) Instead of either of these, assume that it has a solution of the form $y = Ce^{rt}$ and determine the value of $r$ by substituting this solution into the equation. After finding the value of $r$, give the solution to the ODE.

   (b) Solve the IVP $\dfrac{dy}{dt} + 0.5y = 0, \quad y(0) = 4.7$

2. Consider the situation of Example 2.5(b) with the following change: Suppose that the 0.3 pounds per gallon fluid is coming in at a rate of 5 gallons per minute, rather than 7 gallons per minute. (The mixed fluid is still being drained from the tank at 7 gallons per minute.) The goal here is to determine the amount $A$ of salt in the tank at any time $t$.

   (a) Give an expression for the amount of *fluid* in the tank at any time $t$.

   (b) Give an expression for the concentration of salt in the tank at any time $t$.

   (c) Give the initial value problem to be solved in order to determine the amount of salt in the tank as a function of time.

   (d) The differential equation is linear. What are the functions $p(t)$ and $q(t)$?

   (e) Solve the differential equation, using the integrating factor method. Graph the solution and make sure it behaves as expected.

### Reduction of Order

The term **reduction of order** usually refers to a method for finding a second solution to a second order ODE from one solution that is already known. We will use the term more generally, for any process in which one or more ODEs is turned into one or more other ODEs of smaller order. This can be done in a variety of ways, the simplest of which is illustrated in the next few exercises.

3. In this exercise we'll use reduction of order to solve $u'' + 2u' = 0$, where the independent variable is $x$. This equation would likely not show up in any application, but it provides us with an easy introduction to how reduction of order works.

   (a) We begin by letting $v = u'$ where $v$, like $u$, is a function of $x$. What then is $u''$? Substitute the appropriate expressions in $v$ in for $u''$ and $u'$, then solve the resulting ODE for $v$. For simplicity, assume that $v \geq 0$. (Make sure you see why I am allowing this assumption!)

   (b) Now that you have found $v$, substitute $u'$ for $v$ and solve the new ODE. *Note that the original ODE is second order, so your solution should have two arbitrary constants.*

4. A classic problem in the study of PDEs is the equilibrium distribution of heat in a circular disk. In the course of solving that problem one obtains the ODE

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R = 0. \tag{1}$$

   Note that $r$ and $R$ are two different variables! $R$ is the dependent variable, and is a function of the independent variable $r$. Later we will see how to solve this equation for $n \neq 0$, but here we will solve for $n = 0$.

(a) Write the equation with $n = 0$.

(b) Let $S = \dfrac{dR}{dr}$. What then is $\dfrac{d^2R}{dr^2}$, in terms of $S$?

(c) Substitute what you were given and what you determined in (b) into your equation from (a) to obtain a first order ODE with dependent variable $S$.

(d) Solve your equation from (c), solving for $S$ eventually.

(e) Replace $S$ in your answer to (d) with $\dfrac{dR}{dr}$ and solve the resulting first order ODE for $R$. *Note that the original ODE* (1) *is second order, so your solution should have two arbitrary constants.*

Here you used reduction of order to start with a second order ODE but make a substitution that gives us a first order equation.

**Logistic Growth**

When we assume that a population will increase exponentially for all time, the differential equation for the number $N$ of individuals at time $t$ is

$$\frac{dN}{dt} = rN, \tag{2}$$

where $r$ is a constant that represents the growth rate. (See Example 2.5(a).) This model is somewhat unrealistic, however - usually we expect some upper limit to the population due to the fact that as it gets large it will begin to be constrained by some factor like the food available or disease. Thus the growth rate should approach zero at some point, or even become negative if the population becomes too large. On the other hand, the growth rate should be relatively constant for very small numbers $N$ of individuals. These conditions can be incorporated into our model by including a factor $1 - \dfrac{N}{K}$ for some other constant $K > 0$:

$$\frac{dN}{dt} = r\left(1 - \frac{N}{K}\right)N. \tag{3}$$

Equation (3) is one of several forms of what is called the **logistic equation**.

5. (a) Equation (3) is autonomous. Determine all equilibrium solutions (in terms of $K$), and classify each as stable or unstable. Sketch a phase diagram and some solution curves.

   (b) Discuss the significance of the constant $K$.

   (c) What effect should changing $r$ have on solution curves? *Be as specific as possible.* (**Hint:** Think in terms of the direction field for the equation.)

6. (a) Solve equation (3) for $K = 3000$ and $N(0) = 500$. You don't need to know the value of $r$ to do this, but $r$ will appear in your solution.

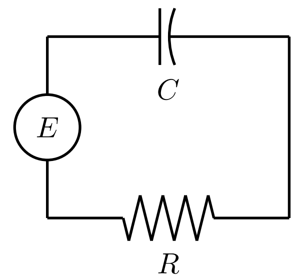
   (b) Suppose that $N(4) = 2000$, Use this to determine the value of $r$.

   (c) Using your solution to (a) with the value of $r$ determined in (b), determine when the population will reach $N = 2500$.

### RC Circuits

In Section 2.5 there is a discussion of RL circuits, ones containing a voltage source, resistor and inductor. Another simple circuit of interest is one containing a voltage source, resistor and capacitor, called an RC circuit. Capacitors are devices that store something called **charge**, which we'll denote as $q$. The units of charge are **coulombs**. The ability of a capacitor to store charge is quantified by a characteristic called **capacitance**, denoted by $C$. The units of capacitance are **farads**.

To the right is a schematic diagram of an RC circuit, which we will also assume has a switch that allows current to begin flowing at some time. The differential equation that models the charge $q$ (in coulombs) "on" (stored by) the capacitor at any time $t$ (in seconds) is




$$R\frac{dq}{dt} + \frac{1}{C}q = E(t),$$

where $R$ is resistance in ohms and $C$ is capacitance in farads, and $E(t)$ is the voltage, which may or may not be constant.

Finally, we note that the derivative $\frac{dq}{dt}$, the rate at which the charge on the capacitor is changing with respect to time, is the current in the circuit.

7. A 12 volt battery is attached to a circuit containing a $0.5\mu F$ (microfarad, $10^{-6}$ farad) and an $8k\Omega$ (kilo-ohm, $10^3$ ohm) resistor. At time zero, when the circuit is closed by a switch, the capacitor has a charge of $10^{-9}$ coulombs.

   (a) Give the charge $q$ on the capacitor and the current $i$ in the circuit as functions of time $t$ in seconds after closing the switch. *Note that constants and variables have to be in terms of volts, ohms, farads in order to get results in terms of coulombs and amperes.*

   (b) Plot the charge on the capacitor and the current in the circuit as two separate graphs. Indicate clearly any asymptotes.

8. Consider the same situation as the previous exercise, but with the 12 volt battery replaced by a variable voltage source $E(t) = 10\cos 240\pi t$. Repeat parts (a) and (b) of the previous exercise, but do not expect asymptotic behavior of either the charge or the current.

### Falling Body With Air Resistance

9. Here is another situation which is basically reduction of order, but disguised a little. Recall (from Section 0.2) that the differential equation governing the motion of a falling object (or one that has been projected upward) is

$$\frac{d^2h}{dt^2} = -32, \tag{4}$$

   where $h$ is the height of the object at any time $t$. For the value $-32$ on the right hand side, $h$ is measured in feet and $t$ in seconds. Now $\frac{d^2h}{dt^2}$ is the acceleration due to gravity. Remembering that acceleration is the derivative of velocity, $(4)$ can be rewritten as

$$\frac{dv}{dt} = -32. \tag{5}$$

   The negative sign here is based on a coordinate system where up is positive. For the sake of simplicity, let's consider a falling body (so it never goes upward), and let's take down to be positive. $(5)$ then becomes

$$\frac{dv}{dt} = 32. \tag{6}$$

$(4)$ and $(5)$ are based on the assumption that there is no air resistance, but now lets remove that assumption. A reasonable alternative premise is that the air resistance is proportional to the velocity, but in the opposite (upward, so negative in our new cordinate system) direction. Letting the constant of proportionality be $k > 0$, $(6)$ then becomes

$$\frac{dv}{dt} = 32 - kv. \tag{7}$$

(a) Equation $(7)$ is autonomous; what is the equilibrium solution? (Your answer will contain the constant $k$.) Is it stable, or unstable? Sketch the phase diagram and and a graph of some solution curves. The equilibrium solution is what people are referring to when they talk about **terminal velocity**.

(b) Solve $(7)$ using one of the methods from this chapter. Take the limit of your solution as $t \to \infty$ and make sure it matches your equilibrium solution from (a).

(c) Give the equation for the velocity of an object that begins its motion by being dropped with no initial velocity, with the assumption that air resistance is proportional to velocity.

# 3 Second Order Linear ODEs

**Learning Outcome:**

3. Solve second order linear, constant coefficient ODEs and IVPs. Understand the nature of solutions to such ODEs and IVPs.

   **Performance Criteria:**

   (a) Solve an Euler equation.

   (b) Solve a second order, linear, constant coefficient, homogeneous ODE.

   (c) Set up and solve second order initial value problems modeling spring-mass systems and RLC circuits.

   (d) Sketch or identify the graph of the solution to an IVP for an undamped mass on a spring with no forcing function.

   (e) Write a function $y = A\sin\omega t + B\cos\omega t$ in the alternate form $y = C\sin(\omega t + \phi)$. From this, determine the amplitude, period, frequency, angular frequency and phase shift.

   (f) Determine from the coefficients of a second order, constant coefficient homogeneous ODE whether the system it models is (i) underdamped, (ii) critically damped, (iii) overdamped, or (iv) undamped.

   (g) Without finding the solution to the differential equation, sketch the graph of a solution of an overdamped or underdamped homogeneous second order, linear, constant coefficient ODE for given initial conditions. .

   (h) Find a particular solution to a second order linear, constant coefficient ODE using the method of undetermined coefficients.

   (i) Evaluate a differential operator for a given function.

   (j) Solve a second order linear, constant coefficient IVP.

   (k) Identify the transient and steady-state parts of the solution to a damped system with forced vibration.

In this course we are focusing on differential equations that can be solved by analytical ("pencil-and-paper") techniques. Many differential equations cannot be solved this way, and numerical methods must be employed to obtain solutions. (See Appendix B for an introduction to solving ODEs by numerical techniques.) Our chances of being able to solve an ODE analytically are much greater if it is linear.

A second order linear ODE has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x), \tag{1}$$

and when $f(x) = 0$ it is a *homogeneous* linear differential equation. Here $a_2$, $a_1$ and $a_0$ are functions of the independent variable $x$. In this chapter we will focus almost entirely on second order linear differential equations in which all the coefficients are constants and the independent variable is time $t$, rather than $x$. So our equations will generally have the form

$$ay'' + by' + cy = f(t), \tag{2}$$

where $a \neq 0$, $b$ and $c$ are constants and $f$ is a function that we will refer to as the **forcing function**. (2) is called a *second order, linear, constant coefficient ODE*. Recall that (1) and (2) are **homogeneous** when $f = 0$ for all $x$ or $t$.

The one other type of (linear) equation we will see in this chapter is a variety called an **Euler equation**. For such equations $a_2(x) = ax^2$, $a_1(x) = bx$ and $a_0(x) = c$, where $b$ and $c$ are constants, and $f(x) = 0$. Thus an Euler equation is one with the form

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0 \tag{3}$$

Equations of this form arise when solving certain partial differential equations. In the first section of the chapter we will solve equations of the forms

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0 \qquad \text{and} \qquad ay'' + by' + cy = 0, \tag{4}$$

both of which are clearly homogeneous. These equations will always have two solutions $y_1$ and $y_2$, and the general solution will be a linear combination

$$y = C_1y_1 + C_2y_2 \tag{5}$$

of the two solutions. ($C_1$ and $C_2$ are of course constants.) It should at this point be no surprise that the general solution to equations of either of the forms (4) contain two arbitrary constants!

Our main focus as the chapter goes on will be solving initial value problems of the form

$$ay'' + by' + cy = f(t), \qquad y(0) = y_0,\ y'(0) = y_0', \tag{6}$$

with $a$, $b$ and $c$ being constants, $a \neq 0$. In addition, the initial values $y_0$ and $y_0'$ are constants as well. The method we will use to solve the IVP will consist of four steps:

(1) We will first solve the homogeneous equation obtained by replacing $f(t)$ with zero. This will give us a solution of the form (5), called the **homogenous solution**. We will denote it by $y_h$.

(2) Next we'll find something called a **particular solution**, denoted by $y_p$, for the ODE in (6). We will do this by a method called the **method of undetermined coefficients**.

(3) The **general solution** to the equation $ay'' + by' + cy = f(t)$ is

$$y = y_h + y_p = C_1y_1(t) + C_2y_2(t) + y_p(t), \tag{7}$$

the sum of the homogenous solution and the particular solution.

(4) The initial conditions $y(0) = y_0$ and $y'(0) = y_0'$ are used to determine the values of the constants $C_1$ and $C_2$. This gives us the final solution to the initial value problem (6).

Initial value problems of the form

$$ay'' + by' + cy = f(t), \qquad y(0) = y_0,\ y'(0) = y_0', \tag{6}$$

model certain electrical circuits and simple mechanical vibration. We will proceed through a series of variations on this initial value problem, developing an understanding of how the particular model describes the system and how the solution $y(t)$ behaves:

- In Section 1.2 we saw how a system consisting of a mass on a spring is modeled by the second order ODE
$$m\frac{d^2y}{dt^2} = -ky \qquad \text{or} \qquad m\frac{d^2y}{dt^2} + ky = 0,$$
which is $ay'' + by' + cy = f(t)$ with $b = 0$ and $f(t) = 0$. When $b = 0$ we refer to the system as **undamped**, and when $f(t) = 0$ the system is **free**. We begin our study of applications with free, undamped systems in Section 3.2.

- In Section 3.3 we will add damping, but still no forcing function. That is, we'll have $b \neq 0$ and $f(t) = 0$. In this case we'll see that there are three possible scenarios - the system is just one of (a) **underdamped**, **overdamped**, or **critically damped**. In Section 3.3 we will also introduce an electrical circuit for which the mathematical model is exactly the same as a mass on a spring.

- Next, in Section 3.4, we will consider systems that are both **damped** (so $b \neq 0$ and **forced** (so $f(t) \neq 0$). We will see in that section how to find **particular solutions** to forced systems.

- Differential operators are introduced in Section 3.5. These give us a convenient way to understand the nature of solutions to forced systems.

- In Section 3.6, we'll put together everything from the previous sections to solve initial value problems with second order, linear, constant coefficient differential equations. We'll consider such IVPs in applied settings, and we'll examine the nature of solutions to such IVPs.

As a specific example of an ODE of the form $ay'' + by' + cy = f(t)$, let's consider
$$y'' + 5y' + 6y = 5\sin 3t.$$

We want to know not only how to solve such ODEs and their associated IVPs, but also to know what kind of behavior to expect from solutions, based on the original ODE or IVP. Here is a summary of some of the ideas you will run across, which were previously brought up in Chapter 2:

- The left hand side of the ODE represents some sort of "system" like a mass on a spring or a simple electric circuit. The function on the right represents some kind of input to the system, which we call a **forcing function**, since it forces movement (for a mass on a spring) or current (for an electrical circuit) in the system.

- The solution function $y$ is the **response** of the system, and will often consist of exponential or trigonometric functions, or products of the two. The solution may have several terms, some of which may "die out" (go to zero) as time goes on, and others that may not. The parts that die out are called **transient** parts of the solution, and parts that are constant or periodic are called **steady-state** parts. These are also referred to as the transient and steady-state response of the system.

## 3.1 Homogeneous Second-Order Equations

**Performance Criterion:**

3. (a) Solve an Euler equation.
   (b) Solve a second order, linear, constant coefficient, homogeneous ODE.

In this section we will solve second order homogeneous equations of the forms

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0 \qquad \text{and} \qquad ay'' + by' + cy = 0, \tag{1}$$

where $a \neq 0$, $b$ and $c$ are constants. In both cases, solutions are obtained by guessing what the general form of a solution might be and then substitution that guess into the ODE and "making it work." The first equation in $(1)$ is called an **Euler equation** and the second is a second order, linear, constant coefficient, homogeneous ODE. The bulk of this chapter is about constant coefficient equations. In this section we'll first see how to solve Euler equations, then look at homogeneous constant coefficient equations, whose solutions take a variety of forms.

Consider the Euler equation

$$x^2y'' + 2xy' - 6y = 0.$$

Note that if $y$ was some power of $x$ then $y'$ would be one power lower and $y''$ two powers lower. If we then substituted our $y$, $y'$ and $y''$ into the ODE we would have three terms of the same power due to the multiplication of $y'$ by $x$ and $y''$ by $x^2$ in the ODE. Thus there might be some hope that the three terms add up to zero.

◇ **Example 3.1(a):** Solve the ODE $x^2y'' + 2xy' - 6y = 0$ by guessing a solution of the form $y = x^p$ and determining the value of $p$.

**Solution:** First we compute $y' = px^{p-1}$ and $y'' = p(p-1)x^{p-2}$. Substituting these into the ODE we get

$$\begin{aligned}
x^2p(p-1)x^{p-2} + 2xpx^{p-1} - 6x^p &= 0 \\
p(p-1)x^p + 2px^p - 6x^p &= 0 \\
x^p\big[p(p-1) + 2p - 6\big] &= 0 \\
x^p(p^2 + p - 6) &= 0
\end{aligned}$$

Now the quantity $x^p$ is only zero when $x = 0$, leading to the trivial solution $y = 0$. Because we would like nonzero solutions, it must be the case that $p^2 + p - 6 = 0$. Solving this gives us $p = -3, 2$, so both $y = x^{-3}$ and $y = x^2$ are solutions. The general solution is then $y = C_1x^{-3} + C_2x^2$.

We now contemplate the second order linear, *constant coefficient*, homogeneous ODE

$$y'' + 5y' + 6y = 0.$$

In this case we are looking for some function $y = y(t)$ (we will be interested in applications in which time is the independent variable) for which we can multiply the function and its first and second derivatives by constants, add the results and get zero. This indicates that $y$ must essentially be equal to its first and second derivatives, which only occurs for exponential functions. Therefore we guess that the solution has the form $y = e^{rt}$, where $r$ is some constant.

◇ **Example 3.1(b):** Solve $y'' + 5y' + 6y = 0$, assuming that the independent variable is $t$.

**Solution:** We substitute the guess $y = e^{rt}$ into the ODE, resulting in

$$r^2 e^{rt} + 5re^{rt} + 6e^{rt} = (r^2 + 5r + 6)e^{rt} = 0$$

Now the quantity $e^{rt}$ is never zero, so $r^2 + 5r + 6 = 0$. Solving this gives us $r = -3, -2$, so both $y = e^{-3t}$ and $y = e^{-2t}$ are solutions. It is not hard to show that if $y_1$ and $y_2$ are both solutions to a homogeneous ODE and $C_1$ and $C_2$ are constants, then $y = C_1 y_1 + C_2 y_2$ is also a solution. So in this case, the general solution is $y = C_1 e^{-3t} + C_2 e^{-2t}$.

---

When using the above method to solve

$$ay'' + by' + cy = 0 \tag{2}$$

we will arrive at $(ar^2 + br + c)e^{rt} = 0$, leading to $ar^2 + br + c = 0$. We will refer to the equation $ar^2 + br + c = 0$ as the **auxiliary equation** (also called the **characteristic equation**) associated with the ODE $ay'' + by' + cy = 0$, and we will call solutions to this equation **roots** of the equation. The following summarizes what we saw in the above example, which is only one of several possible forms that a solution to (2) can take.

> When the auxiliary equation for a second order, linear, constant coefficient homogeneous equation has two real roots $r_1$ and $r_2$, the solution to the ODE is
>
> $$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Video Example - watch from 1:20 to 3:10

The most efficient method for finding the roots of the auxiliary equation $r^2 + 5r + 6 = 0$ from Example 3.1(b) is to factor the left side, but we could have used the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2}$$

instead. We will at times want or need to use the quadratic formula, and at other times factoring or another method may be more efficient for finding the roots of the auxiliary equation. Regardless of what we might do in practice, an examination of (2) is useful for determining the various possibilities when obtaining the roots to the auxiliary equation:

- When $b^2 - 4ac > 0$ there will be two real roots, as in Example 3.1(b).
- When $b^2 - 4ac = 0$ there will only be one real root, which is a rather special case.
- When $b^2 - 4ac < 0$ and $b \neq 0$ the roots will be complex conjugates. That is, they will be complex numbers of the form $r = k + \lambda i$ and $r = k - \lambda i$, where $i = \sqrt{-1}$.

- When $b^2 - 4ac < 0$ and $b = 0$, the roots will be two purely imaginary numbers $r = \lambda i$ and $r = -\lambda i$. (This is just the previous case, with $k = 0$.)

Video Discussion

Each of the above cases results in different forms of the solution to the homogeneous equation $ay'' + by' + cy = 0$, and the methods for solving the auxiliary equations are generally different as well, although the quadratic formula *could* be used in all cases. We've already taken care of the first case with Example 3.1(b).

We now consider the second case above. Suppose that we were to try the method of Example 3.1(b) for the ODE $y'' + 6y' + 9y = 0$. We would only get one value for $r$, $-3$. By the same reasoning as used before we then have the solution $y = C_1e^{-3t} + C_2e^{-3t}$; because these are like terms, this can be written $y = Ce^{-3t}$, where $C = C_1 + C_2$. But (for reasons we'll go into in more depth later) the general solution to a second order ODE must be the sum of two "different" functions, each multiplied by arbitrary constants, so something is wrong! The following example shows that there is in fact another solution besides $e^{-3t}$.

◇ **Example 3.1(c):** Verify that $y = te^{-3t}$ is also a solution to the ODE $y'' + 6y' + 9y = 0$.

**Solution:** Using the product and chain rules to compute the derivatives,

$$y' = t(e^{-3t})' + t'e^{-3t} = -3te^{-3t} + e^{-3t}$$

$$y'' = -3(te^{-3t})' - 3e^{-3t} = -3(-3te^{-3t} + e^{-3t}) - 3e^{-3t} = 9te^{-3t} - 6e^{-3t}$$

Substituting into the ODE,

$$\begin{aligned} y'' + 6y' + 9y &= \big(9te^{-3t} - 6e^{-3t}\big) + 6\big(-3te^{-3t} + e^{-3t}\big) + 9\big(te^{-3t}\big) \\ &= 9te^{-3t} - 6e^{-3t} - 18te^{-3t} + 6e^{-3t} + 9te^{-3t} \\ &= 0 \end{aligned}$$

The general solution to the ODE $y'' + 6y' + 9y = 0$ is then $y = C_1e^{-3t} + C_2te^{-3t}$. Any time that our auxiliary equation has a repeated root $r$ (both solutions are the same), one of the solutions is $e^{rt}$, and the other solution is $te^{rt}$. We'll see in Section 4.2 how this solution is obtained.

When the auxiliary equation for a second order, linear, constant coefficient homogeneous equation has only one real root $r$, the solution to the ODE is

$$y = C_1e^{rt} + C_2te^{rt}.$$

Video Example

We now consider the case where $b^2 - 4ac < 0$ and $b = 0$, resulting in two purely imaginary roots $r = \pm\lambda i$. We'll consider the homogeneous equation $y'' + 9y = 0$ that we looked at previously, which can be rewritten as $y'' = -9y$. Through our familiarity with derivatives and the chain rule, we guessed (correctly) that both $y = \sin 3t$ and $y = \cos 3t$ are solutions, and it is not hard to show that

$y = C_1 \sin 3t + C_2 \cos 3t$ is a solution. Now how would it work to try the method of Example 3.1(b) for this equation? Letting $y = e^{rt}$ we have

$$y'' + 9y = r^2 e^{rt} + 9e^{rt} = (r^2 + 9)e^{rt} = 0$$

so we need to solve $r^2 + 9 = 0 \Rightarrow r^2 = -9$. If we allow complex solutions, the solution to this equation is $r = \pm 3i$, so the solution to the ODE is $y = Ae^{3it} + Be^{-3it}$, where $A$ and $B$ are arbitray constants. (We could have used $C_1$ and $C_2$ for the constants as we have been doing, but we are "saving" them, as you'll see.) To continue we will need the following:

**Euler's Formula:** $e^{i\theta} = \cos\theta + i\sin\theta$

To see where Euler's Formula comes from, see Appendix B.5. We will also use the two basic trig identities:

$$\cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta$$

Combining the two solutions $e^{3it}$ and $e^{-3it}$, we get

$$\begin{aligned} y &= Ae^{3it} + Be^{-3it} \\ &= A\big(\cos 3t + i\sin 3t\big) + B\big[\cos(-3t) + i\sin(-3t)\big] \\ &= A\cos 3t + Ai\sin 3t + B\cos 3t - Bi\sin 3t \\ &= (A+B)\cos 3t + (A-B)i\sin 3t \\ &= C_1\cos 3t + C_2\sin 3t. \end{aligned}$$

Here $A$ and $B$ are constants, and $C_1 = A + B$, $C_2 = (A - B)i$. Now it seems that $C_2$ should be a complex number, which is a bit disconcerting. However, it is possible that $A$ and $B$ are complex as well, in such a way that perhaps $C_2$ turns out to be real! You will find that this method "works," regardless. The following summarizes what we have just seen.

When the auxiliary equation for a second order, linear, constant coefficient homogeneous equation has two purely imaginary roots $r = \pm\lambda i$, the solution to the ODE is

$$y = C_1 \sin\lambda t + C_2\cos\lambda t.$$

Let's use a specific example to examine the final situation, where the auxiliary equation has two complex roots $r = k \pm \lambda i$.

◇ **Example 3.1(d):** Solve $y'' + 10y' + 28y = 0$.

**Solution:** Guess $y = e^{rt}$, so $y' = re^{rt}$ and $y'' = r^2 e^{rt}$. Then

$$y'' + 10y' + 28y = r^2e^{rt} + 10re^{rt} + 28e^{rt} = (r^2 + 10r + 28)e^{rt} = 0$$

Again, $e^{rt}$ cannot equal zero, so we solve $r^2 + 10r + 28 = 0$; this is done in this case using the quadratic formula:

$$r = \frac{-10 \pm \sqrt{10^2 - 4(28)}}{2} = -5 \pm 2i\sqrt{2}$$

Therefore

$$\begin{aligned} y &= Ae^{(-5+2i\sqrt{2})t} + Be^{(-5-2i\sqrt{2})t} \\ &= Ae^{-5t}e^{2i\sqrt{2}\,t} + Be^{-5t}e^{-2i\sqrt{2}\,t} \\ &= e^{-5t}\Big[\big(A\cos(2\sqrt{2}\,t) + Ai\sin(2\sqrt{2}\,t)\big) + \big(B\cos(-2\sqrt{2}\,t) + Bi\sin(-2\sqrt{2}\,t)\big)\Big] \\ &= e^{-5t}\Big[\big(A\cos(2\sqrt{2}\,t) + Ai\sin(2\sqrt{2}\,t)\big) + \big(B\cos(2\sqrt{2}\,t) - Bi\sin(2\sqrt{2}\,t)\big)\Big] \\ &= e^{-5t}\big[(A+B)\cos(2\sqrt{2}\,t) + (A-B)i\sin(2\sqrt{2}\,t)\big] \\ &= e^{-5t}[C_1\cos(2\sqrt{2}\,t) + C_2\sin(2\sqrt{2}\,t)] \end{aligned}$$

The solution to $y'' + 10y' + 28y = 0$ is $y = e^{-5t}\big[C_1\cos(2\sqrt{2}\,t) + C_2\sin(2\sqrt{2}\,t)\big]$.

---

In general, we have the following.

> When the auxiliary equation for a second order, linear, constant coefficient homogeneous equation has two complex roots $r = k \pm \lambda i$, the solution to the ODE is
>
> $$y = e^{kt}(C_1 \sin \lambda t + C_2 \cos \lambda t).$$

Example 3.1(d) and the discussion before it show how sine and cosine functions arise when $r_1$ and $r_2$ are complex numbers, but you need not show all those steps when solving. Unless asked to show more, we will just set up and solve the equation $ar^2 + br + c = 0$, then give the solution that arises from the form of the roots. In the Chapter 3 Summary you will find a flowchart detailing the process of finding the solution to a second order, linear, constant coefficient, homogeneous differential equation. *You will need to "memorize" the forms of the solution for the various forms of* $r_1$ *and* $r_2$, but if you do enough exercises, you should know them by the time you are done.

**Section 3.1 Exercises** To Solutions

1. Solve each Euler equation.

   (a) $x^2y'' - 4xy' + 4y = 0$ (b) $x^2y'' + 4xy' + 2y = 0$ (c) $3x^2y'' - xy' + y = 0$

2. Solve each ODE, assuming the independent variable is $t$. Use exact values except for (i) - use decimals rounded to the hundredth's place there.

   (a) $y'' - 2y' - 3y = 0$ (b) $y'' + 2y' + 10y = 0$ (c) $y'' + 10y' + 25y = 0$

   (d) $y'' + 6y' + 11y = 0$ (e) $y'' + 3y' + 2y = 0$ (f) $y'' + 2y = 0$

   (g) $y'' + 2y' + y = 0$ (h) $y'' + 16y = 0$ (i) $y'' + 3.1y' + 4.5y = 0$

3. (a) Give the auxiliary equations for the ODEs $y'' + 25y = 0$ and $y'' + 25y' = 0$.

   (b) Solve the ODE $y'' + 25y' = 0$ using the method of this section.

4. Give the general form of the solution (that is, give one of the forms found in the boxes in this section) to $ay'' + by' + cy = 0$ under each of the following conditions:

   (a) $b = 0$ (b) $b^2 - 4ac > 0$ (c) $b^2 - 4ac < 0,\ b \neq 0$ (d) $b^2 - 4ac = 0$

5. (a) Solve the Euler equation $x^2y'' - 3xy' + 4y = 0$. You should obtain only one solution.

   (b) We know that there should be two solutions. Show that $y = x^2 \ln x,\ \ x > 0,$ is also a solution.

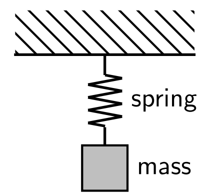
## 3.2 Free, Undamped Vibration

**Performance Criteria:**

3. (c) Set up and solve second order initial value problems modeling spring-mass systems and RLC circuits.
   (d) Sketch or identify the graph of the solution to an IVP for an undamped mass on a spring with no forcing function.
   (e) Write a function $y = A\sin\omega t + B\cos\omega t$ in the alternate form $y = C\sin(\omega t + \phi)$. From this, determine the amplitude, period, frequency, angular frequency and phase shift.

Let's return to the following example:

◇ **Example 1.1(a):** Suppose that a mass is hanging on a spring that is attached to a ceiling, as shown to the right. If we lift the mass, or pull it down, and let it go, it will begin to oscillate up and down. Its height (relative to some fixed reference, like its height before we lifted it or pulled it down) varies as time goes on from when we start it in motion. We say that *height is a function of time*.




In Section 1.2 we derived the differential equation

$$m\frac{d^2y}{dt^2} + ky = 0 \tag{1}$$

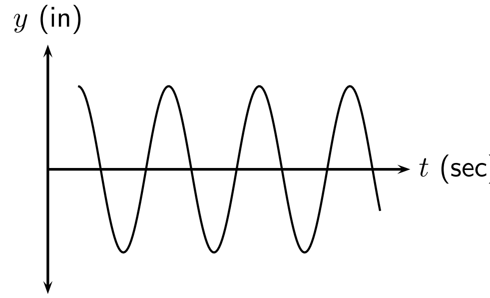
that governs the motion of the mass. Here $y$ is the height (from equilibrium) of the mass at any time $t$ after it is set in motion by some means, *with positive being upward*. $m$ is the mass of the mass (the first of these being a quantity, and the second being an object) and $k$ is the spring constant. We will assume that there is no external force acting on the mass once it is set in motion - in such cases we call the vibration **free**. We will also consider the system to be **undamped**, which means that there is nothing hindering the motion of the mass once it begins moving up and down.

First let's remind ourselves of how we can solve such ODEs:

◇ **Example 3.2(a):** Solve Equation 1 for a mass of 10.3 and a spring constant of 28.7 (both with appropriate units).

**Solution:** The auxiliary equation corresponding to the ODE is $10.3r^2 + 28.7 = 0$. Subtracting 28.7 from both sides and dividing by 10.3 gives $r^2 = -2.786....$ If we then take the square root of both sides and round to the nearest hundredth, we have $r = \pm 1.67i$. Therefore the solution to the ODE is $y = C_1 \sin 1.67t + C_2 \cos 1.67t$.

Intuitively it is reasonable that, once set in motion, a mass on a spring with no damping will oscillate forever with the same amplitude. The graph of such motion would look something like what is shown to the right; this is referred to as **harmonic motion**. It may not be clear how the solution to the above differential equation, which contains a sum of sine and cosine functions, gives a graph like the one shown. We'll soon see a computation that makes it clear how this happens.



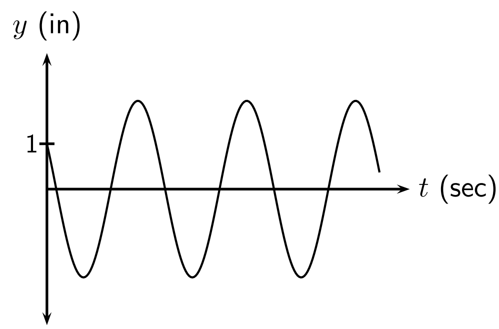

A mass hanging on a spring can be set in motion by doing one of three things:

- moving the mass away from its equilibrium position and letting it go
- giving the mass some initial velocity upward or downward from its initial position
- moving the mass away from its initial position *and* giving it an initial velocity

If we move the mass upward or downward, that will give $y(0)$ equal to some value other than zero, and if we impart an initial velocity the value of $y'(0)$ will be nonzero.

◇ **Example 3.2(b):** Suppose that the mass from Example 3.2(a) is set in motion by lifting it up one inch and giving it an initial velocity of 2 inches per second downward, both at time zero. Give the initial conditions in function form and sketch a graph of what you expect the motion to look like. Then determine the solution to the initial value problem and graph it to check your graph.

**Solution:** The condition of raising the mass up by one inch is given by $y(0) = 1$, and the initial velocity of two inches per second downward is given by $y'(0) = -2$, with the negative indicating that the velocity is in the downward direction. Because the mass starts above its equilibrium point, we expect the $y$-intercept of the graph to be at positive one. The condition of being given an initial velocity downward means that the slope of the tangent line to the graph at $t = 0$ will be negative. We therefore expect a graph something like that shown to the right.




Applying the first initial condition with the solution from Example 3.2(a) gives $C_2 = 1$. Taking the derivative of the solution obtained in Example 3.2(a) gives

$$y' = 1.67C_1 \cos 1.67t - 1.67C_2 \sin 1.67t.$$

Substituting the initial velocity initial condition and $C_2 = 1$ into this gives us $C_1 = -1.20$ (the zero indicates accuracy to the hundredth's place), so the solution to the IVP is

$$y = -1.20 \sin 1.67t + \cos 1.67t.$$

The graph of this function agrees with that shown above.

It is sometimes useful to change an expression of the form $-1.20 \sin 1.67t + \cos 1.67t$ into a single sine function with a phase shift, and here is what we use to do it:

**$C\sin(\omega t+\phi)$ Form**

$$A\sin\omega t + B\cos\omega t = C\sin(\omega t+\phi), \quad \text{where} \quad C=\sqrt{A^2+B^2}$$

$$\text{and} \quad \phi = \tan^{-1}\frac{B}{A} \quad \text{if} \quad A>0, \qquad \phi = \pi + \tan^{-1}\frac{B}{A} \quad \text{if} \quad A<0$$

$$\text{If} \quad A=0, \quad \text{then} \quad B\cos\omega t = B\sin\left(\omega t + \tfrac{\pi}{2}\right)$$

*Note that radian mode must be used when using a calculator to compute $\phi$!*

◇ **Example 3.2(c):** Change the solution $y = -1.20\sin 1.67t + \cos 1.67t$ into the form $y = C\sin(\omega t+\phi)$, where $C$, $\omega$ and $\phi$ are all decimals rounded to the hundredth's place.

$$C = \sqrt{(-1.20)^2+1^2} = 1.56, \qquad \phi = \pi + \tan^{-1}\frac{B}{A} = 2.44$$

Thus the solution can be written as $y = 1.56\sin(1.67t+2.44)$.

---

We see that the amplitude of the motion for the situation from Examples 3.2(a), (b) and (c) is 1.56, greater than the initial position of one unit. That is due to the initial velocity - you will see in the exercises what happens if there is no initial velocity.

There are some important parameters associated with a function $y = C\sin(\omega t+\phi)$:

**Amplitude, frequency, period and phase shift**

For a function $y = C\sin(\omega t+\phi)$,

- $C$ is the **amplitude**
- $\omega$ is the **angular frequency**
- $T = \dfrac{2\pi}{\omega}$ is the **period**
- $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi}$ is the **frequency**
- $-\dfrac{\phi}{\omega}$ is the **phase shift**
- $\omega = 2\pi f$

◇ **Example 3.2(d):** For the function $y = 1.56\sin(1.67t+2.44)$, give the amplitude, period, frequency and phase shift.

**Solution:** The amplitude is $1.56$, the period is $T = \dfrac{2\pi}{1.67} = 3.76$, the frequency is $f = \dfrac{1}{T} = 0.64$ and the phase shift is $-\dfrac{2.44}{1.67} = -1.46$.

---

With a bit of careful thought, the following should be clear: *Only the amplitude and and phase shift are influenced by the initial conditions.* The period, frequency and angular frequency in this case are all determined by $m$ and $k$, the parameters of the system.

## Section 3.2 Exercises

To Solutions

1. Suppose that the mass is set in motion by moving it upward by 2.5 cm and releasing it with no initial velocity.

   (a) Sketch what you think the graph of $y$ versus $t$ will look like, taking care with the fact that positive $y$ is upward. Make the amplitude of the motion clear on your graph.

   (b) Express the initial conditions mathematically by giving values for $y(0)$ and $y'(0)$.

2. Repeat Exercise 1 for a mass that is set in motion by hitting it upward (as it hangs at equilibrium), giving it an initial speed of 3 inches per second.

3. Repeat Exercise 1 for a mass that is set in motion by giving it an initial speed of 8 cm per second downward from a point 2 cm above equilibrium. *The amplitude is not two* - make it clear whether it is more or less than two.

4. A mass of $\frac{3}{4}$ is attached to a spring with a spring constant of 15. It is set into motion by raising it 2.5 cm and releasing it. (See Exercise 1.)

   (a) Set up an initial value problem consisting of a differential equation and two initial conditions.

   (b) Solve the initial value problem to find the position function $y$. Give all numbers as decimals, rounded to the hundredth's place.

   (c) Graph your solution using some technology and compare with your answer to Exercise 1. They should agree; if they don't, figure out what is wrong and fix it!

   (d) Put your solution in the form $C\sin(\omega t + \phi)$. Graph the resulting function and make sure the graph agrees with what you got for (c). If not, try to find and correct your error.

   (e) Give the amplitude, angular frequency, period, frequency and phase shift of the solution.

5. (a) Consider again a mass of $\frac{3}{4}$ attached to a spring with a spring constant of 15, as in Exercise 4. Solve the initial value problem obtained with the initial conditions of Exercise 3, where the initial speed was 8 cm downward from a point 2 cm above equilibrium. Again give all numbers as decimals, rounded to the hundredth's place.

   (b) Graph your solution using technology and compare with your answer to Exercise 3.

   (c) Put your solution in the form $y = A\sin(\omega t + \phi)$. Check it by graphing it and your solution to part (a) together; they should be the same!

   (d) Give the amplitude, angular frequency, period, frequency and phase shift of the solution.

6. A mass of $\frac{4}{10}$ on a spring with spring constant $4$ is given an initial velocity of $9$ cm/sec upward from an initial position of $4$ cm below equilibrium.

   (a) Give the initial value problem.

   (b) Find the equation of motion of the mass, $y(t)$, in the form $y = A\sin(\omega t + \phi)$.

   (c) Give the amplitude, angular frequency, period, frequency and phase shift of the solution.

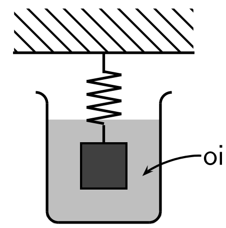
## 3.3 Free, Damped Vibration

**Performance Criteria:**

3. (c) Set up and solve second order initial value problems modeling spring-mass systems and RLC circuits.

   (f) Determine from the coefficients of a second order, constant coefficient homogeneous ODE whether the system it models is (i) underdamped, (ii) critically damped, (iii) overdamped, or (iv) undamped.

   (g) Without finding the solution to the differential equation, sketch the graph of a solution of an overdamped or underdamped homogeneous second order, linear, constant coefficient ODE for given initial conditions.




Consider again a mass on a spring, but suppose that we submerge the mass in an oil bath, as shown to the right (think about an oil-damped shock absorber). As the mass moves up and down there is now an additional force acting on it, the resistance of the oil. We will make the assumption that the force is directly proportional to the velocity but in the opposite direction; that is, for some *positive* constant $\beta$ (the Greek letter *beta*), the force of resistance is given by $-\beta\,\frac{dy}{dt}$.

Suppose that we also have some variable external force acting on the mass as well. (Think force exerted upward on a shock absorber by the road, as a car drives along.) If this force is some function $f(t)$, then the net force on the mass at any time $t$ is

$$F_{\text{net}} = m\,\frac{d^2y}{dt^2} = f(t) - \beta\,\frac{dy}{dt} - ky.$$

If we rearrange this equation we get

$$m\,\frac{d^2y}{dt^2} + \beta\,\frac{dy}{dt} + ky = f(t), \tag{1}$$

the second order differential equation that models the motion of a spring-mass system with damping and an external forcing function $f$.

Now suppose that we have an electric circuit consisting of a resistor, an inductor and a capacitor in series with each other, along with (perhaps) a voltage source. We will refer to such a circuit as an **RLC circuit**, shown schematically to the right. The differential equation that models an RLC circuit is derived from Kirchoff's voltage law which tells us that the sum of the voltage drops across each of the three components is equal to the voltage imposed on the system by the voltage source. By Ohm's law the voltage drop across the resistor is $V = iR$, where $i$ is the current and $R$ is the resistance of the resistor.

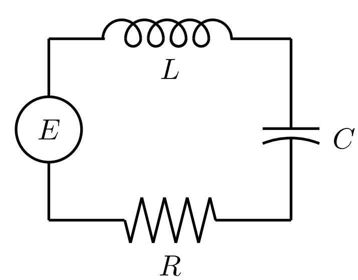



Figure 3.1

The voltage drops across the inductor and the capacitor are $L\dfrac{di}{dt}$ and $\dfrac{1}{C}q$, where $q$ is the charge on the capacitor. From Kirchoff's voltage law we now have

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t), \tag{2}$$

where $E(t)$ is the voltage as a function of time. (It may of course be constant, such as in the case of DC voltage supplied by a battery.)

But the current $i$ is the rate at which charge is passing by a point in the circuit: $i = \dfrac{dq}{dt}$ Thus $\dfrac{di}{dt} = \dfrac{d}{dt}\left(\dfrac{dq}{dt}\right) = \dfrac{d^2q}{dt^2}$ and the above equation becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)\ , \qquad (3),$$

where $L$ is the inductance of the inductor in henries, $R$ is the resistance of the resistor in ohms, and $C$ is the capacitance of the capacitor in farads. *All are positive quantities; this will be important!* If we would rather work with current rather than charge we can differentiate both sides of $(3)$ to get

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'(t)\ , \qquad (4),$$

Note that both $(3)$ and $(4)$ are completely analogous to equation $(1)$, which models a spring-mass system. This illustrates the principle, first encountered in Section 2.5, that different physical situations are often modeled by the same differential equation.

In this section we will consider the situation in which there is no forcing function. That is, the right hand sides of $(1)$, $(3)$ and $(4)$ are zero. Of course something needs to happen to get the mass moving or to get current to flow in the circuit, and each can be accomplished in a variety of ways. For example, the spring and the capacitor both have the ability to store energy by compressing or stretching the spring, or by storing charge in the capacitor (with the positivity or negativity of that charge being analogous to compressing or stretching the spring). As that energy is released it will cause the mass to move or current to flow in the circuit. Here is a summary of initial conditions we can have for the spring, which we previously gave in Section 3.2:

- the mass can be displaced and let go, with no initial velocity
- the mass can be given some velocity at its resting position
- the mass can be displaced *AND* given some initial velocity upward or downward

The analogous conditions for the electric circuit are as follows:

- there is no current in the circuit, but there is an initial charge on the capacitor
- there is no charge on the capacitor, but there is an initial current
- there is both an initial charge on the capacitor and an initial current

Our main objective in this section is to understand the behavior of the solution function $y(t)$ of equation (1), (3) or (4) when $f(t) = 0$ or $E(t) = 0$, and how that behavior varies depending on the *parameters* $m$, $k$ and $\beta$ or $R$, $L$ and $C$. You will be looking at some exercises to do this. Realize that the values given for the parameters may not be realistic - they were chosen in such a way as to make the mathematics reasonable.

**Section 3.3 Exercises** To Solutions

For all exercises in this section you will be working with the equation

$$m\frac{d^2y}{dt^2}+\beta\frac{dy}{dt}+ky=f(t), \tag{1}$$

for various values of $m$, $\beta$ and $k$, **but always with** $f(t)=0$**.**

1. (a) Solve the initial value problem consisting of Equation $(1)$ with $m=5$, $\beta=6$ and $k=80$, and initial conditions $y(0)=2$, $y'(0)=-6$. Give your answer in the form $y=Ce^{at}\sin(\omega t+\phi)$ and all numbers in decimal form, rounded to the nearest tenth. (Note that $5.0007$ rounded to the nearest tenth is $5.0$, *not* $5$! What is the difference?)
   (b) Graph the solution to the IVP on your calculator. Adjust the viewing window to get about three cycles of the motion displayed fairly large. Sketch your graph.
   (c) Graph $y=2.3e^{-0.6t}$ and $y=-2.3e^{-0.6t}$ together with the solution you graphed in (b). Add them to your sketch *as dashed curves*.

What you have just seen is an example of what is called **underdamped vibration**. There is damping, but it is small enough to allow the mass to move up and down while the vibration decays.

2. (a) Solve the initial value problem consisting of Equation $(1)$ with $m=5$, $\beta=50$ and $k=80$, and initial conditions $y(0)=2$, $y'(0)=6$.
   (b) Sketch the graph of your solution over a time period long enough to show what is happening over time. Make your vertical scale such that the general shape of the graph can be seen.
   (c) Compare and contrast your solution to this IVP with the solution to the IVP from Exercise 1, in terms of amplitude over time and oscillation.

For your answer to 2(c) you should have noted that the solution of the IVP in Exercise 1 oscillates as it decays, and the solution of the IVP in Exercise 2 does not. We say that the situation from Exercise 2 is **overdamped vibration**. The initial conditions do not affect whether the vibration will be underdamped or overdamped (can you see why not?), so the only difference is the value of $\beta$.

3. (a) Determine a value of $\beta$ that is the "dividing line" between equations have solutions that oscillate as they decay and those that do not oscillate. Explain how you determined it.
   (b) Solve the IVP with $m=5$, $k=80$ and the value of $\beta$ you obtained in (a), along with the initial conditions from Exercise 2. Graph the solution using some technology and sketch the graph.
   (c) Suppose now that $m=5$ and $k=80$. How would you expect the solution to $(1)$ to behave if $\beta$ was slightly smaller than the value obtained in (a), and why. How would you expect the solution to behave for $\beta$ slightly larger than the value obtained in (a)? **Answer these questions in complete sentences.**

For your answer to 3(c) you should have said that if $\beta$ is a little less than the value that you found in (a) the solution will behave like that in Exercise 1, and if $\beta$ is a little more than that value the solution will behave like that in Exercise 2. The situation in Exercises 3(a) and 3(b) is called the **critically damped** case, meaning that if there is just a little less damping the behavior will be oscillatory, but if the damping has that critical value or greater, the motion will not be oscillatory.

4. Describe a method for determining from the coefficients $m$, $\beta$ and $k$ whether the solution will be underdamped, overdamped, or critically damped.

5. Sketch the graph of the solution to a 2nd-order, constant-coefficient homogeneous ODE with the given damping conditions and initial conditions.

   (a) Underdamped, $y(0) < 0,\ y'(0) > 0.$

   (b) Overdamped, $y(0) < 0,\ y'(0) < 0.$

   (c) Overdamped, $y(0) > 0,\ y'(0) < 0.$ (I think there may be a couple different appearances possible here - we'll investigate this more later.)

   (d) Undamped (not *under*damped, but *un*damped), $y(0) > 0,\ y'(0) > 0.$

6. Consider a circuit like that shown in Figure 3.1, but without the voltage source $E = 0$. The components are a 0.15 henry inductor, a 500 ohm resistor and a $2 \times 10^{-5}$ farad capacitor, the initial charge on the capacitor is $1.3 \times 10^{-6}$ coulombs and there is no initial current. (This can be achieved by having an open switch in the circuit and closing the switch at time zero.)

   (a) Is the system overdamped, underdamped, or critically damped?

   (b) Use equation (3) to determine the charge on the capacitor at any time $t$. Round all constants to three significant figures and give correct units with your answer.

   (c) If you didn't already in part (b), put your answer in $y = Ce^{k}t\sin(\omega t + \phi)$ form.

   (d) Give the current in the circuit at any time $t$, again giving units with your answer and rounding to two significant digits.

## 3.4 Particular Solutions, Part One

**Performance Criteria:**

3. (h) Find a particular solution to a second order linear, constant coefficient ODE using the method of undetermined coefficients.

In the previous section we found out how to solve equations of the form

$$ay'' + by' + cy = f(t) \tag{1}$$

when $f(t) = 0$, but in practice we are usually interested in situations where $f(t) \neq 0$. For those situations we will use a technique called the **method of undetermined coefficients** to find a solution. To do this we (again!) substitute a guess, which we will call the trial particular solution, into the ODE. This trial solution will contain one or more unknown constants, whose values can be determined by finding the result obtained when the trial solution is substituted into the left hand side of the ODE and then setting it equal to the known right hand side $f(t)$. Like terms are then equated, and the constants determined. The resulting solution is called the **particular solution** to $(1)$. Later we will see how the particular solution is combined with the homogeneous solution to give the general solution to $(1)$.

Let's consider the ODE

$$y'' + 9y = 5e^{-2t}.$$

If $y$ had the form $y = Ae^{-2t}$ for some constant $A$, then $y''$ would have the same form. Perhaps there is then a choice of $A$ for which $y'' + 9y$ will equal $5e^{-2t}$. The next example shows us that is in fact the case.

◇ **Example 3.4(a):** Find a value $A$ for which $y = Ae^{-2t}$ is a solution to $y'' + 9y = 5e^{-2t}$.

**Solution:** First we observe that

$$y = Ae^{-2t} \quad \Longrightarrow \quad y' = -2Ae^{-2t} \quad \Longrightarrow \quad y'' = 4e^{-2t}.$$

Substituting these into the left side of $(1)$ we get

$$y'' + 9y = 4Ae^{-2t} + 9Ae^{-2t} = 13Ae^{-2t}.$$

We now set this equal to the right hand side of the ODE to get $13Ae^{-2t} = 5e^{-2t}$. The only way that this can be true is if the coefficients of $e^{-2t}$ are equal: $13A = 5 \;\Rightarrow\; A = \frac{5}{13}$. Therefore $y = \frac{5}{13}e^{-2t}$ is a solution to $y'' + 9y = 5e^{-2t}$.

---

A particular solution to a differential equation is any solution that does not contain arbitrary constants, so $y = \frac{5}{13}e^{-2t}$ is a particular solution to the ODE $y'' + 9y = 5e^{-2t}$. We sometimes subscript the dependent variable with the letter $p$ to indicate a particular solution. For the above we would then write $y_p = \frac{5}{13}e^{-2t}$. This distinction is made because, as we will soon see, there are other solutions as well.

Next we will examine the ODE $y'' + 7y' + 10y = 5t^2 - 8$. We might guess that the particular solution will be a fourth degree polynomial, because then $y''$ would be a second degree polynomial. It turns out that there is no harm in trying a fourth degree polynomial (you will try it in the exercises), but in fact a second degree polynomial is adequate. The next example demonstrates this.

◇ **Example 3.4(b):** Find the coefficients $A$, $B$ and $C$ for which $y_p = At^2 + Bt + C$ is a solution to $y'' + 7y' + 10y = 5t^2 - 8$.

**Solution:** First we compute the needed derivatives:

$$y_p = At^2 + Bt + C \qquad \Longrightarrow \qquad y_p' = 2At + B \qquad \Longrightarrow y_p'' = 2A.$$

Next we substitute these values into the left side of the ODE and *group by powers of* $t$:

$$\begin{aligned} y_p'' + 7y_p' + 10y_p &= 2A + 7(2At + B) + 10(At^2 + Bt + C) \\ &= 2A + 14At + 7B + 10At^2 + 10Bt + 10C \\ &= 10At^2 + (14A + 10B)t + (2A + 7B + 10C) \end{aligned}$$

Setting this equal to the right hand side of the ODE gives us

$$10At^2 + (14A + 10B)t + (2A + 7B + 10C) = 5t^2 + 0t - 8,$$

and equating coefficients of powers of $t$ (including the constant term) results in the three equations

$$10A = 5, \qquad 14A + 10B = 0, \qquad 2A + 7B + 10C = -8.$$

From the first equation we see that $A = \frac{1}{2}$. Substituting that value into the second equation and solving for $B$ results in $B = -\frac{7}{10}$. Finally, when we substitute the values we have found for $A$ and $B$ into the last equation and solve for $C$ we get $C = -\frac{41}{100}$. The particular solution to $y'' + 7y' + 10y = 5t^2 - 8$ is then

$$y_p = \tfrac{1}{2}t^2 - \tfrac{7}{10}t - \tfrac{41}{100}.$$

---

It is important to note that even though the right hand side $5t^2 - 8$ contains only a $t^2$ term and a constant term, we need to include a $t$ term in our guess for the particular solution. If we had instead guessed a particular solution of the form $y_p = At^2 + B$ and substituted it into the ODE we would have obtained the two equations

$$10A = 5 \qquad \text{and} \qquad 14A = 0,$$

both of which cannot be true at the same time!

So far we have found a particular solution to $ay'' + by' + cy = f(t)$ when $f(t)$ is an exponential function (Example 3.4(a)) and a polynomial (Example 3.4(b)). The only other type of function we will consider for $f(t)$ is a trigonometric function; suppose we have the ODE

$$y'' + 4y' + 3y = 5\sin 2t. \tag{2}$$

Based on what we have seen so far, our first inclination might be that we should consider a particular solution of the form $y_p = A\sin 2t$. Let's try it:

$$y_p = A\sin 2t \qquad \Longrightarrow \qquad y_p' = 2A\cos 2t \qquad \Longrightarrow \qquad y_p'' = -4A\sin 2t$$

so

$$\begin{aligned} y_p'' + 4y_p' + 3y_p &= -4A\sin 2t + 8A\cos 2t + 3A\sin 2t \\ &= -A\sin 2t + 8A\cos 2t. \end{aligned}$$

Here we need $A = 0$ because there is no cosine term on the right hand side of (2), but then we'd have no sine term either!

So what do we do? Well, the "trick" is to let $y_p$ have both sine and cosine terms even though the right side of the ODE has only a sine term.

◇ **Example 3.4(c):** Determine values for $A$ and $B$ so that $y_p = A\sin 2t + B\cos 2t$ is the particular solution to $y'' + 4y' + 3y = 5\sin 2t$.

**Solution:** We see that

$$y_p' = 2A\cos 2t - 2B\sin 2t, \qquad y_p'' = -4A\sin 2t - 4B\cos 2t.$$

so

$$\begin{aligned} y_p'' + 4y_p' + 3y_p &= (-4A\sin 2t - 4B\cos 2t) \\ &\quad + 4(2A\cos 2t - 2B\sin 2t) + 3(A\sin 2t + B\cos 2t) \\ &= (-A - 8B)\sin 2t + (8A - B)\cos 2t \end{aligned}$$

Thus, in order for the left hand side to equal the right hand side, it must be the case that $8A - B = 0$ because there is no cosine term in the right hand side, and $-A - 8B = 5$, so that the sine terms are equal. Solving the first equation for $B$ and substituting into the second equation results in $A = -\frac{1}{13}$. Substituting this back into the second equation gives $B = -\frac{8}{13}$. Our particular solution is then $y_p = -\frac{1}{13}\sin 2t - \frac{8}{13}\cos 2t$.

---

At this point we have the following guesses for a particular solution to a differential equation of the form $ay'' + by' + cy = f(t)$ when using the method of undetermined coefficients:

- If $f$ is a polynomial of degree $n$, then

$$y_p = A_n t^n + A_{n-1}t^{n-1} + \cdots + A_2 t^2 + A_1 t + A_0$$

  Note that all powers of $t$ less than or equal to the degree of $f(t)$ are included.

- If $f(t) = Ce^{kt}$, then $y_p = Ae^{kt}$.

- If $f(t) = C_1\sin kt + C_2\cos kt$, then $y_p = A\sin kt + B\cos kt$. *Even if one of $C_1$ or $C_2$ is zero, the trial $y_p$ must still contain both the sine and cosine terms.*

In Section 4.3 we will see that there is a bit more to be added to this story, but for now the above summarizes what we have seen so far.

**Section 3.4 Exercises** To Solutions

1. For each of the following, give the form the particular solution must have.

   (a) $y'' + 3y' + 2y = 5t - 1$  (b) $y'' + 6y' + 9y = \cos 2t$

   (c) $y'' + 9y = 2e^{5t}$  (d) $y'' - 4y' - 5y = 6\sin t$

   (e) $2y'' + 3y' + y = 7$  (f) $y'' + 3y = 2t + 3e^{-t}$

2. Determine the particular solution for each of the ODEs in Exercise 1.

3. Suppose that you thought that the ODE $y'' + 3y' + 2y = 5t - 1$ should have a particular solution of $y_p = At^3 + Bt^2 + Ct + D$. (Note that this is the ODE from Exercise 1(a).) Substitute this into the ODE and see what happens for this guess. Does it give you the correct particular solution?

4. You would think that the particular solution to $y'' + 3y' + 2y = 6e^{-t}$ would have the form $y_p = Ae^{-t}$, but that is not the case. In Section 4.3 we will see what our guess for the particular solution should be. For now, try substituting the given particular solution into the ODE to see what happens.

5. Solve each of the following homogeneous ODEs, assuming the independent variable for each is $t$.

   (a) $y'' + 4y' + 29y = 0$
   (b) $2y'' + 11y' + 5y = 0$
   (c) $y'' + 6y' + 9y = 0$
   (d) $y'' + 3y = 0$

## 3.5 Differential Operators

**Performance Criterion:**

3. (i) Evaluate a differential operator for a given function.

### An Example

In this section we will begin by exploring a specific second order ODE in order to illustrate some ideas we will capitalize on in order to solve linear, constant coefficient, second order ODEs. The ODE that we will be considering is

$$y'' + 9y = 5e^{-2t}, \tag{1}$$

which we found to have a particular solution of $y_p = \frac{5}{13}e^{-2t}$. (See Example 3.1(a).) This was obtained by substituting a guess of $y = Ae^{-2t}$ for $y$ in $y'' + 9y$ and setting the result equal to $5e^{-2t}$. The following example shows that $y = \frac{5}{13}3^{-2t}$ is not the only solution to (1).

◇ **Example 3.5(a):** Show that $y = C\sin 3t + \frac{5}{13}e^{-2t}$, where $C$ is *any* constant, is a solution to the differential equation (1).

**Solution:** Taking derivatives we get

$$y' = 3C\cos 3t - \tfrac{10}{13}3^{-2t} \qquad \text{and} \qquad y'' = -9C\sin 2t + \tfrac{20}{13}e^{-2t}.$$

Substituting in to the left hand side of the ODE we get

$$\begin{aligned} y'' + 9y &= -9C\sin 2t + \tfrac{20}{13}e^{-2t} + 9(C\sin 3t + \tfrac{5}{13}e^{-2t}) \\ &= -9C\sin 2t + \tfrac{20}{13}e^{-2t} + 9C\sin 3t + \tfrac{45}{13}e^{-2t}) \\ &= \tfrac{65}{13}e^{-2t} \\ &= 5e^{-2t} \end{aligned}$$

Thus $y = C\sin 3t + \frac{5}{13}e^{-2t}$ is a solution to $y'' + 9y = 5e^{-2t}$.

To examine further what is going on here it is convenient to develop some terminology and notation.

### Differential Operators

A function can be thought of as a "mathematical machine" that takes in a number and, in return, gives out a number. There are other mathematical machines that take in things other than numbers, like functions or vectors, usually giving out things like what they take in. These sorts of "machines" are often referred to as **operators**. A simple example of an operator that you are quite familiar with is the derivative operator. When we take the derivative of a function, the result is another function. To indicate the action of a derivative on a function $y = y(t)$ we will write $\dfrac{dy}{dt}$ as

$$\frac{d}{dt}(y),$$

which is like the function notation $f(x)$, with $\frac{d}{dt}$ taking the place of $f$ and $y$ taking the place of $x$. The derivative operator has a very special property that you should be familiar with from calculus. If $a$ and $b$ are any constants and $y_1$ and $y_2$ are functions of $t$, then

$$\frac{d}{dt}(ay_1 + by_2) = \frac{d}{dt}(ay_1) + \frac{d}{dt}(by_2) = a\frac{d}{dt}(y_1) + b\frac{d}{dt}(y_2). \tag{2}$$

Operators that "distribute over addition" and "pass through constants" like this are called **linear operators**.

You might guess that the second derivative, and other higher derivatives, are linear operators as well, and that is correct. We are particularly interested in operators that are created by multiplying a function and some of its derivatives by constants and adding them all together. It is customary to denote such operators with the letter $D$, for **differential operator**. An example would be $D = 3\frac{d^2}{dt^2} + 5\frac{d}{dt} - 4$, whose action on a function $y = y(t)$ is defined by

$$D(y) = 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} - 4y. \tag{3}$$

Let's look at a specific example of how this operator works.

◇ **Example 3.5(b):** For the operator $D$ defined by (3), find $D(y)$ when $y = t^2 - 3t$.

**Solution:**

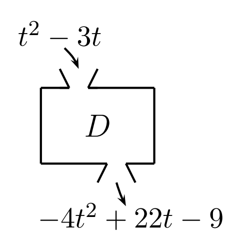
$$\begin{aligned} D(y) &= 3\frac{d^2}{dt^2}(t^2 - 3t) + 5\frac{d}{dt}(t^2 - 3t) - 4(t^2 - 3t) \\ &= 3(2) + 5(2t - 3) - 4(t^2 - 3t) \\ &= 6 + 10t - 15 - 4t^2 + 12t \\ &= -4t^2 + 22t - 9 \end{aligned}$$

When we combine several mathematical objects by multiplying each by a constant and adding (or subtracting) the results, we obtain what is called a **linear combination** of those objects. The operator $D$ defined by (3) is a linear combination of a function and its first two derivatives. (We can think of the function itself as the "zeroth derivative," making all the things being combined derivatives.) You have seen linear combinations in other contexts; any polynomial function like

$$f(x) = 3x^4 - 7x^3 + \frac{1}{3}x^2 - x + 5.83$$

is a linear combination of $1, x, x^2, x^3, x^4, \ldots$. Those of you who have had a course in linear algebra have seen linear combinations of vectors.

Again, we can think of an operator as a machine that takes in a function and gives out some resulting function that is based somehow on the input function. This is illustrated to the right for Example 3.5(b). The next example demonstrates that a differential operator formed as a linear combination of derivatives is a linear operator.




◇ **Example 3.5(c):** Show that the operator $D$ defined by

$$D(y) = 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} - 4y$$

is a linear operator by showing that it satisfies (2).

**Solution:** To determine whether $D$ is linear we need to apply it to $ay_1 + by_2$:

$$\begin{aligned}
D(ay_1 + by_2) &= 3\frac{d^2}{dt^2}(ay_1 + by_2) + 5\frac{d}{dt}(ay_1 + by_2) - 4(ay_1 + by_2) \\
&= 3\left(a\frac{d^2y_1}{dt^2} + b\frac{d^2y_2}{dt^2}\right) + 5\left(a\frac{dy_1}{dt} + b\frac{dy_2}{dt}\right) - 4ay_1 - 4by_2 \\
&= 3a\frac{d^2y_1}{dt^2} + 3b\frac{d^2y_2}{dt^2} + 5a\frac{dy_1}{dt} + 5b\frac{dy_2}{dt} - 4ay_1 - 4by_2 \\
&= 3a\frac{d^2y_1}{dt^2} + 5a\frac{dy_1}{dt} - 4ay_1 + 3b\frac{d^2y_2}{dt^2} + 5b\frac{dy_2}{dt} - 4by_2 \\
&= a\left(3\frac{d^2y_1}{dt^2} + 5\frac{dy_1}{dt} - 4y_1\right) + b\left(3\frac{d^2y_2}{dt^2} + 5\frac{dy_2}{dt} - 4y_2\right) \\
&= aD(y_1) + bD(y_2)
\end{aligned}$$

Because $D(ay_1 + by_2) = aD(y_1) + bD(y_2)$, $D$ is a linear operator.

At the second line above we have applied the fact that the first and second derivative are linear operators. With a bit of thought it should be clear that this, along with the distributive property, is what makes a linear combination of derivatives a linear operator.

### Back to the Example

We return now to considering the ODE

$$y'' + 9y = 5e^{-2t}, \tag{1}$$

for which we have shown that both

$$y = \tfrac{5}{13}e^{-2t} \qquad \text{and} \qquad y = C\sin 3t + \tfrac{5}{13}e^{-2t} \tag{4}$$

are solutions. If we now let $D$ be the operator defined by $D(y) = y'' + 9y$, then (4) says that

$$D\Big(\tfrac{5}{13}e^{-2t}\Big) = 5e^{-2t} \qquad \text{and} \qquad D\Big(C\sin 3t + \tfrac{5}{13}e^{-2t}\Big) = 5e^{-2t}.$$

Looking a little more closely, we see that

$$D\big(C\sin 3t\big) = \frac{d^2}{dt^2}\big(C\sin 3t\big) + 9\big(C\sin 3t\big) = -9C\sin 3t + 9C\sin 3t = 0.$$

This explains why $D$ applied to the sum of $C\sin 3t$ and $\frac{5}{13}e^{-2t}$ is a solution; by linearity of differential operators,

$$D\Big(y_1 + \tfrac{5}{13}e^{-2t}\Big) = D\big(y_1\big) + D\Big(\tfrac{5}{13}e^{-2t}\Big) = 0 + 5e^{-2t} = 5e^{-2t}$$

for any function $y_1$ for which $D\big(y_1\big) = 0$ (like $C\sin 3t$, for example). This gives us the following:

Let $D$ be a linear differential operator with independent variable $t$. If $y_2$ is a solution to $D(y_2) = f(t)$ and $y_1$ is a solution to $D(y_1) = 0$, then it is also the case that $D(y_1 + y_2) = f(t)$.

The general form of equation that we are most interested in is

$$ay'' + by' + cy = f(t), \tag{5}$$

where $a$, $b$ and $c$ are constants. Our goal is to find a **general solution** to this equation, meaning a solution that encompasses all possible solutions. Such a solution will consist of a particular solution to (5) that contains no arbitrary constants plus the family of *all* possible solutions to

$$ay'' + by' + cy = 0. \tag{6}$$

The solution to (5) without arbitrary constants is of course the **particular solution** to the equation, and the family of all possible solutions to (6) is the **homogeneous solution**. We saw in Section 3.1 how to find the homogeneous solution $y_h$ and in Section 3.2 we saw how to find the particular solution $y_p$. The general solution is then $y = y_h + y_p$.

As an example, we know that the ODE

$$y'' + 9y = 5e^{-2t}$$

has homogeneous solution $y_h = C_1 \sin 3t + C_2 \cos 3t$ and particular solution $y_p = \frac{5}{13}e^{-2t}$, so the general solution is

$$y = y_h + y_p = C_1 \sin 3t + C_2 \cos 3t + \tfrac{5}{13}e^{-2t}.$$

For reasons you will see in Section 4.3, we will always find the homogeneous solution first and then the particular solution.

**Section 3.5 Exercises** To Solutions

1. Let the differential operator $D$ be defined on a function $y = y(t)$ by

$$D(y) = \frac{d^2}{dt^2}(y) + 3\frac{d}{dt}(y) + 2y.$$

   Find $D(y)$ for each of the following functions $y$.

   (a) $y = t^2 + 7t$ (b) $y = 5e^{-2t}$ (c) $y = 5\cos 2t$ (d) $y = \frac{5}{2}t - \frac{17}{4}$

2. What does your answer to Exercise 1(d) tell us about the ODE $y'' + 3y' + 2y = 5t - 1$?

3. Although you don't realize why at this point, your answer to Exercise 1(b) is somewhat special.

   (a) Does the same thing happen for $y = Ce^{-2t}$, where $C$ is some constant other than 5? If so, for what value or values of $C$?

   (b) Does the same thing happen if $y = e^{kt}$, where $k$ is some constant other than $-2$? If so, for what value or values of $k$?

4. Let the differential operator $D$ be defined on a function $y = y(t)$ by

$$D(y) = \frac{d^2}{dt^2}(y) + 6\frac{d}{dt}(y) + 9y.$$

Find $D(y)$ for each of the following functions $y$.

(a) $y = e^{-3t}$ (b) $y = te^{-3t}$ (c) $y = 5e^{-3t} - 2te^{-3t}$

5. Let $S$ be an operator on functions. $S$ is linear if

$$S(af + bg) = aS(f) + bS(g), \tag{7}$$

where $a$ and $b$ are any constants and $f$ and $g$ are any functions of the sort that $S$ can act on. $(7)$ is equivalent to the two separate conditions that

$$S(af) = aS(f) \qquad \text{and} \qquad S(f + g) = S(f) + S(g), \tag{8}$$

where $a$, $f$ and $g$ are as before. That is, if both conditions in $(8)$ hold for $S$, then it is a linear operator. Now let's define a specific operator $S$ by $S(f(t)) = f(t) + 3$ for any function $f(t)$.

(a) What is $S(\cos t)$? $S(t^2 + 5t - 1)$?

(b) What is $S(4\cos t)$? What is $4S(\cos t)$? Are both results the same? What does that tell us about $S$, in terms of linearity?

(c) Find $S(\cos t + e^{2t})$ and $S(\cos t) + S(e^{2t})$.

## 3.6 Initial Value Problems and Forced, Damped Vibration

**Performance Criteria:**

3. (j) Solve a second order linear, constant coefficient IVP.
   (c) Set up and solve second order initial value problems modeling spring-mass systems and RLC circuits.
   (k) Identify the transient and steady-state parts of the solution to a damped system with forced vibration.

Now that we know how to find the general solution to $ay'' + by' + cy = f(t)$ we are ready to solve initial value problems whose ODEs are of this form. Here is how the process goes:

**Solving the IVP** $\quad ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_0'$

1) Find the homogeneous solution $y_h$ to $ay'' + by' + cy = 0$.
2) Use the method of undetermined coefficients to find the particular solution $y_p$ of the ODE $ay'' + by' + cy = f(t)$.
3) Construct the general solution $y = y_h + y_p$.
4) Apply the initial conditions to the general solution to find the values of the arbitrary constants.

We have already covered the first three steps of the above. It is very important to remember that *the initial conditions are applied to the general solution* to find the values of the arbitrary constants. A common mistake by students is to find the values of the constants based on only the homogeneous solution - this is incorrect.

Let's look at an example:

◇ **Example 3.6(a):** Solve the IVP

$$y'' + 4y' + 3y = 5\sin 2t, \quad y(0) = 2, \quad y'(0) = -1$$

**Solution:** We must first solve the homogeneous equation $y'' + 4y' + 3y = 0$. The roots of the auxiliary equation are $r_1 = -1$ and $r_2 = -3$ (of course it doesn't matter which is which), so the homogeneous solution is $y_h = C_1e^{-t} + C_2e^{-3t}$. Our trial particular solution is $y_p = A\sin 2t + B\cos 2t$. This gives us

$$y_p' = 2A\cos 2t - 2B\sin 2t, \qquad y_p'' = -4A\sin 2t - 4B\cos 2t.$$

so

$$\begin{aligned}
\text{LHS} &= (-4A\sin 2t - 4B\cos 2t) + 4(2A\cos 2t - 2B\sin 2t) + 3(A\sin 2t + B\cos 2t) \\
&= (-A - 8B)\sin 2t + (8A - B)\cos 2t
\end{aligned}$$

Thus, in order for the left hand side to equal the right hand side, it must be the case that $8A - B = 0$ because there is no cosine term in the right hand side, and $-A - 8B = 5$, so that the sine terms are equal. Solving the first equation for $B$ and substituting into the second equation results in $A = -\frac{1}{13}$. Substituting this back into the second equation gives $B = -\frac{8}{13}$. Our particular solution is then $y_p = -\frac{1}{13}\sin 2t - \frac{8}{13}\cos 2t$, and the general solution is

$$y = y_h + y_p = C_1e^{-t} + C_2e^{-3t} - \tfrac{1}{13}\sin 2t - \tfrac{8}{13}\cos 2t$$

The derivative of the general solution is

$$y' = -C_1e^{-t} - 3C_2e^{-3t} - \tfrac{2}{13}\cos 2t + \tfrac{16}{13}\sin 2t$$

Applying the initial conditions gives us the two equations

$$C_1 + C_2 - \tfrac{8}{13} = 2 \qquad \text{and} \qquad -C_1 - 3C_2 - \tfrac{2}{13} = -1$$

Adding these and solving for $C_2$ gives us $C_2 = -\frac{23}{26}$. Substituting this into either equation gives $C_1 = \frac{91}{26}$. The solution to the IVP is then

$$y = \tfrac{91}{26}e^{-t} - \tfrac{23}{26}e^{-3t} - \tfrac{1}{13}\sin 2t - \tfrac{8}{13}\cos 2t$$

It is often the case that finding the constants $C_1$ and $C_2$ comes down to solving a system of two equations in two unknowns, as it did here.

**Section 3.6 Exercises** To Solutions

1. You may have found the particular solution to each of the following ODEs in Exercise 2 of Section 3.4. Give the general solution to each.

   (a) $y'' + 3y' + 2y = 5t - 1$
   (b) $y'' + 6y' + 9y = 5\cos 3t$
   (c) $y'' + 9y = 2e^{5t}$
   (d) $y'' - 4y' - 5y = 6\sin t$
   (e) $2y'' + 3y' + y = 7$
   (f) $y'' + 3y = 2t + 3e^{-t}$

2. Solve each of the following IVPs by the process described in the box at the start of the section.

   (a) $y'' + 9y = 4\sin t\,, \quad y(0) = 2\,, \quad y'(0) = 4$
   (b) $y'' + 4y' + 4y = 5e^{3t}\,, \quad y(0) = 0\,, \quad y'(0) = 0$
   (c) $y'' - 10y' + 25y = 30t + 3\,, \quad y(0) = 2\,, \quad y'(0) = 8$

3. Solve each Euler equation.

   (a) $x^2y'' - 6y = 0$
   (b) $4x^2y'' + 4xy' - y = 0$

4. Consider the second order initial value problem

$$y'' + 2y' + 10y = 9.4\sin t\,, \qquad y(0) = 5\,, \qquad y'(0) = 0,$$

which could model either a spring-mass system or an RLC circuit. In this exercise you will find the solution to this initial value problem, and you will investigate its behavior. Recall the process for solving such an equation:

- Find the homogeneous solution $y_h$. It will contain two constants.
- Use undetermined coefficients (guessing) to find the particular solution $y_p$ to the equation.
- Add $y_c$ and $y_p$ to find the general solution to the equation.
- Apply the two initial conditions to determine the values of the unknown constants. **Be sure to conclude by writing your final solution to the IVP**.

(a) Carry out the above process for the given IVP. **Give all numbers as decimals, rounded to the tenth's place.**

(b) Graph the solution on your calculator or an online tool like *Desmos*, for $t = 0$ to $t = 10$. Sketch your graph.

(c) Look carefully at your solution (the equation itself, not the graph). Recall that the transient part of a solution is any part that goes to zero as time goes on. Any part that does not go to zero over time is called the steady-state part of the solution, or just the steady-state solution. Give the transient and steady-state parts of your solution, telling clearly which is which.

(d) Graph the steady-state solution together with the complete solution that you already graphed. The complete solution should approach the steady state solution as time goes on. Add the graph of the steady-state solution to your graph from (b).

### 3.7 Chapter 3 Summary

With the exception of the methods for solving Euler equations, this chapter was primarily concerned with solving initial value problems of the form

$$ay'' + by' + cy = f(t), \qquad y(0) = y_0,\ y'(0) = y_0', \tag{1}$$

with $a$, $b$ and $c$ being constants, $a \neq 0$. Here is the procedure for solving the initial value problem (1):

**Solving the IVP** $ay'' + by' + cy = f(t),\ y(0) = y_0,\ y'(0) = y_0'$

1) Find the homogeneous solution $y_h$ to $ay'' + by' + cy = 0$. *There is a flowchart on the next page, outlining the process for finding the homogeneous solution.*
2) Use the method of undetermined coefficients (see below) to find the particular solution $y_p$ of the ODE $ay'' + by' + cy = f(t)$.
3) Construct the general solution $y = y_h + y_p$.
4) Apply the initial conditions to the general solution to find the values of the arbitrary constants.

**Undetermined Coefficients**

Consider the constant coefficient ODE $ay'' + by' + cy = f(t)$, and assume $y_h$ contains no terms that are constant multiples of $f(t)$. The trial particular solution $y_p$ is chosen as follows.

- If $f$ is a polynomial of degree $n$, then

$$y_p = A_n t^n + A_{n-1} t^{n-1} + \cdots + A_2 t^2 + A_1 t + A_0$$

- If $f(t) = Ce^{kt}$, then $y_p = Ae^{kt}$.
- If $f(t) = C_1 \sin kt + C_2 \cos kt$, then $y_p = A \sin kt + B \cos kt$. *Even if one of $C_1$ or $C_2$ is zero, $y_p$ still contains both the sine and cosine terms.*

We will see in the next chapter how the method of undetermined coefficients needs to be modified when $y_h$ contains any term that is a constant multiple of $f$.

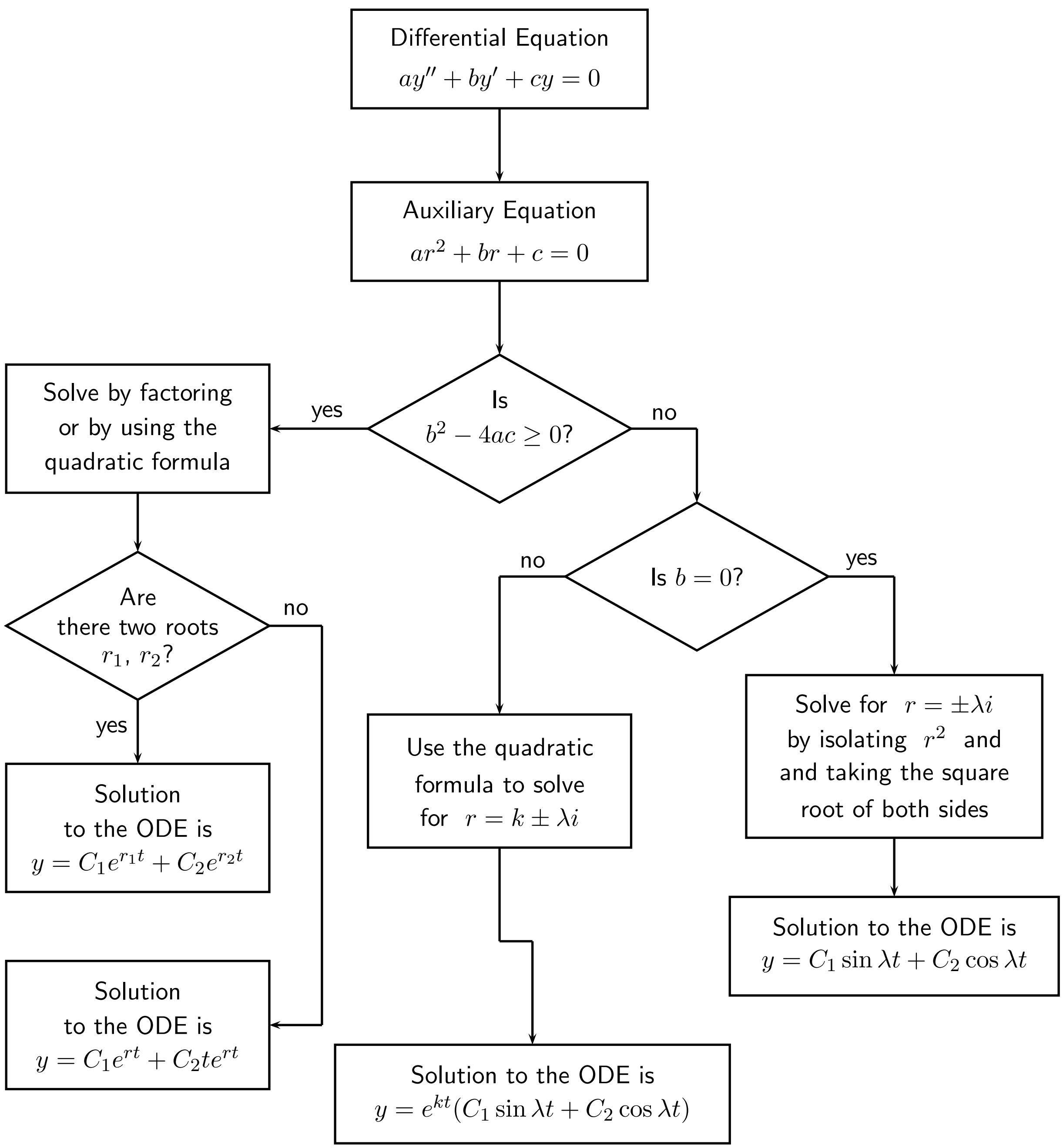
A couple of additional comments are in order:

- The homogeneous solution has the form $y_h = C_1 g(t) + C_2 h(t)$, where $C_1$ and $C_2$ are arbitrary constants and $g$ and $h$ are "different" functions (in a sense that will be made more precise in the next chapter) that are each solutions to the homogeneous ODE $ay'' + by' + cy = 0$ by themselves. *Every possible solution to the homogeneous ODE looks like $y_h = C_1 g(t) + C_2 h(t)$ for the same functions $g$ and $h$.*

- The particular solution to the ODE in $(1)$ contains no arbitrary constants, which is why it is called "particular." Another method for finding the particular solution is called **variation of parameters**. The interested reader can find an explanation of this method on the internet or in any introductory differential equations text.

Here is a flowchart for finding homogeneous solutions:

**Solving Second Order, Linear, Constant Coefficient, Homogeneous ODEs**



### 3.8 Chapter 3 Exercises

1. Solve each Euler equation.

   (a) $3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$  (b) $6x^2\dfrac{d^2y}{dx^2} + 11x\dfrac{dy}{dx} + y = 0$

2. In Exercise 4 of the Chapter 2 Exercises we saw the Euler equation

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2R = 0 \tag{1}$$

   which arises in the study of the equilibrium distribution of heat in a circular disk. As pointed out before, $r$ and $R$ are two different variables; $R$ is the dependent variable, and is a function of the independent variable $r$. In Chapter 2 we solved the equation for the case $n = 0$. Solve it for any integer $n \neq 0$.

3. The ODE $ay'' + by' + cy = 0$ is only second order if $a \neq 0$. We saw in Section 3.3 what the solution to the ODE looks like when $b = 0$. In this exercise and the next we will solve, by two different methods, an equation in which $c = 0$

   (a) In this exercise we will solve $2y'' + 3y' = 0$. begin by making the substitution $y' = x$ (so what then is $y''$?) and solving the resulting first order ODE for $x$.

   (b) To determine $y$ you will now need to integrate your answer to (a). DO that, remembering that your final solution needs to contain two arbitrary constants.

4. Solve $2y'' + 3y' = 0$ by assuming $y = e^{rt}$ and following a process like that done for the various scenarios in Section 3.1.

# 4 More on Second Order Differential Equations

**Learning Outcome:**

4. Understand independence of solutions to ODEs, and know how to use reduciton of order to find second solutions. Understand the nature of solutions to second order linear, constant coefficient ODEs and IVPs modeling spring-mass systems or RLC circuits, including resonance and beats.

**Performance Criteria:**

(a) Demonstrate that two functions $f$ and $g$ are dependent by giving nonzero constants $c_1$ and $c_2$ for which $c_1 f(x) + c_2 g(x) = 0$.

(b) Use the Wronskian to determine whether two solutions to a second order linear ODE are independent.

(c) Given one solution to a second order homogeneous ODE, use reduction of order to find a second solution.

(d) Determine the particular solution to a differential equation of the form $ay'' + by' + cy = f(t)$ when the homogeneous solution has the same form as $f(t)$.

(e) Determined whether a forced, undamped system will exhibit resonance, beats, or neither. Determine the solution for such a system.

(f) For a spring-mass system or electric circuit, demonstrate an understanding of the relationships between

- the physical situation (presence and type of damping and/or forcing)
- the form of the ODE, including the function $f$
- the the analytic and graphical nature of the solution (in particular, the presence and appearance of transient and steady-state parts of the solution)

The bulk of our efforts in Chapter 3 were focused on solving second order ODEs of the form

$$ay'' + by' + cy = f(t). \qquad (1)$$

There are three issues that came up that we put off at the time:

- When solving the homogeneous equation $ay'' + by' + cy = 0$ we usually found two "different" solutions, but when solving an equation like $y'' + 6y' + 9y = 0$ we only found one solution, $y = e^{-3t}$.
- In some cases when we attempted to find a particular solution to $(1)$ the "standard" guess for a trial particular solution failed to give us a result.
- We neglected to address situation in which $b = 0$ and $f(t) \neq 0$, which we call forced, undamped vibration.

Regarding the first item, there are two questions we will address:

(1) What do we mean by "different" solutions?

(2) In addition to the solution $y = e^{-3t}$ to $y'' + 6y' + 9y = 0$ that we found using the auxiliary equation, we also saw that $y = te^{-3t}$ is a solution. How is such a solution found?

The first question above addresses the concept of **linear indepndence** of solutions. If you have had a course in linear algebra you should be familiar with the idea in that context. This is addressed in Section 4.1. The second question is answered by a method called **reduction of order**, which we'll see in Section 4.2.

In Section 4.3 we will return to the finding of particular solutions. When one of the solutions to the homogeneous equation $ay'' + by' + cy = 0$ associated with the ODE

$$ay'' + by' + cy = f(t). \tag{1}$$

has the same form as $f(t)$, our previously used guesses for particular solutions will not yield a result, so we must modify our trial particular solution in a way described in Sectoin 4.3.

Finally, we'll go back to undamped systems, but with nonzero forcing functions $f(t)$, which are often sine or cosine functions. This will give rise to two phenomena called **beats** and **resonance**. These things will be studied in Section 4.4.

## 4.1 Linear Independence of Solutions

**Performance Criteria:**

4. (a) Demonstrate that two functions $f$ and $g$ are dependent by giving nonzero constants $c_1$ and $c_2$ for which $c_1f(x)+c_2g(x)=0$.

   (b) Use the Wronskian to determine whether two solutions to a second order linear ODE are independent.

We begin with two questions:

(1) When solving the ODE $y''+3y'+2y=0$, we assumed a solution of the form $y=e^{rt}$ for some constant $r$ and found that $r$ must equal $-1$ or $-2$. We then assumed that every solution to the ODE is of the form $y=C_1e^{-t}+C_2e^{-2t}$. How do we know that this is the case?

(2) When solving $y''+2y'+y=0$ we found only one solution, $y=e^{-t}$. We then demonstrated that $y=te^{-t}$ is also a solution, and we assumed that the general solution to the ODE is $y=C_1e^{-t}+C_2te^{-t}$. How might one know or find the second solution without it being given?

In this section we will develop some language and see some theorems that answer the first question, and in the next section we'll see a way to use **reduction of order** (see Chapter 2 exercises) that gives an answer to the second question.

### Linearly Independent Solutions

**Linearly Independent Functions**

Two functions $f$ and $g$ are **linearly dependent** on an interval $[a,b]$ if there exist two *non-zero* constants $c_1$ and $c_2$ for which

$$c_1f(x)+c_2g(x)=0 \qquad \text{for every } x \text{ in } [a,b]. \tag{1}$$

If (1) is true only when both $c_1$ and $c_2$ are zero, then $f$ and $g$ are **linearly independent** on $[a,b]$.

The expression $c_1f(x)+c_2g(x)$ above is called a **linear combination** of $f$ and $g$.

◇ **Example 4.1(a):** Show that the two functions $y=e^{-2t}$ and $y=e^{-t}$ are linearly independent for all values of $t$.

**Solution:** Suppose that $c_1e^{-2t}+c_2e^{-t}=0$ for some $c_1$ and $c_2$. Then $e^{-2t}(c_1+c_2e^{t})=0$, but $e^{-2t}$ is never zero so it must be the case that $c_1+c_2e^t=0$, which implies that $c_1=-c_2e^t$. Because $e^t$ is not constant, this can only be true if $c_1=c_2=0$. Therefore $y=e^{-2t}$ and $y=e^{-t}$ are linearly independent.

There may be times that it is difficult to tell, using the definition above, whether two functions are linearly independent. In those cases we can use a new function created from the two functions, called the **Wronskian**, to determine whether the functions are linearly independent.

**The Wronskian of Two Functions**

The **Wronskian** $W$ of two functions $f$ and $g$ is

$$W(x) = f(x)g'(x) - f'(x)g(x).$$

Those of you who have had linear algebra may recognize the Wronskian as the determinant of the $2 \times 2$ matrix

$$\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix}$$

Our interest is in determining whether two solutions to an ODE are linearly independent. Note that any linear second order homogeneous ODE with independent variable $t$ can (almost always, and definitely in the case that $p$ and $q$ are constant) be written in the form

$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

The following tells how the Wronskian is used to determine whether two solutions to an equation of the form $(2)$ are linearly independent.

**The Wronskian and Linearly Independent Solutions**

Two solutions $f$ and $g$ of $(2)$ are linearly independent on the interval $(a, b)$ if there exists some point $x$ in the interval for which $W(x) \neq 0$.

◇ **Example 4.1(b):** Show that the two solutions $y = \sin 3t$ and $y = \cos 3t$ of $y'' + 9y = 0$ are linearly independent for all values of $t$.

**Solution:** The Wronskian of these two functions is

$$W(t) = (\sin 3t)(\cos 3t)' - (\cos 3t)(\sin 3t)' = -\sin^2 3t - \cos^2 3t = -1,$$

which is clearly not zero for *any* value of $t$. Therefore $y = \sin 3t$ and $y = \cos 2t$ are linearly independent for all values of $t$.

We conclude with why we are interested in all of this.

**General Solutions to $y'' + p(t)y' + q(t)y = 0$**

If $y_1$ and $y_2$ are linearly independent solutions to $y'' + p(t)y' + q(t)y = 0$, then the general solution is

$$y = C_1 y_1 + C_2 y_2$$

for arbitrary constants $C_1$ and $C_2$.

Note that the above says that the general solution is a linear combination of the two solutions $y_1$ and $y_2$.

In the exercises you will show that $y = e^{-t}$ and $y = te^{-t}$ are linearly independent solutions to $y'' + 2y' + y = 0$, so the above tells us that the general solution is then $y = C_1 e^{-t} + C_2 te^{-t}$. In the next section we'll find out where the solution $y = te^{-t}$ comes from.

## Section 4.1 Exercises

To Solutions

1. For each pair of functions, give nonzero constants $c_1$ and $c_2$ for which $c_1 f(x) + c_2 g(x) = 0$ for all real numbers $x$ *if possible*. Note that when this can be done, the two functions are dependent.

   (a) $f(x) = 3x^2 - 5,\ \ g(x) = 2x + 1$

   (b) $f(x) = 4x + 2,\ \ g(x) = 2x + 1$

   (c) $f(x) = 3e^{5x},\ \ g(x) = -2e^{5x}$

   (d) $f(x) = 3e^{5x},\ \ g(x) = e^{2x}$

2. Use the facts that $\cos(-x) = \cos x$ and $\sin(-x) = -\sin(x)$ for the following.

   (a) Give nonzero constants $c_1$ and $c_2$ such that $c_1 \cos x + c_2 \cos(-x) = 0$. Are $\cos x$ and $\cos(-x)$ linearly independent?

   (b) Repeat part (a) for $\sin x$ and $\sin(-x)$.

3. For each pair of functions in Exercise 1 that you could not find nonzero constants $c_1$ and $c_2$ for which $c_1 f(x) + c_2 g(x) = 0$, give the Wronskian and one value of $x$ for which it is not zero.

4. Find the Wronskian for $y_1 = e^{kt}$ and $y_2 = te^{kt}$ (where $k$ is any nonzero constant) and give a value of $t$ for which it is not zero. What does this tell us about the functions $y_1$ and $y_2$?

5. Use the Wronskian to determine whether $e^x$ and $e^{-x}$ are linearly independent.

## 4.2 Reduction of Order

**Performance Criteria:**

4. (c) Given one solution to a homogeneous second order ODE, use reduction of order to find a second solution.

Recall the questions with which we begin the previous section:

(1) When solving the ODE $y'' + 3y' + 2y = 0$, we assumed a solution of the form $y = e^{rt}$ for some constant $r$ and found that $r$ must equal $-1$ or $-2$. We then assumed that every solution to the ODE is of the form $y = C_1e^{-t} + C_2e^{-2t}$. How do we know that this is the case?

(2) When solving $y'' + 2y' + y = 0$ we found only one solution, $y = e^{-t}$. We then demonstrated that $y = te^{-t}$ is also a solution, and we assumed that the general solution to the ODE is $y = C_1e^{-t} + C_2te^{-t}$. How might one know or find the second solution without it being given?

The result in the box at the bottom of page 122 answers the first question. In this section we take up the second question.

**Reduction of order** is a method for finding a second solution to a second order differential equation when one solution is already known. Our main interest in this is finding the second solution when we have repeated roots, so we will not go into the method in excessive detail. Perhaps the best way to introduce the method is through an example. The two key ideas are these:

- We will assume that if $y_1 = y_1(t)$ is a solution, then the second solution has the form $y_2(t) = u(t)y_1(t)$ for some function $u$. We then substitute $y_2$ into the ODE, which results in a new ODE for $u$.
- The new ODE for $u$ will contain $u''$ and $u'$ terms, but no $u$ term. If we let $v(t) = u'(t)$ then $v'(t) = u''(t)$, and making these two substitutions we get a first order equation in $v$. (This is where the name *reduction of order* comes from - we've reduced a second order equation to a first order equation.) We solve that for $v$, then solve $u'(t) = v(t)$ to get $u$.

Now let's get to that example!

◇ **Example 4.2(a):** Use the solution $y_1(t) = e^{-t}$ and reduction of order to find a second solution to $y'' + 3y' + 2y = 0$.

**Solution:** We begin by assuming $y_2 = u(t)e^{-t}$. Then (using the product rule),

$$y_2' = -u(t)e^{-t} + u'(t)e^{-t} \qquad \text{and} \qquad y_2'' = u(t)e^{-t} - 2u'(t)e^{-t} + u''(t)e^{-t}.$$

Substituting into the ODE we get

$$\begin{aligned} y_2'' + 3y_2' + 2y_2 &= \big[u(t)e^{-t} - 2u'(t)e^{-t} + u''(t)e^{-t}\big] + 3\big[-u(t)e^{-t} + u'(t)e^{-t}\big] + 2u(t)e^{-t} \\ &= u(t)e^{-t} - 2u'(t)e^{-t} + u''(t)e^{-t} - 3u(t)e^{-t} + 3u'(t)e^{-t} + 2u(t)e^{-t} \\ &= u''(t)e^{-t} + u'(t)e^{-t} \\ &= e^{-t}(u''(t) + u'(t)) \end{aligned}$$

Setting the result equal to zero (because we want $y_2 = u(t)e^{-t}$ to be a solution to $y''+3y'+2y = 0$) and noting that $e^{-t}$ is never zero, we must have $u''(t)+u'(t) = 0$. Here we let $v(t) = u'(t)$, so $v'(t) = u''(t)$ and this last ODE becomes $v'(t)+v(t) = 0$. This equivalent to $v'(t) = -v(t)$, so $v(t) = C_1e^{-t}$.

We now replace $v(t)$ with $u'(t)$ to obtain $u'(t) = C_1e^{-t}$. The solution to this is $u(t) = C_2e^{-t}+C_3$; for reasons to be given later, we can take $C_2$ to be any non-zero value and $C_3$ can have any value. We'll take $C_2 = 1$ and $C_3 = 0$. Therefore $y_2(t) = u(t)e^{-t} = e^{-t}e^{-t} = e^{-2t}$. Disregarding the constant (because we will replace it when adding this solution to the one given), we have the second solution $y_2 = e^{-2t}$.

---

Forming the linear combination of the given solution and the one that we found using it, we get $y = ae^{-t} + be^{-2t}$ for constants $a$ and $b$. We now examine the way that the constants $C_2$ and $C_3$ were handled in the above. Let's see what would have happened if we had not let $C_2 = 1$ and $C_3 = 0$. In that case we would have had

$$y_2 = u(t)y_1(t) = (C_2e^{-t} + C_3)e^{-t} = C_2e^{-2t} + C_3e^{-t}.$$

When we then form a linear combination of $y_1$ and $y_2$ using constants $A$ and $B$, we'll get

$$\begin{aligned} y &= Ae^{-t} + B(C_2e^{-2t} + C_3e^{-t}) \\ &= Ae^{-t} + BC_2e^{-2t} + BC_3e^{-t} \\ &= (A + BC_3)e^{-t} + BC_2e^{-2t} \\ &= ae^{-t} + be^{-2t}, \end{aligned}$$

where $a = A + BC_3$ and $b = BC_2$. If we keep the constants $C_2$ and $C_3$, they essentially get "absorbed" into the constants for the linear combination of the two solutions.

In Example 4.2(a) there was no need to use reduction of order to determine a second solution $y_2 = e^{-2t}$ from the first solution $y_1 = e^{-2t}$; we could arrive at both solutions via the auxiliary equation method. However, the above example demonstrates how the method works. In the exercises you will encounter ODEs for which you will again be asked to find a second solution by this method when it is unnecessary, but you will also use it for situations where the second solution (and maybe the first as well) can't be obtained by methods we have used so far. You will also use reduction of order to find the second solution to $y'' + 2y' + y = 0$, knowing the first solution $y_1 = e^{-t}$, which is obtained by the auxiliary equation method.

**Section 4.2 Exercises** To Solutions

1. Consider the ODE $y'' + 8y' + 15y = 0$.

   (a) Given that one solution is $y_1 = e^{-5t}$, use reduction of order to find another solution.

   (b) Use the auxiliary equation to find both solutions, to check your answer to (a).

2. Using the auxiliary equation method with $y'' + 2y' + y = 0$, we get the single solution $y_1 = e^{-t}$. Use reduction of order to obtain the second solution $y_2 = te^{-t}$.

3. Given that one solution to $2x^2y'' + xy' - 3y = 0$ is $y_1 = \frac{1}{x}$, find a second solution $y_2$.

4. Given that one solution to $x^2y'' + 2xy' - 2y = 0$ is $y_1 = x$, find a second solution $y_2$.

## 4.3 Particular Solutions, Part Two

**Performance Criteria:**

4. (d) Determine the particular solution for a differential equation of the form $ay'' + by' + cy = f(t)$ when the homogeneous solution has the same form as $f(t)$.

At this point you have seen the entire process for solving initial value problems for second order, linear, constant coefficient differential equations. In this section we see one difficulty that can arise, and how to handle such situations. We begin with an example.

◇ **Example 4.3(a):** Determine the values of $A$ and $B$ for which

$$y_p = A\sin 3t + B\cos 3t$$

is the particular solution to the ODE $y'' + 9y = 2\sin 3t$.

**Solution:** As usual, we begin by finding the derivatives of $y_p$:

$$y_p' = 3A\cos 3t - 3B\sin 3t \qquad \Longrightarrow \qquad y_p'' = -9A\sin 3t - 9B\cos 3t.$$

We then have

$$\text{LHS} = y_p'' + 9y_p = -9A\sin 3t - 9B\cos 3t + 9(A\sin 3t + B\cos 3t) = 0.$$

Thus there are no values of $A$ and $B$ for which $y = A\sin 3t + B\cos 3t$ is a solution to $y'' + 9y = 2\sin 3t$.

---

The problem here is that the homogeneous solution to $y'' + 9y = 2\sin 3t$ is $y_h = C_1\sin 3t + C_2\sin 3t$. Thus we cannot hope to obtain $2\sin 3t$ when applying the operator $D = \dfrac{d^2}{dt^2} + 9$ to $y = A\sin 3t + B\cos 3t$, as the result is always zero. However, we will find that a different guess for $y_p$ will give us the particular solution that we seek:

◇ **Example 4.3(b):** Determine the values of $A$ and $B$ for which

$$y_p = At\sin 3t + Bt\cos 3t$$

is the particular solution to the ODE $y'' + 9y = 2\sin 3t$.

**Solution:** We *carefully* use the product rule to find the derivatives of $y_p$:

$$y_p' = 3At\cos 3t + A\sin 3t - 3Bt\sin 3t + B\cos 3t$$

and

$$y_p'' = -9At\sin 3t + 3A\cos 3t + 3A\cos 3t - 9Bt\cos 3t - 3B\sin 3t - 3B\sin 3t.$$

Grouping the like terms of the second derivative gives us

$$y_p'' = -9At\sin 3t - 9Bt\cos 3t - 6B\sin 3t + 6A\cos 3t.$$

Substituting into the left side of the ODE gives us

$$\begin{aligned} y_p'' + 9y_p &= -9At\sin 3t - 9Bt\cos 3t - 6B\sin 3t + 6A\cos 3t + 9(At\sin 3t + Bt\cos 3t) \\ &= -6B\sin 3t + 6A\cos 3t. \end{aligned}$$

In order for this to equal $2\sin 3t$ we must have $A = 0$ and $B = -\frac{1}{3}$, so the particular solution to $y'' + 9y = 2\sin 3t$ is $y_p = -\frac{1}{3}t\cos 3t$.

---

We already knew the homogeneous solution, so the general solution to $y'' + 9y = 2\sin 3t$ is

$$y = C_1\sin 3t + C_2\cos 3t - \tfrac{1}{3}t\cos 3t.$$

We can now make an amendment to the listing at the end of Section 3.4 to get the overall summary for guesses to use for particular solutions.

**Undetermined Coefficients**

Consider the constant coefficient ODE $ay'' + by' + cy = f(t)$, and assume $y_h$ contains no terms that are constant multiples of $f(t)$. The trial particular solution $y_p$ is chosen as follows.

- If $f$ is a polynomial of degree $n$, then

$$y_p = A_nt^n + A_{n-1}t^{n-1} + \cdots + A_2t^2 + A_1t + A_0$$

- If $f(t) = Ce^{kt}$, then $y_p = Ae^{kt}$.
- If $f(t) = C_1\sin kt + C_2\cos kt$, then $y_p = A\sin kt + B\cos kt$. *Even if one of $C_1$ or $C_2$ is zero, $y_p$ still contains both the sine and cosine terms.*

When $y_h$ contains any term that is a constant multiple of $f$, $y_p$ will be as above but multiplied by the smallest power of $t$ for which no terms of $y_p$ are of the same form as any terms of $y_h$.

◇ **Example 4.3(c):** Find the trial particular solution to $y'' + y' - 6y = 5t - 3$.

**Solution:** The homogeneous solution is $y_h = C_1e^{-3t} + C_2e^{2t}$, so the trial particular solution is $y_p = At + B$.

---

◇ **Example 4.3(d):** Find the trial particular solution to $y'' + y' - 6y = 7\cos 5t$.

**Solution:** The homogenous solution is the same as in Example 4.3(c), so trial particular solution is $y_p = A\sin 5t + B\cos 5t$.

---

◇ **Example 4.3(e):** Find the trial particular solution to $y'' + y' - 6y = 4e^{2t}$.

**Solution:** Again the homogeneous solution is $y_h = C_1e^{-3t} + C_2e^{2t}$. $f(t)$ has the same form as one of the terms of the homogeneous solution, the trial particular solution is $y_p = Ate^{2t}$.

---

We conclude this section by further examining homogeneous and particular solutions to an ODE

$$ay'' + by' + cy = f(t). \tag{1}$$

Let's denote the left side of $(1)$ using the operator notation $D(y)$. We have found that the homogeneous solution consists of a linear combination of two functions $g(t)$ and $h(t)$ that are both, by themselves, solutions to $D(g) = 0$ and $D(h) = 0$. By a linear combination we mean

$$y_h = C_1g(t) + C_2h(t),$$

where $C_1$ and $C_2$ are *ANY* constants. When $D$ is applied to the particular solution $y_p$ the result is $D(y_p) = f(t)$. The general solution is

$$y = C_1g(t) + C_2h(t) + y_p(t).$$

Applying $D$ to the solution then gives

$$D(C_1g + C_2h + y_p) = C_1D(g) + C_2D(h) + D(y_p) = 0 + 0 + f(t) = f(t).$$

Note the use of the fact that $D$ is a linear operator in this computation.

◇ **Example 4.3(f):** The general solution to $(1)$ is $y = C_1e^{-2t} + C_2e^{-t} + 4\cos 5t$. Which of the following are solutions to $ay'' + by' + cy = 0$?

(a) $y = 5e^{-t}$ (b) $y = 7e^{-2t} + 4\cos 5t$ (c) $y = 7e^{-2t} + 5e^{-t}$

**Solution:** The homogeneous solution is the part containing the arbitrary constants, $y_h = C_1e^{-2t} + C_2e^{-t}$. It is a solution to $ay'' + by' + cy = 0$ for all choices of $C_1$ and $C_2$, so the functions in (a) and (c) are both solutions. The function in (b) is a solution to (1), but not to $ay'' + by' + cy = 0$ because $D$ applied to $7e^{-2t}$ is zero, but when applied to the particular solution $y_p = 4\cos 5t$ the result is $f(t)$, not zero.

---

◇ **Example 4.3(g):** The general solution to $(1)$ is $y = C_1e^{-2t} + C_2e^{-t} + 4\cos 5t$. Which of the following are solutions to $(1)$?

(a) $y = 4\cos 5t$ (b) $y = 8\cos 5t$ (c) $y = 7e^{-2t} + 5e^{-t} + 4\cos 5t$

**Solution:** We again recognize that the homogeneous solution is $y_h = C_1e^{-2t} + C_2e^{-t}$ and the particular solution is $y_p = 4\cos 5t$. Because the particular solution by itself is a solution to (1), the function in (a) is a solution. Unlike the homogeneous solution, a constant in the particular solution *is not* arbitrary, so the function in (b) is not a solution. (Test it to see for sure?) The function in (c) is a solution, because it is simply the general solution with the arbitrary constants having the specific values $C_1 = 7$ and $C_2 = 5$.

---

## Section 4.3 Exercises

To Solutions

1. Below are each of the ODEs from Examples 4.3(c), (d) and (e). In each case, substitute the given trial particular solution from the example into the ODE to determine the value(s) of any constant(s).

   (c) $y'' + y' - 6y = 5t - 3, \qquad y_p = At + B$

   (d) $y'' + y' - 6y = 7\cos 5t, \qquad y_p = A\sin 5t + B\cos 5t$

   (e) $y'' + y' - 6y = 4e^{2t}, \qquad y_p = Ate^{2t}$

2. Suppose that when you were finding the particular solution to $y'' + y' - 6y = 4e^{2t}$ you didn't notice that $4e^{2t}$ was of the same form as one of the terms of $y_h$. Try a particular solution of $y_p = Ae^{2t}$ and see what happens. It will try to tell you that something is wrong!

3. Solve each of the following IVPs by the process described in the box at the start of Section 3.4.

   (a) $y'' + y' - 6y = 1 + 8t - 6t^2\,, \qquad y(0) = 2\,, \qquad y'(0) = -3$

   (b) $y'' + 7y' + 10y = 6e^{-2t}\,, \qquad y(0) = 2\,, \qquad y'(0) = -11$

   (c) $y'' + 4y = 3\sin 2t\,, \qquad y(0) = \frac{1}{2}\,, \qquad y'(0) = \frac{5}{2}$

4. The functions below are solutions to second order linear, constant coefficient initial value problems. Give the steady-state and transient parts of each.

   (a) $y = -\frac{2}{3}\sin 3t + \frac{5}{3}\cos 3t$

   (b) $y = e^{-3t}(4\sin t + 7\cos t) + \frac{3}{4}\cos 7t$

   (c) $y = \frac{3}{5}\sin 5t - \frac{6}{5}\cos 5t + \frac{7}{2}e^{-2t}$

   (d) $y = 3te^{-5t} - 7e^{-5t} + e^{-t}$

5. Suppose that the ODE
$$ay'' + by' + cy = f(t) \tag{1}$$
has general solution
$$y = e^{-2t}\big(C_1\sin 3t + C_2\cos 3t\big) + 5e^{-2t}.$$
Which of the following are then solutions to $(1)$?

   (a) $y = 7e^{-2t}\sin 3t + 5e^{-2t}$

   (b) $y = e^{-2t}\big(3\sin 3t - 2\cos 3t\big) + 5e^{-2t}$

   (c) $y = e^{-2t}\big(3\sin 3t - 2\cos 3t\big)$

   (d) $y = e^{-2t}\big(3\sin 3t - 2\cos 3t\big) + 4e^{-2t}$

   Which of the following are solutions to
$$ay'' + by' + cy = 0? \tag{2}$$

   (e) $y = 7e^{-2t}\sin 3t + 5e^{-2t}$

   (f) $y = e^{-2t}\big(3\sin 3t - 2\cos 3t\big)$

   (g) $y = 7e^{-2t}\sin 3t$

   (h) $y = e^{-2t}\big(C_1\sin 3t + C_2\cos 3t\big) + 4e^{-2t}$

## 4.4 Forced, Undamped Vibration

**Performance Criterion:**

4. (e) Determined whether a forced, undamped system will exhibit resonance, beats, or neither. Determine the solution for such a system.

Suppose that we have either

- a mass on a spring with no damping, subject to a sinusoidal external force, or
- an inductor and a capacitor (no resistor) in series with a voltage source that is putting out a sinusoidal current.

How would we expect the functions describing the position of the mass or the charge on the capacitor to behave? That is what you will investigate in the exercises for this section. You might want to take a guess as to what you would expect, before the mathematics of the situation answers the question.

**Section 4.4 Exercises** To Solutions

1. For this exercise and each of the following, work in decimals, rounding all values to the nearest tenth.

   (a) Solve the initial value problem

   $$x'' + 4.84x = 8\cos 5t\ , \qquad x(0) = x'(0) = 0$$

   (b) Graph the solution using some technology, sketch the graph. Be sure to get a viewing window that is appropriate. Put a scale on your graph.

   (c) Discuss the situation of transient and steady-state solutions. Why should we expect this before even solving the differential equation?

2. (a) Solve the initial value problem

   $$x'' + 4.84x = 8\cos 2.2t\ , \qquad x(0) = x'(0) = 0$$

   (b) Graph the solution using some technology, sketch the graph.

   (c) The phenomenon you are observing here is called **resonance**. In either the mechanical or electrical case, as the amplitude gets larger and larger, something will fail - the spring or one of the electrical components. What is it about the situation that is causing this to happen?

Note that the angular frequency of $2.2$ that appears in the solution to the IVP comes from the homogeneous equation, so it depends only on the spring-mass or LC system, not on the forcing function. That frequency is sometimes called the **natural frequency** of the system.

3. (a) Solve the initial value problem

   $$x'' + 4.84x = 8\cos 2t\ , \qquad x(0) = x'(0) = 0$$

   (b) Graph your solution from $t = 0$ to $t = 75$. Sketch the graph.

(c) The phenomenon you are observing here is called **beats** - in electronics this is **amplitude modulation**. All I know about this is that the AM in AM radio stands for amplitude modulation (FM is frequency modulation)! Ask your local EET instructor for details. Look carefully at how this initial value problem compares with the other two. What do you suppose it is that is causing the beats?

(d) A trig identity can help us get a little better insight into the solution. You should be able to write your solution in the form $x(t) = A(\cos\omega_0 t - \cos\omega t)$. Use the identity

$$\cos u - \cos v = 2\sin\left(\frac{v-u}{2}\right)\sin\left(\frac{u+v}{2}\right)$$

to rewrite your solution. The new form of the solution is trying to talk to you. Can you see what it is trying to tell you?

(e) Graph $y = 19\sin(0.1t)$ and $y = -19\sin(0.1t)$ together with the graph of the solution, and sketch what you see. Can you *now* see what the solution to (d) is trying to tell you?

## 4.5 Chapter 4 Summary

**Performance Criteria:**

4. (f) For a spring-mass system or electric circuit, demonstrate an understanding of the relationships between

    - the physical situation (presence and type of damping and/or forcing)
    - the form of the ODE, including the function $f$
    - the the analytic and graphical nature of the solution (in particular, the presence and appearance of transient and steady-state parts of the solution)

In this section we will attempt to summarize all that we have seen in Chapters 3 and 4. In particular, we want to recognize from an ODE, the solution to an ODE, or the graph of the solution to an ODE whether it models a situation

- in which the system is undamped, under-damped, critically damped or over-damped
- with or without an external forcing function
- for which the solution has transient or steady-state parts, or both
- resulting in or exhibiting resonance or beats

The type of differential equation that we are talking about here is one of the form

$$ay'' + by' + cy = f(t) \tag{1}$$

where the coefficients $a, b$ and $c$ are constant parameters based on the physical properties of the system we are considering:

- In a spring-mass system $a$ is the mass, $b$ is the coefficient of damping, and $c$ is the spring constant.
- In an electric circuit, $a$ is the inductance, $b$ is the resistance and $c$ is the reciprocal of the capacitance.

It is clear that without a mass and a spring there is no spring-mass system, so for that situation neither $a$ nor $c$ can be zero. For the electric circuit situation it is reasonable to consider a system with only a resistance and inductance, but that can be treated as a first order ODE in a manner you have already seen. For an RLC circuit of the sort we wish to consider, none of the values $a$, $b$ or $c$ are zero, although we will consider the case $b = 0$ as a theoretical possibility.

### The Left Side of the ODE

*The left hand side of the ODE describes the system itself.* In the case of a spring-mass system, it is the spring, the mass, and the damping. In an electric circuit it is the resistor, inductor and capacitor. *The system doesn't cause motion or current, it just shapes it by the way it reacts to the forcing function and/or initial conditions.* There will not be any motion or current unless there are nonzero initial conditions, a forcing function $f$ or $E$, or both. What is of real concern to us on the left hand side of the equation (1) is the role of $b$, which controls the damping. Here is a summary of that:

- When $b = 0$ the system is undamped.
- When $b^2 - 4ac < 0$ the system is under-damped. Oscillation will occur, but any oscillation due to the initial conditions will decay. (The solution will have a transient part if either of the initial conditions is nonzero.) Any steady-state behavior will be due to the forcing function $f$.
- When $b^2 - 4ac > 0$ the system will be over-damped, and there will be no oscillation due to the system itself. The system will again have a transient part, and any steady-state part of the solution is again due to the external forcing function $f$.
- When $b^2 - 4ac = 0$ the system is critically damped, and the solution will behave similarly to the over-damped situation. The transient part will have a $te^{-kt}$ term $(k > 0)$, but still decays over time because $e^{-kt}$ decays faster than $t$ grows.

Any quick investigation of an ODE of the form (1) should perhaps begin by observing whether a damping term is present. If it is, computation of $b^2 - 4ac$ should follow to determine which of the last three cases above we are dealing with.

**Initial Conditions**

In order to solve a second order ODE, we must have two initial conditions. For a spring-mass system the meaning of the initial conditions is pretty straightforward. The initial position tells us whether the mass is raised or pulled down at time zero. In either case, there is potential energy due to either gravity (for $y(0) > 0$) or the spring (for $y(0) < 0$) that is converted to kinetic energy of motion when the mass is let go. $y'(0)$ is the initial velocity imparted to the mass; it is negative if the initial velocity is downward, and positive if the initial velocity is upward. For an RLC circuit, $q(0)$ is the initial charge on the capacitor, which has electric potential that can cause current, and $i(0) = q'(0)$ is the initial current.

The effect of initial conditions is not lasting, unless the system is undamped. For any damped system, the initial conditions will lead to a transient part of the solution. For an undamped system, initial conditions will lead to a steady-state part of the solution.

**The Forcing Function $f$**

The function $f$, on the right hand side of (1), is the forcing function that is imposed on the system. In the case of the spring-mass system it might be something like effect of bumps in the road for a shock absorber, or perhaps the effect of some vibration added by a motor. For an electrical circuit, it is the voltage source that is supplying the circuit. Often $f$ will be, in reality, a periodic function made up of sine or cosine functions. For this reason it is sufficient to understand the behavior of the system when $f$ is a single trig function.

$f$ provides input to the system over time, unlike the initial conditions, which only supply input right at time zero. It, of course, leads to the particular solution to (1). At this point we will only consider decaying exponential or trigonometric forcing functions.

- A decaying exponential function is itself transient, so in a mathematical sense when $f$ is such a function it leads to a transient part of the solution. (That part of the solution is the particular solution.) For an undamped system such a forcing function will also act to cause a steady-state part of the solution as well, even in the absence of initial values. (In that case it acts, in a sense, like an initial velocity.)
- When $f$ is a periodic function like a trig function, it will provide input to the system forever. Because of this, it will usually lead to a periodic steady-state part of the solution. The one exception is for an undamped system, where we find the following behaviors:

- When the frequency of the forcing function is significantly different from the natural frequency (sometimes called the **resonant frequency**) of the system, the result is a steady-state solution with two parts, one with the resonant frequency and one with the frequency of the forcing function.
- When the frequency of the forcing function is the same as the resonant frequency of the system, the forcing function will cause part the solution to be trigonometric functions with linearly increasing amplitude. This is the condition we call **resonance**.
- When the frequency of the forcing function is close to the resonant frequency of the system, it will cause vibration with increasing amplitude when the forcing function is in phase with the vibration. Eventually the forcing function will become out of phase with the natural vibration, canceling it out. It will then get back in phase, then out, over and over. The result is the phenomenon called **beats**.

**Exercises on the next page.**

## Chapter 4 Exercises

To Solutions

1. Here are some equations of the sort we have been discussing:

(i) $y'' + 3y' + 2y = 0$

(ii) $y'' + 4y = \sin(9t)$

(iii) $y'' + 10y' + 25y = 0$

(iv) $y'' + 25y = 3.1\sin(5t)$

(v) $y'' + 9y = 0$

(vi) $y'' + 5y' + 7y = 7.4\sin(2.4t)$

(vii) $y'' + 3y' + 5y = 0$

(viii) $y'' + 16y = 7\cos(3.8t) - 4\sin(3.8t)$

   (a) Which equations model an undamped system? Which model an under-damped system? Critically damped? Over-damped?

   (b) Which equations will have solutions with a transient part? Which will have solutions with a steady-state part?

   (c) Which equations will have solutions that exhibit resonance? Which will have solutions exhibiting beats?

2. Consider the following *solutions* to differential equations of the type we have been discussing (second order, constant coefficient).

(i) $y = C_1e^{-t} + C_2e^{-3t}$

(ii) $y = C_1e^{-2t} + c_2te^{-2t}$

(iii) $y = C_1\cos(5.1t) + C_2\sin(5.1t)$

(iv) $y = e^{-1.2t}[C_1\cos(5t) + C_2\sin(5t)]$

(v) $y = e^{-0.4t}[C_1\cos(2t) + C_2\sin(2t)] - 1.3\cos(7t)$

(vi) $y = C_1\cos(3t) + C_2\sin(3t) + 0.13\cos(8t) - 1.46\sin(8t)$

(vii) $y = C_1\cos(3t) + C_2\sin(3t) + 0.13t\cos(3t) - 1.46t\sin(3t)$

(viii) $y = A\sin(0.1t)\sin(6.1t)$

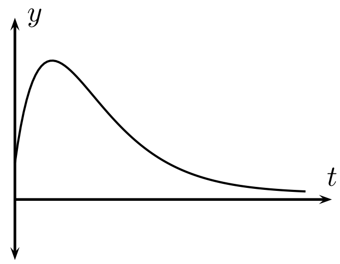
   (a) Identify the transient and steady-state parts of each solution. (Some may not have both.)

   (b) Which solutions are for differential equations of the form $ay'' + by' + cy = 0$?

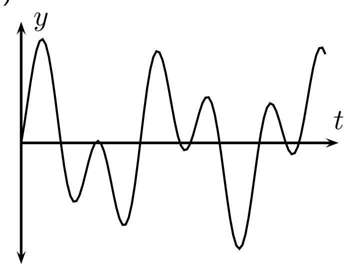
   (c) Which solutions are for undamped systems? Which are for under-damped systems? Critically damped? Over-damped?

3. Below are some graphs of solutions to ODEs of the form $ay'' + by' + cy = f(t)$, where either, or both, of $b$ or $f(t)$ may be zero.

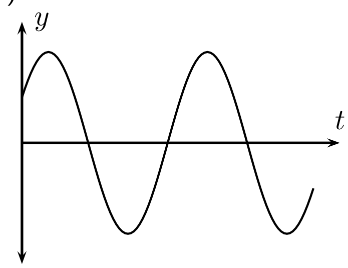
(i)



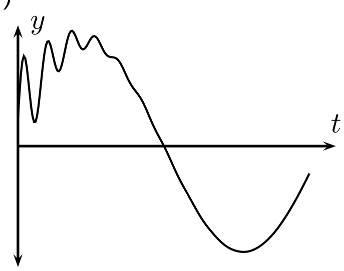

(ii)



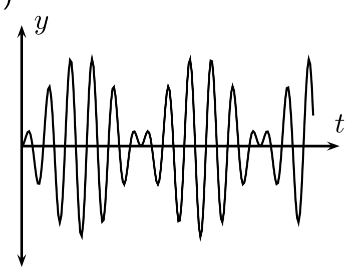

(iii)



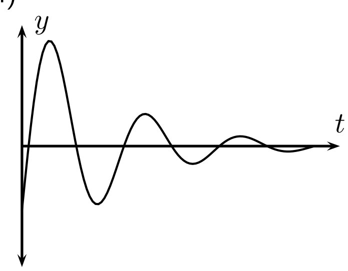

(iv)



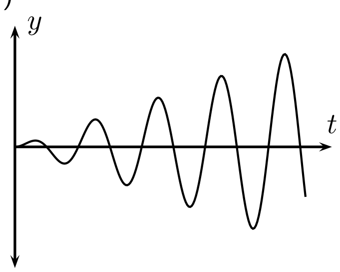

(v)




(vi)




(vii)




(a) Which graphs are for solutions to undamped systems? Under-damped systems? Critically or over-damped systems? (You should not be able to tell the graphs for critically damped or over-damped apart.)

(b) Which graphs are for ODEs of the form $ay'' + by' + cy = 0$?

4. None of the ODEs in Exercise 1 have a solution equation given in Exercise 2, or solution graph given in Exercise 3. However, we *CAN* match up the *FORMS* of the ODEs, solution equations, and graphs of solutions. For example, equation (iii) from Exercise 1 matches with solution (ii) from Exercise 2 and graph (i) from Exercise 3. Find eight other sets of three like this; one graph will have to be used more than once.

# 5 Boundary Value Problems

**Learning Outcome:**

5. Set up and solve boundary value problems.

   **Performance Criteria:**

   (a) Solve a boundary value problem for the deflection of a horizontal beam.

   (b) Give the boundary conditions for a horizontal beam.

   (c) Predict the shape of the deflection curve for a horizontal beam that is supported in a given manner.

   (d) Determine whether a function is an eigenfunction of a differential operator. If it is, give the corresponding eigenvalue.

   (e) Give eigenfunctions of the first or second derivative, for a given eigenvalue.

   (f) Solve a boundary value problem for eigenvalues and the corresponding eigenfunctions.

   (g) Give the boundary conditions for a vertical column.

   (h) Find the buckling modes (non-trivial solutions) for a vertical column.

   (i) Find the critical loads for a vertical column.

   (j) Give the pinning conditions resulting in each of the buckling modes of a vertical column.

All of the applications that we have studied so far have involved some quantity that is a function of time; that is, time has been the independent variable. Arbitrary constants have arisen in the process of solving the associated ODEs, and we have used given initial conditions to determine the values of the constants. In this chapter we look at **deflection** (bending) of horizontal beams and vertical columns. For horizontal beams the deflection is a function of the distance along the beam or column. The independent variable is then a one dimensional position variable, as discussed in Section 1.4. As seen in Section 1.7, we use boundary conditions, rather than initial conditions, to determine the values of the arbitrary constants.

We will designate the variable $x$ to denote the distance along the beam (or column) from one end or the other. Due to the weight of the beam there will be some deflection $y$ off of the horizontal line the beam would follow if it had no weight. The deflection will be different at different points along the beam, so $y = y(x)$. That is, the amount of deflection depends on where one is looking along the length of the beam. The solution function is obtained from a fourth order ODE having four boundary conditions. Solving such problems is relatively straightforward.

The situation will be significantly different for vertical columns, in a way that might be somewhat surprising. We'll see that such a column will remain straight as more and more weight is added to it until, at some weight (called the **first critical load**), it suddenly deflects ("buckles"). It will then either deform or break as a the load is increased. However, if we prevent the middle of the column from deflecting, each half will deflect at a load (called the **second critical load**) that is *four times* the first critical load.

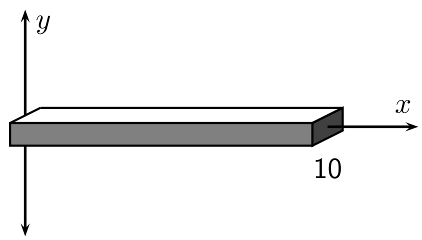
Solving the boundary value problems associated with vertical columns requires solving what we call an **eigenvalue problem**, which is more nuanced that the boundary value problems associated with horizontal beams. We will devote two sectoins of this chapter to eigenvalue problems and vertical columns. We then conclude the chapter with a look at perhaps the simplest applicatoin of partial differential equation, heat distribution in a rod. The method of solution leads us two two types of ODEs, one of which is an eigenvlalue problem.

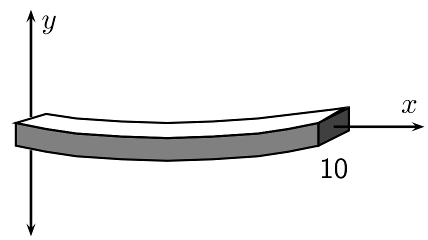
## 5.1 Deflection of Horizontal Beams

**Performance Criteria:**

5. (a) Solve a boundary value problem for the deflection of a horizontal beam.
   (b) Give the boundary conditions for a horizontal beam.
   (c) Predict the shape of the deflection curve for a horizontal beam that is supported in a given manner.

In this section we will take a look at the differential equations associated with beams that are suspended horizontally in some way. The beams themselves will not be horizontal over their entire lengths, because the force of gravity will cause some bending. The first thing to understand is the mathematical setup. Suppose that we have a beam of length 10 feet. We put the cross-sectional center of its left end at the origin of an $x$-$y$ coordinate plane, and the cross-sectional center of its right end at the point $(10, 0)$. The longitudinal axis of symmetry of the beam then runs along the $x$-axis from $x = 0$ to $x = 10$; see the figure below and to the left.




Now the beam will deflect (a fancy term for "sag") in some way, due to any weight it is supporting, *including its own weight*. The shape it takes will depend on the manner in which it is supported (we will get into that soon), but one possibility is shown in the figure above and to the right. The points along what was the original axis of symmetry of the beam now follow the graph of a function, which we will call $y(x)$. *Note that the domain of the function is just the interval* $[0, 10]$. Our goal will be to find the mathematical equation of the function.

Let us still consider a 10 foot beam, but we will represent it with just the curve described by the deflection of the longitudinal axis of symmetry. (*From now on, when we talk about the beam, we really mean the deflected original longitudinal axis of symmetry of the beam.*) Suppose also that the ends of the beam are what we call **embedded**. This means that they are not only supported at both ends, but the ends are also held horizontal by being "clamped" somehow. A good image to keep in mind is a beam that is stuck into two opposing walls of a structure. See the diagram to the right.

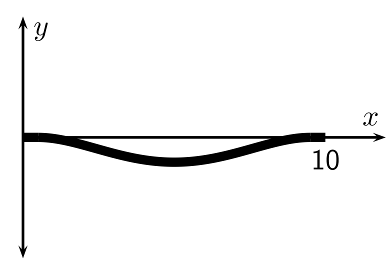



Figure 5.1(a)

The theory behind obtaining a differential equation to model a horizontal beam is beyond the scope of this class. Suffice it to say that it involves ideas from the area of statics, like the "bending moment" of the beam, and the properties of the material from which the beam is built. The differential equation itself is fourth order:

$$EI\frac{d^4y}{dx^4} = w(x) \tag{1}$$

Here $E$ is Young's modulus of elasticity for the material from which the beam is made, $I$ is the moment of inertia of a cross-section of the beam and $w(x)$ is the load per unit length of the beam. If the beam has uniform cross-section and the only weight that it is supporting is its own weight, then $w(x)$ is a constant. *We will consider only that situation.*

Let's consider the situation shown in Figure 5.1(a), where both ends are embedded. Because the ODE (1) is fourth order, we will need *four* boundary conditions to determine all of the constants that will arise in solving it. We first recognize that because the two ends are supported, there will be no deflection at either end. Therefore $y(0) = y(10) = 0$. This will be the case for any horizontal beam that is supported at both ends. Next we consider the fact that the ends of the beam are embedded horizontally into a wall. The embedding causes both ends to be horizontal right at the points where they leave the walls they are embedded in, so the slope of the beam is zero at those points. Mathematically we express this by $y'(0) = y'(10) = 0$. When we put the ODE together with these boundary conditions we get a boundary value problem. Suppose that for our ten foot beam beam $E = 10$, $I = 5$ and $w(x) = 100$. The boundary value problem that we have is then

$$50\frac{d^4y}{dx^4} = 100, \qquad y(0) = 0,\ y'(0) = 0, \qquad y(10) = 0,\ y'(10) = 0 \tag{2}$$

We solve this by simply taking a succession of antiderivatives and finding constants along the way, when we are able to.

◇ **Example 5.1(a):** Solve the boundary value problem (2) above.

**Solution:** We begin by dividing both sides by $50$ to get $\dfrac{d^4y}{dx^4} = 2$. Our task now is to keep integrating both sides until we find $y = y(x)$. Integrating once gives $\dfrac{d^3y}{dx^3} = 2x + C_1$, and integrating again gives $\dfrac{d^2y}{dx^2} = x^2 + C_1x + C_2$. Next we find that $\dfrac{dy}{dx} = \frac{1}{3}x^3 + \frac{1}{2}C_1x^2 + C_2x + C_3$, and applying the initial condition $y'(0) = 0$ gives $C_3 = 0$. Substituting this value and integrating one more time we get $y = \frac{1}{12}x^4 + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_4$, and applying the boundary condition $y(0) = 0$ results in

$$y = \tfrac{1}{12}x^4 + \tfrac{1}{6}C_1x^3 + \tfrac{1}{2}C_2x^2. \tag{3}$$

We now apply the initial condition $y(10) = 0$ to get $0 = \frac{10{,}000}{12} + \frac{1000}{6}C_1 + \frac{100}{2}C_2$, and the initial condition $y'(10) = 0$ to get $0 = \frac{1000}{3} + \frac{100}{2}C_1 + 10C_2$. To solve this system we multiply the first equation by $12$ and the second by $6$ to get the system to the left below, which can be solved in the manner shown in the other steps:

$$\begin{aligned} 2000C_1 + 600C_2 &= -10{,}000 \\ 300C_1 + 60C_2 &= -2000 \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} 20C_1 + 6C_2 &= -100 \\ -30C_1 - 6C_2 &= 200 \\ \hline -10C_1 &= 100 \\ C_1 &= -10 \end{aligned}$$

Substituting this value for $C_1$ and solving for $C_2$ gives us $C_2 = -\frac{40}{3}$. Putting these values into (3), the solution to the IVP is $y = \frac{1}{12}x^4 - \frac{5}{3}x^3 - \frac{20}{3}x^2$.

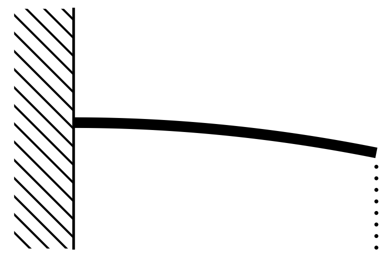
---

Use your calculator or an online grapher like *Desmos* to graph the solution from $x = 0$ to $x = 10$, using a $y$ scale that allows you to actually see the deflection of the beam. Does the result surprise you? (It should!) One annoying feature of the differential equation is that it is based on taking *down* to be the positive direction. To see what the actual shape of the beam will be, multiply your solution by $-1$, *then* graph it. Now the result should look something like Figure 5.1(a).

Suppose again that we have a 10 foot beam, but now it is supported by a fulcrum at each end, and each end is free to pivot around the fulcrum. We will call the beam **simply supported** in this case. Civil engineers might call this "pinned-pinned." See the diagram to the right. In this case, we will have $y(0) = y(10) = 0$, just like the embedded case. However, we can see that we will not have $y'(0) = 0$ or $y'(10) = 0$.


So how do we get two more boundary conditions? Note that the downward force of gravity in the interior and the upward force of the supports at the ends bend the beam into a concave up shape. (Remember concavity from differential calculus?) The upward concavity here means that it must be the case that $y''(x) > 0$ for values of $x$ between, *but not equal to* 0 and 10. However, there are no opposing forces to bend the beam *right at its ends*. Thus there is no concavity right at the ends of the beam, resulting in the conditions $y''(0) = y''(10) = 0$.

The final situation we will consider for now is a beam that is embedded at the left end and free at the right end, as shown. The left end of the beam is embedded, so we know the values $y(0) = y'(0) = 0$. We know neither the displacement nor the slope of the right end, but what we do know there is that there is no concavity there, so $y''(10) = 0$. This gives us three boundary conditions, but of course we need four. The last condition comes from some theory we won't go into here, but it is $y'''(10) = 0$.




Let us now summarize the the possible boundary conditions for a horizontal beam:

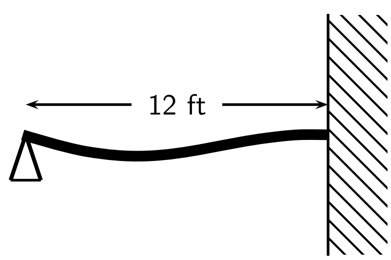
- At an embedded end both $y$ and $y'$ are zero.
- At a simply supported end $y$ and $y''$ are zero.
- At a free end $y''$ and $y'''$ are zero.

I expect you be able to give any of those conditions - you should be able to figure all of them out each time you need them, without memorization, with the possible exception of the third derivative just discussed.

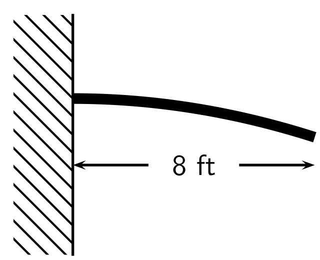
**Section 5.1 Exercises** To Solutions

1. For each beam pictured below, list the boundary conditions. *Assume that the height of the left end of each is zero.*

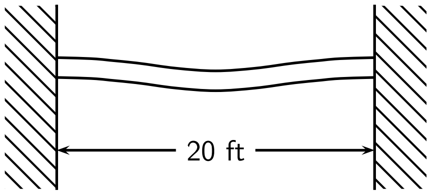
(a)



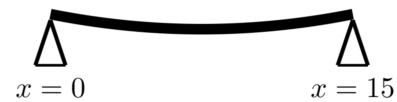

(b)



(c)




(d)




2. Suppose that you have a beam that is 8 feet long. For each of the scenarios given, determine first whether it would make any sense physically to have the beam supported in the manner given. If not, explain why. If it does make sense, give the four boundary conditions.

   (a) Left end embedded horizontally, right end simply supported.

   (b) Left end simply supported, right end free.

   (c) Both ends free.

3. Find the deflection function $y = y(x)$ for an eight foot beam that is embedded at both ends, carrying a constant load of $w(x) = 150$ pounds per foot. Suppose also that $E = 30$ and $I = 80$, in the appropriate units.

   (a) Give the appropriate boundary value problem (differential equation plus boundary conditions).

   (b) Solve the differential equation and apply the boundary conditions in order to determine the constants. What is the final solution?

   (c) Graph your solution on the appropriate $x$ interval, using a $y$ scale that allows you to actually see the deflection of the beam. Remember to multiply the right side of your solution from (b) by negative one so that it appears the same way that the beam will.

   (d) Where do you believe the maximum deflection should occur? Find the deflection there - you need not give units with your answer, since I have been somewhat vague about the units of the constants $E$ and $I$.

4. (a) Sketch a graph of the deflection of a beam that is embedded at its left end and free at its right end.

   (b) Suppose that the beam is 10 feet long, with values of $w_0$, $E$ and $I$ of 100, 10 and 5, respectively. Solve the boundary value problem.

   (c) Graph your solution and compare with your sketch in part (a). Of course they should be the same.

   (d) What is the maximum deflection of the beam, and where does it occur?

5. Repeat parts (a)-(d) of Exercise 4 for a 10 foot beam that is simply supported on both ends.

6. Repeat steps (a)-(c) of Exercise 4 for an eight foot beam, with the same parameters as in Exercise 3, that is embedded on the left end and simply supported on the right end. Then do the following:

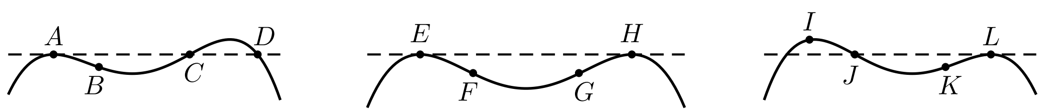
   (d) It was intuitively clear where the maximum deflection occurred for the two previous situations, but it is not so clear in this case. Take a guess as to about where you think it should occur for this case. Then use the graph on your calculator, along with the trace function, to determine where the maximum deflection occurs, and how much it is.

7. The graphs below are those of some fourth degree polynomials. The points labeled A, D, E, H, I and L are maxima for their respective functions, and the points labeled B, C, F, G, J and K are inflection points. For each of the following boundary situations, give the endpoints of a section of graph that has the shape the deflection curve would take. *Assume that both ends of the beam are supported at the same level, and that the dashed lines are horizontal.*

   (a) Simply supported at the left end, embedded at the right end.

   (b) Embedded at both ends.

   (c) Simply supported at both ends.



## 5.2 Second-Order Boundary Value Problems, Eigenfunctions and Eigenvalues

**Performance Criteria:**

5. (d) Determine whether a function is an eigenfunction of a differential operator. If it is, give the corresponding eigenvalue.

   (e) Give eigenfunctions of the first or second derivative, for a given eigenvalue.

We begin by returning to a boundary value problem that we saw in Section 1.7. It is similar to a sort of problem that comes up often in applications. The main thing that distinguishes this from an initial value problem is that the independent variable is postion, $x$, rather than time. Another difference we will usually see in boundary value problems is that we are given values of the function at two different values of the independent variable, in this case at zero and $\pi$.

◇ **Example 5.2(a):** Solve the boundary value problem

$$y'' + \tfrac{1}{4}y = 0, \qquad y(0) = 3,\ y(\pi) = -4.$$

**Solution:** The auxiliary equation for the differential equation is $r^2 + \frac{1}{4} = 0$, which has the solution $r = \pm\frac{1}{2}i$. This gives us the solution

$$y = C_1 \sin\tfrac{1}{2}x + C_2 \cos\tfrac{1}{2}x \tag{1}$$

to the differential equation. To find the values of the constants we apply the boundary conditions $y(0) = 3,\ y(\pi) = -4$. For the boundary condition $y(0) = 3$ we substitute $x = 0$ and $y = 3$ into $(3)$ to get

$$3 = C_1 \sin\tfrac{1}{2}(0) + C_2 \cos\tfrac{1}{2}(0).$$

This gives us $C_2 = 3$. Substituting $x = \pi$ and $y = -4$ into $(1)$ gives us $C_1 = -4$. Therefore the solution to the boundary value problem is $y = -4\sin\frac{1}{2}x + 3\cos\frac{1}{2}x$.

### Differential Operators, Again

Recall from Section 3.5 that a mathematical object that "works on" a function to produce another function is called an **operator**, and the derivative is probably the simplest example of an operator. Of course the second derivative is an operator as well. In that section we also showed that we can combine derivatives to get other operators.

◇ **Example 5.2(b):** The second derivative $\dfrac{d^2}{dx^2}$ is an operator. You should be quite familiar with its action:

$$\frac{d^2}{dx^2}\big(5x^3 + 7x^2 - 2x + 4\big) = 30x + 14$$

◇ **Example 5.2(c):** We can create new operators by forming something called a *linear combination* of derivatives. As an example, we can define an operator

$$D = 3\frac{d^2}{dt^2} + 5\frac{d}{dt} - 4$$

by its action on a function $y = y(t)$:

$$D(y) = 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} - 4y.$$

So, for example, if $y = e^{-2t}$,

$$D\big(e^{-2t}\big) = 3\frac{d^2}{dt^2}\big(e^{-2t}\big) + 5\frac{d}{dt}\big(e^{-2t}\big) - 4e^{-2t} = 12e^{-2t} - 10e^{-2t} - 4e^{-2t} = -2e^{-2t}$$

You saw such operators when we studied second order linear ODEs.

---

◇ **Example 5.2(d):** You may know that when we multiply the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ times the vector $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ we get

$$A\vec{\mathbf{u}} = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} (-4)(1) + (-6)(3) \\ (3)(1) + (5)(3) \end{bmatrix} = \begin{bmatrix} -22 \\ 18 \end{bmatrix}$$

Similarly, for the vector $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$,

$$A\vec{\mathbf{v}} = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

We can think of $A$ as an operator that acts on vectors with two components to create other vectors with two components.

---

Recall that the derivative operator is what we call a **linear operator**. What this means is that if $f$ and $g$ are functions, and $c$ is a constant, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx}(x) + \frac{dg}{dx}(x) \qquad \text{and} \qquad \frac{d}{dx}[cf(x)] = c\frac{df}{dx}(x)$$

*This behavior is not unique.* If $A$ is a matrix, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are vectors, and $c$ is a scalar (constant),

$$A(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = A\vec{\mathbf{u}} + A\vec{\mathbf{v}} \qquad \text{and} \qquad A(c\vec{\mathbf{u}}) = cA\vec{\mathbf{u}}$$

Linear operators have these two properties, of "distributing over addition" and "passing through constants." (This is where the the language "linear" in *linear algebra* comes from.) Many operators used in applications are linear operators.

**Eigenfunctions and Eigenvalues**

Let's go back to the differential equation $y'' + \frac{1}{4}y = 0$ from Example 5.2(a). Note that we can arrange the differential equation as

$$\frac{d^2y}{dx^2} = -\tfrac{1}{4}y. \tag{2}$$

When we seek a solution to this differential equation, the equation tells us that we are looking for a function $y = y(x)$ whose second derivative is one-fourth the function itself. We looked at such equations in Section 1.2, and established by guessing and checking that a function of the form

$$y = C_1 \sin \tfrac{1}{2}x + C_2 \cos \tfrac{1}{2}x \tag{3}$$

is a solution for *any* values of $C_1$ and $C_2$. Of course we now know how to solve (2) using its auxiliary equation, and we know also that *every* solution to (2) must have the form (3). The fact that the action of the second derivative operator the function (3) is to simply multiply the the function by $-\frac{1}{4}$ is something fairly special. That's not the case for most other functions when the second derivative operator "works on" them. Here's an example of a more complicated operator and two functions, one of which has this property that the operator acting on it is the same as multiplying by a number, and the another function for which this is not the case.

◇ **Example 5.2(e):** Let $L$ be the differential operator defined on a function $y = y(x)$ by

$$L(y) = (x^2 - 1)\,\frac{d^2y}{dx^2} + 2x\,\frac{dy}{dx}.$$

Apply this operator to the functions $p(x) = x^2 - 5x + 2$ and $q(x) = 5x^3 - 3x$.

**Solution:** We see that

$$\begin{aligned} L\big(x^2 - 5x + 2\big) &= (x^2-1)\frac{d^2}{dx^2}\big(x^2 - 5x + 2\big) + 2x\frac{d}{dx}\big(x^2 - 5x + 2\big) \\ &= (x^2-1)(2) + 2x(2x-5) \\ &= 6x^2 - 10x - 2 \end{aligned}$$

and

$$\begin{aligned} L\big(5x^3 - 3x\big) &= (x^2-1)\frac{d^2}{dx^2}\big(5x^3 - 3x\big) + 2x\frac{d}{dx}\big(5x^3 - 3x\big) \\ &= (x^2-1)(30x) + 2x(15x^2 - 3) \\ &= 60x^3 - 36x \end{aligned}$$

There is nothing special about the result when the operator $L$ of the previous example is applied to $p(x) = x^2 - 5x + 2$, but we see that

$$L\big[q(x)\big] = L\big(5x^3 - 3x\big) = 60x^3 - 36x = 12(5x^3 - 3x) = 12q(x).$$

Note that the ultimate effect of $L$ on $q$ is to multiply it by twelve. When an operator operates on a function and the result is to simply multiply the function by a constant, we call the function an **eigenfunction** of the operator:

**Eigenfunctions and Eigenvalues**

Let $A$ be an operator that operates on functions and let $y$ be a *nonzero* function for which there is a constant $\lambda$ such that

$$Ay = \lambda y.$$

Then $y$ is an **eigenfunction** of the operator $A$, with corresponding **eigenvalue** $\lambda$. Note that $\lambda = 0$ is allowable, but $y = 0$ is not.

◇ **Example 5.2(f):** For the operator $L$ of Example 5.2(e), give an eigenfunction and the corresponding eigenvalue.

**Solution:** Because $L\big[q(x)\big] = 12q(x)$, $q(x) = 5x^3 - 3x$ is an eigenfunction of $L$ with eigenvalue 12.

---

◇ **Example 5.2(g):** Consider again the second derivative $\dfrac{d^2}{dx^2}$, and note that

$$\frac{d^2}{dx^2}(\sin \tfrac{1}{2}x) = -\tfrac{1}{4}\sin \tfrac{1}{2}x.$$

The effect of the derivative on $\sin \frac{1}{2}x$ is to simply multiply the function by $-\frac{1}{4}$, so $\sin \frac{1}{2}x$ is an eigenfunction for the operator $\dfrac{d^2}{dx^2}$, with corresponding eigenvalue $-\frac{1}{4}$.

---

◇ **Example 5.2(h):** Note that in Example 5.2(d), the result of $A$ times $\vec{\mathbf{v}}$ was simply $-1$ times $\vec{\mathbf{v}}$. We say that $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is an **eigenvector** (instead of eigenfunction) for the matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$, with eigenvalue $-1$.

---

◇ **Example 5.2(i):** Because the derivative of a constant is zero, which is also zero times the function, every nonzero constant function is an eigenfunction of the first derivative operator $\dfrac{d}{dx}$, with corresponding eigenvalue zero. This emphasizes that even though the zero function isn't allowed as an eigenfunction, eigenfunctions *are* allowed to have eigenvalues of zero.

---

### Eigenfunction Problems

Here we see how eigenfunctions are important to us in our study of differential equations. The differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0 \tag{4}$$

is called **Legendre's Differential Equation** and arises when modeling steady-state heat distribution in a solid medium using polar coordinates. If we let $n(n+1)=\lambda$, move the term $n(n+1)y=\lambda y$ to the right side, apply the derivative outside the brackets on the left (product rule!) and negate both sides, $(4)$ becomes

$$(x^2-1)\,\frac{d^2y}{dx^2}+2x\,\frac{dy}{dx}=\lambda y. \tag{5}$$

If we then let $L$ be the operator of Example 5.2(e) defined by

$$L(y)=(x^2-1)\,\frac{d^2y}{dx^2}+2x\,\frac{dy}{dx}.$$

then $(5)$ becomes the eigenfunction/eigenvalue equation

$$L(y)=\lambda y. \tag{6}$$

Solving $(4)$ then is equivalent to finding eigenfunctions and eigenvalues of the operator $L$. This is what we mean by solving the eigenvalue problem $(5)$.

We now make some observations related to Example 5.2(g), where we saw that $y=\sin\frac{1}{2}x$ is an eigenfunction for the second derivative with eigenvalue $-\frac{1}{4}$, and Example 5.2(a). First, for any constant $C$,

$$\frac{d^2}{dx^2}(C\sin\tfrac{1}{2}x)=-\tfrac{1}{4}C\sin\tfrac{1}{2}x=-\tfrac{1}{4}(C\sin\tfrac{1}{2}x). \tag{7}$$

This indicates that *any constant multiple of an eigenfunction is also an eigenfunction, with the same eigenvalue*. This holds for eigenvectors as well; we don't really think of such multiples as new eigenfunctions or eigenvectors. We also see that

$$\frac{d^2}{dx^2}(\cos\tfrac{1}{2}x)=-\tfrac{1}{4}\cos\tfrac{1}{2}x, \tag{8}$$

showing that an operator can have more than one eigenfunction (beyond just constant multiples) with the same eigenvalue. This also holds for matrices and eigenvectors.

Finally, combining $(7)$ and $(8)$ gives us that any function of the form

$$y=C_1\sin\tfrac{1}{2}x+C_2\cos\tfrac{1}{2}x$$

is an eigenfunction of the second derivative with eigenvalue $-\frac{1}{4}$. This indicates that solving the differential equation $y''+\frac{1}{4}y=0$ amounts to finding eigenfunctions of the second derivative with eigenvalue $-\frac{1}{4}$.

**Section 5.2 Exercises** To Solutions

1. In this exercise you will be considering the first derivative operator $\dfrac{d}{dx}$.

   (a) The function $y=e^{3x}$ is an eigenfunction for the operator. What is the corresponding eigenvalue?

   (b) Give the eigenfunction of the operator that has eigenvalue $-5$.

   (c) Based on your answers to parts (a) and (b), what is the general form of any eigenfunction of the first derivative operator and what is the corresponding general eigenvalue?

2. Now consider the second derivative operator $\dfrac{d^2}{dx^2}$. There are three general forms of the eigenfunctions for this operator, depending on whether the eigenvalues are positive, negative or zero.

   (a) Give a *specific* function that has eigenvalue zero; that is, the second derivative of the function is zero.

   (b) Give the most general form of function whose eigenvalue is zero.

   (c) Give two different functions (neither of them being a multiple of the other) that are eigenfunctions of the second derivative with eigenvalue $-4$.

   (d) Give two different functions, neither of them being a multiple of the other, that are eigenfunctions of the second derivative with eigenvalue $-3$.

   (e) Give two different functions, neither of them being a multiple of the other, that are eigenfunctions of the second derivative with eigenvalue $-\lambda^2$, where $\lambda$ is a positive real number.

   (f) Give the eigenfunctions that will have eigenvalue nine; there are two of them!

   (g) Give the general form of the eigenfunctions of the operator that have positive eigenvalues, and give the general eigenvalue.

3. Let $D$ be the operator $D = \dfrac{d^2}{dt^2} + 2\dfrac{d}{dt} - 3$, whose action on a function $y = y(t)$ is defined by $Dy = \dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 3y$.

   (a) Show that $y = e^{-2t}$ is an eigenfunction for this operator, and determine the corresponding eigenvalue.

   (b) In general, any function of the form $e^{kt}$ is an eigenfunction for $D$. Determine the general eigenvalue.

   (c) Give two values of $k$ for which $e^{kt}$ is an eigenfunction of $D$ with eigenvalue zero.

   (d) Give two values of $k$ for which $e^{kt}$ is an eigenfunction of $D$ with eigenvalue five.

4. A very important ODE in many applications is $y'' + \lambda^2 y = 0$. Note that this can be rearranged to get $y'' = -\lambda^2 y$, which says that any $y$ that is a solution to the differential equation is an eigenfunction with eigenvalue $-\lambda^2$.

   (a) Give the eigenfunctions of the second derivative with eigenvalue $-\lambda^2$.

   (b) (Challenge) Let $\mathcal{S}$ be the set of functions $y = f(x)$ that have continuous second derivatives on the interval $[0, 2\pi]$ and for which $f(0) = f(2\pi) = 0$. Determine *ALL* eigenfunctions of the second derivative with eigenvalue $-\lambda^2$ that are in $\mathcal{S}$.

5. Let $L$ be the differential operator defined on a function $y = y(x)$ by

$$L(y) = x\,\frac{d^2y}{dx^2} + (1-x)\,\frac{dy}{dx}.$$

Determine which of the following are eigenfunctions of $L$. For those that are, give the corresponding eigenvalue.

(a) $y = x^2 + 3x$

(b) $y = 1 - x$

(c) $y = x^3 - 9x^2 + 18x - 6$

(d) $y = 2x + 3$

(e) $y = x^2 - 4x + 2$

6. Example 5.2(d) showed that the function $P_3(x) = 5x^3 - 3x$ is an eigenfunction of the operator $L$ defined by

$$L(y) = (x^2-1)\,\frac{d^2y}{dx^2} + 2x\,\frac{dy}{dx},$$

with eigenvalue 12. The function $P_3(x)$ is called a Legendre polynomial. There are more Legendre polynomials, each of which is an eigenfunciton for $L$ - here are a few of them:

$$P_1(x) = x, \qquad P_2(x) = 3x^2 - 1, \qquad P_4(x) = 35x^4 - 30x^2 + 3, \qquad P_5(x) = 63x^5 - 70x^3 + 15x.$$

(a) Determine the corresponding eigenvalue for each of the eigenfunctions given above.

(b) Make a table of $n$ values for $n = 1, 2, 3, 4, 5$ and the corresponding eigenvalues.

(c) There should be a pattern to the eigenvalues, but you may find it difficult to figure out. Give it a try, and take a guess as to what the eigenvalue is for $n = 7$. The eigenfunction is $P_7 = 429x^7 - 693x^5 + 315x^3 - 35x$ - apply $L$ to it to check your guess for the eigenvalue.

(d) What is the eigenvalue for the $n$th Legendre polynomial $P_n(x)$?

## 5.3 Eigenvalue Problems, Deflection of Vertical Columns

**Performance Criteria:**

5. (f) Solve a boundary value problem for eigenvalues and the corresponding eigenfunctions.
   (g) Give the boundary conditions for a vertical column.
   (h) Find the buckling modes (non-trivial solutions) for a vertical column.
   (i) Find the critical loads for a vertical column.
   (j) Give the pinning conditions resulting in each of the buckling modes of a vertical column.

When solving the equation

$$\frac{d^2y}{dx^2} = -\tfrac{1}{4}y$$

we see that we are looking for eigenfunctions of the second derivative with eigenvalue $-\frac{1}{4}$. In many applications we are looking for eigenfunctions without knowing what the eigenvalue is. This seems like an impossible problem to solve, but it isn't, as we see in the following example.

◇ **Example 5.3(a):** Solve the boundary value problem

$$y'' + \lambda^2 y = 0, \qquad y(0) = 0, \; y'(2\pi) = 0, \tag{1}$$

where $\lambda$ is a positive value to be determined.

**Solution:** The auxiliary equation for the differential equation is $r^2 + \lambda^2 = 0$, which leads to $r^2 = -\lambda^2$, so $r = \pm\lambda i$. The solution to the ODE is then

$$y = C_1 \sin\lambda x + C_2 \cos\lambda x.$$

Applying the first boundary condition $y(0) = 0$ gives us $C_2 = 0$, so the solution is

$$y = C_1 \sin\lambda x.$$

From this we can compute $y' = C_1\lambda\cos\lambda x$, and applying the second boundary condition gives us

$$C_1\lambda\cos 2\pi\lambda = 0.$$

There are three possibilities here: $C_1 = 0$, $\lambda = 0$, or $\cos 2\pi\lambda = 0$. The first two result in a solution of $y = 0$ for the boundary value problem. This is a valid solution, but quite uninteresting! For this reason we will refer to it as the *trivial solution*. To get a non-trivial solution it must be the case that $\cos 2\pi\lambda = 0$. Now $\cos\theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$. Thus we have

$$\begin{aligned} 2\pi\lambda &= \tfrac{\pi}{2},\; \tfrac{3\pi}{2},\; \tfrac{5\pi}{2}, \ldots \\ \lambda &= \tfrac{1}{4},\; \tfrac{3}{4},\; \tfrac{5}{4}, \ldots \end{aligned} \tag{2}$$

Changing $C_1$ to just $C$, the non-trivial solutions to the boundary value problem $(1)$ are then

$$y = C\sin\tfrac{1}{4}x,\; y = C\sin\tfrac{3}{4}x,\; y = C\sin\tfrac{5}{4}x, \ldots, \tag{3}$$

each corresponding to one of the values of $\lambda$ determined in $(2)$.

As stated, each of the solutions (3) to the BVP (1) corresponds to a particular value of $\lambda$. Let's verify one of those solutions.

◇ **Example 5.3(b):** Verify that $y = C\sin\frac{5}{4}x$ is a solution to the boundary value problem

$$y'' + \lambda^2 y = 0, \qquad y(0) = 0,\ y'(2\pi) = 0 \tag{1}$$

when $\lambda = \frac{5}{4}$.

**Solution:** The differential equation in this case is $y'' + \frac{25}{16}y = 0$. We see that

$$y = C\sin\tfrac{5}{4}x \implies y' = \tfrac{5}{4}C\cos\tfrac{5}{4}x \implies y'' = -\tfrac{25}{16}C\sin\tfrac{5}{4}x, \tag{4}$$

so

$$y'' + \tfrac{25}{16}y = -\tfrac{25}{16}C\sin\tfrac{5}{4}x + \tfrac{25}{16}(C\sin\tfrac{5}{4}x) = 0,$$

showing that $y = C\sin\frac{5}{4}x$ is a solution to the differential equation.

We must now show that $y = C\sin\frac{5}{4}x$ satisfies the boundary conditions. Note that we found $y'$ in (4).

$$y(0) = C\sin\tfrac{5}{4}(0) = 0 \quad \text{and} \quad y'(2\pi) = \tfrac{5}{4}C\cos\tfrac{5}{4}(2\pi) = \tfrac{5}{4}C\cos\tfrac{5\pi}{2} = 0.$$

Because $y = C\sin\frac{5}{4}x$ satisifies both the differential equation and boundary conditions, it is a solution to the BVP (1) when $\lambda = \frac{5}{4}$.

---

We note two ways our solution to the boundary value problem from Example 5.3(a) differs from the initial value problems we've solved, and the particular boundary value problems that we have solved up to now:

- There are infinitely many solutions to this boundary value problem, each corresponding to a specific choice of $\lambda$.
- There is an arbitray constant whose value we cannot determine from the information given.

For the particular application that we will look at in this section, we consider only one of the solutions

$$y = C\sin\tfrac{1}{4}x,\ y = C\sin\tfrac{3}{4}x,\ y = C\sin\tfrac{5}{4}x, ..., \tag{3}$$

at a time. For other applications we need at some point to consider instead the arbitray **linear combination** of solutions

$$y = C_1\sin\tfrac{1}{4}x + C_2\sin\tfrac{3}{4}x + C_3\sin\tfrac{5}{4}x + \cdots \tag{5}$$

When these applications occur (in the solving of partial differential equations), there is additional information that can be used to determine the values of all these constants. Some of you may recognize (5) as a **Fourier series**.

We now examine Example 5.3(a) in the context of eigenvalues and eigenfunctions. Note that the differential equation from the BVP can be written

$$\frac{d^2}{dx^2}(y) = -\lambda^2 x,$$

which is saying we are looking for functions that are eigenfunctions of the second derivative with eigenvalues $-\lambda^2$. This is called an **eigenvalue problem**, and it is typical that we need to find both the eigenvalues and eigenfuctions. (Any of you who have had linear algebra may recall that in that course you had to find both eigenvalues and eigenvectors.) The result of Example 5.3(a) gives us the eigenvalues (using the results of (2))

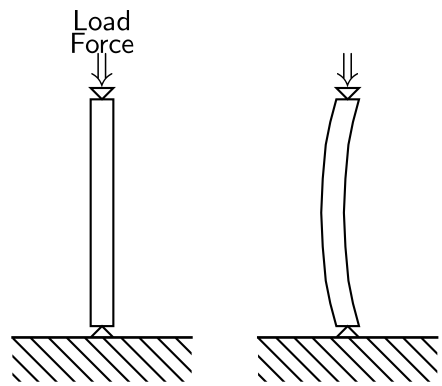
$$-\lambda^2 = -\tfrac{1}{16}, \quad -\tfrac{9}{16}, \quad -\tfrac{25}{16}, ...,$$

with corresponding eigenfunctions

$$y = C\sin\tfrac{1}{4}x, \;\; y = C\sin\tfrac{3}{4}x, \;\; y = C\sin\tfrac{5}{4}x, ...$$

We now look at one application of these ideas that some of you may have encountered in a strengths of materials course.

### Deflection of Vertical Columns

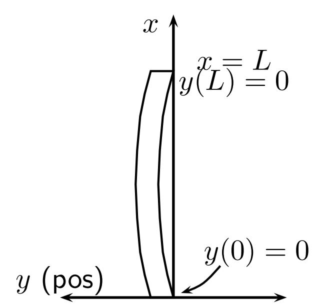



Now we will examine the behavior of a vertical column when a load is applied directly downward on its top, as shown in the first picture to the right. We will begin by considering columns that are **pinned**; this means we will allow the top and bottom of the column to be at angles other than vertical. *We will require the top and bottom of the column to be vertically aligned with each other - later we will consider a situation where we will relax this condition*. So, for example, when enough force is applied downward the column will deflect horizontally as shown in the second picture to the right.

We will set up a coordinate system as shown below and to the right, with $x$ indicating the distance upward from the bottom of the column and $y = y(x)$ representing the *horizontal* deflection of the column at any point $x$. The differential equation governing the deflection of the column is

$$EI\frac{d^2y}{dx^2} = -Py, \tag{1}$$

where the parameters $E$ and $I$ are again the modulus of elasticity and cross-sectional moment of inertia of the column. They are properties that *could* vary along the column (with the variable $x$), but this would be unusual. We'll only consider columns where they do not change. $P$ is the (positive) force exerted downward on the top of the column, and we will look at the effects of different values of $P$, but for purposes of solving the ODE it is a constant. (It is a parameter rather than a variable, but its value is to be determined when solving the ODE.) Because the ODE is second order we will need two conditions to determine the solution. If the length of the column is denoted by $L$, we have the boundary conditions $y(0) = 0$ and $y(L) = 0$. If we rearrange the equation and combine it with the boundary conditions we get the boundary value problem (BVP)




$$\frac{d^2y}{dx^2} + \frac{P}{EI}\,y = 0, \qquad y(0) = 0, \;\; y(L) = 0. \tag{2}$$

We will see that solving this boundary value problem is somewhat different than solving the kind of BVPs we saw for horizontal beams, in the previous section. This is because we can rearrange the ODE (1) to get the eigenvalue problem

$$\frac{d^2y}{dx^2} = -\frac{P}{EI}y. \tag{3}$$

Note that $y$ is an eigenfunction of the second derivative, with eigenvalue $-\frac{P}{EI}$, where $P$, $E$ and $I$ are all positive. As we now know very well, $y$ must be of the form

$$y = C_1 \sin ax + C_2 \cos ax$$

for some constant $a$ yet to be determined. Let's now go through the details for a specific case:

◇ **Example 5.3(c):** Solve the boundary value problem

$$\frac{d^2y}{dx^2} + \frac{P}{EI}\, y = 0, \qquad y(0) = 0, \quad y(20) = 0, \tag{4}$$

with $E = 800$, $I = 150$ and $P > 0$.

**Solution:** Substituting the values for $E$ and $I$ into the ODE gives us

$$\frac{d^2y}{dx^2} + \frac{P}{120,000}\, y = 0 \tag{5}$$

Using our methods from Chapter 3, this equation has auxiliary equation $r^2 + \frac{P}{120{,}000} = 0$ and, because $P > 0$, its roots are $r = \pm i\sqrt{\frac{P}{120{,}000}}$. The solution to (4) is then

$$y = C_1 \sin \sqrt{\tfrac{P}{120{,}000}}\; x + C_2 \cos \sqrt{\tfrac{P}{120{,}000}}\; x$$

Applying the boundary condition $y(0) = 0$ gives us $C_2 = 0$, so the solution is then

$$y = C \sin \sqrt{\tfrac{P}{120{,}000}}\; x.$$

(I've omitted the subscript for simplicity.) Now here is where things start to get interesting! The other boundary condition tells us that

$$0 = C \sin \sqrt{\tfrac{P}{120{,}000}}\,(20) \tag{6}$$

which, in turn, tells us that either $C = 0$ or $\sin \sqrt{\frac{P}{120{,}000}}\,(20) = 0$. The first possibility gives us the "trivial" solution $y = 0$ - it satisfies the differential equation and boundary conditions, but it isn't particularly interesting! Considering the second possibility, $\sin\theta = 0$ for $\theta = 0, \pi, 2\pi, 3\pi, ..., n\pi, ...$ so, for $C \neq 0$, $(6)$ will be true for

$$\sqrt{\tfrac{P}{120{,}000}}\,(20) = 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots,\ n\pi,\ \ldots, \tag{7}$$

the first of which also gives us the trivial solution $y = 0$. Therefore, in theory at least (we'll talk later about what all this means from a practical point of view), we can only have nonzero solutions to the BVP if

$$\sqrt{\tfrac{P}{120{,}000}} = \tfrac{\pi}{20},\ \tfrac{2\pi}{20},\ \tfrac{3\pi}{20},\ \ldots,\ \tfrac{n\pi}{20},\ \ldots$$

This gives us the nonzero solutions

$$y = C \sin \tfrac{\pi}{20}\, x,\ y = C \sin \tfrac{2\pi}{20}\, x,\ y = C \sin \tfrac{3\pi}{20}\, x,\ \ldots,\ y = C \sin \tfrac{n\pi}{20}\, x,\ \ldots$$

to the boundary value problem.

This isn't really the end of the story, but we need to pause to catch our breath and develop some terminology before resuming. At this point what we know about the boundary value problem

$$\frac{d^2y}{dx^2} + \frac{P}{EI}\,y = 0, \qquad y(0) = 0, \quad y(20) = 0, \tag{4}$$

is that $y = 0$ is a solution, called the *trivial solution* (because it is mathematically uninteresting), and we only have nontrivial solutions for discrete values of $\sqrt{\frac{P}{120{,}000}}$; those solutions are the ones the example concluded with. The non-trivial solutions are called **buckling modes** (for reasons you'll soon see). The first non-trivial solution is called the **first buckling mode**, the second is the **second buckling mode**, and so on. The only values of $P$ for which we can have nontrivial solutions are those that satisfy

$$\sqrt{\tfrac{P}{120{,}000}}\,(20) = \pi,\ 2\pi,\ 3\pi,\ \ldots,\ n\pi,\ \ldots. \tag{7}$$

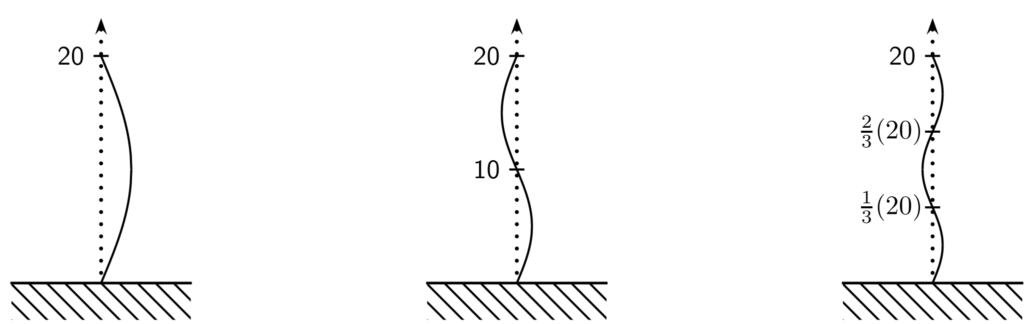
Solving for $P$ gives us

$$\begin{aligned} P &= 120000\left(\frac{\pi}{20}\right)^2,\ 120000\left(\frac{2\pi}{20}\right)^2,\ 120000\left(\frac{3\pi}{20}\right)^2,\ \ldots, 120000\left(\frac{n\pi}{20}\right)^2,\ \ldots \\ &= 300\pi^2,\ 300(2\pi)^2,\ 300(3\pi)^2,\ \ldots, 300(n\pi)^2,\ \ldots \\ &= 300\pi^2,\ 4(300\pi^2),\ 9(300\pi^2),\ \ldots, n^2(300\pi^2),\ \ldots \end{aligned}$$

These values of $P$ are called **critical loads**. Like the buckling modes, they are numbered, so $300\pi^2$ is the **first critical load**, $4(300\pi^2)$ is the **second critical load**, etc. The word "load" refers to the load held up by the column. Note that the second critical load is four (two squared) times the first critical load, the third critical load is nine (three squared) times the first critical load, and so on.

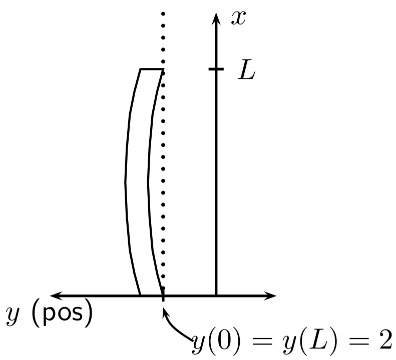
What happens physically is this: When there is no load on the column it is perfectly straight (the solution $y = 0$), and it remains that way as we increase the load, until the first critical load is reached. At that point the column will deflect sideways, taking the shape of the curve $y = C\sin\frac{\pi}{20}x$, shown at the left below. In reality, as the load increases beyond the first critical load, the deflection will remain the same shape, but with increasing amplitude, until the column fails.

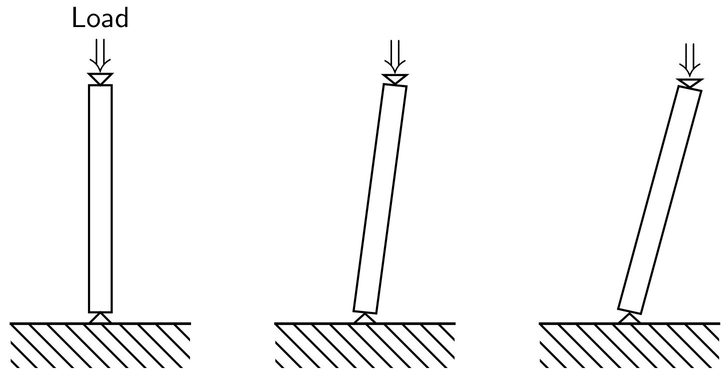
If we were able to prevent the middle point of the column, at $x = 10$, from deflecting, the column would be able to support the second critical load. Because the column is held with $y(10) = 0$, the deflection of the column will take the shape of a full period of the sine function, as shown in the middle picture below - this is the second buckling mode. The act of preventing deflection is sometimes called "pinning." If we pin the column at points one-third and two-thirds of the way along its length, the column would be able to support the third critical load, and the shape of the deflection would be given by the third buckling mode, shown to the right below.



1st buckling mode　　2nd buckling mode　　3rd buckling mode

Let's revisit the boundary conditions $y(0) = y(L) = 0$, where $L$ is the length of the column. We should first note that the only requirement we really had was that the top and bottom of the column were vertically aligned. We could just as well have put $y(0) = y(L) = 2$, as shown in the diagram to the left below; however, the mathematics involved are a bit simpler if we instead use $y(0) = y(L) = 0$, as we did. Physically *we must still insist that the top and bottom be aligned*. Without this restriction, any horizontal shifting of the "ceiling" would result in hinging and a collapse. The beginning of such a collapse is indicated by the three diagrams to the right below.



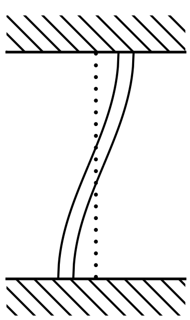

Now suppose the top and bottom were embedded, rather than being pinned. If the column still has length $L$, we then have the boundary conditions

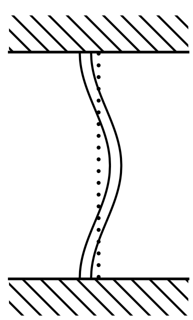
$$y'(0) = y'(L) = 0. \tag{5}$$

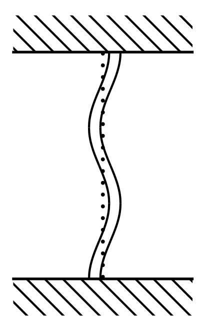
This gives us two boundary conditions, which is what we should need in order to solve the second order ODE

$$\frac{d^2y}{dx^2} + \frac{P}{EI}\,y = 0,$$

and leaves us without any conditions on $y(0)$ and $y(L)$. In this situation, we could conceivably allow the "ceiling" to "drift" laterally without collapse, because the column being held in a vertical alignment at the bottom and top would provide enough rigidity to prevent collapse. The first three buckling modes for this situation are shown below; if the ceiling were prevented from drifting, the second mode would then become the first, the fourth would become the second, and so on. You will investigate this situation, along with the corresponding critical loads, in the exercises.


1st buckling mode


2nd buckling mode


3rd buckling mode

**Section 5.3 Exercises** To Solutions

**Give all answers in exact form. When asked for the first ... buckling modes or critical loads, give only nonzero modes or values.**

1. For each of the following, boundary values are given to go with the ODE $y'' + \lambda^2 y = 0$, which has the solution
$$y = C_1 \sin \lambda x + C_2 \cos \lambda x.$$
Use the method of Example 5.3(a) to determine the first four nonzero values of $\lambda$ for which the boundary value problem has a solution, and give the corresponding four solutions.

   (a) $y(0) = 0,\ y(5) = 0$

   (b) $y'(0) = 0,\ y'(3\pi) = 0$

   (c) $y(0) = 0,\ y'(\pi) = 0$

   (d) $y'(0) = 0,\ y(7) = 0$

   (e) $y'(0) = 0,\ y'(10) = 0$

   (f) $y(0) = 0,\ y(5\pi) = 0$

2. A $12$ foot vertical column is pinned at both ends. For the material it is made of we have $E = 500$ and, from its design, we have $I = 200$, both in the appropriate units.

   (a) Give the first four nonzero buckling modes, showing all steps of solving the IVP to get them.

   (b) Give the first four nonzero critical loads.

   (c) How many times larger is the third nonzero critical load than the first nonzero critical load?

3. Now suppose we have a $6$ foot vertical column with $E = 500$, $I = 200$, and both ends pinned.

   (a) Give the first four nonzero buckling modes, and find the corresponding critical loads.

   (b) Compare your critical loads with those from the $12$ foot length (Exercise 1). How do the corresponding critical loads for the six foot column compare with those for the $12$ foot column? Does that make sense intuitively?

   (c) The first nonzero buckling mode for the six foot column is the same as a six foot section of which buckling mode for the $12$ foot column? Draw a picture showing what is going on here.

4. Consider now a $12$ foot vertical column with $E$ and $I$ values of $500$ and $200$ again, but this time with both ends embedded. Suppose also that the "ceiling" is not allowed to drift, so the top and bottom of the column are vertically aligned.

   (a) Give the first three nonzero buckling modes.

   (b) Give the first three nonzero critical loads.

   (c) How does the third nonzero critical load compare to the first? (That is, how many times larger is it?)

   (d) How does the first nonzero critical load compare with the first nonzero critical load for the $12$ foot column with pinned ends? (See Exercise 1(a).) How about the other critical loads?

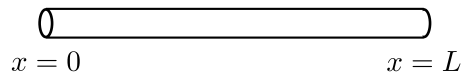
5. Repeat the previous exercise, but with the assumption that the ceiling is allowed to drift. (**Hint:** You should be able to use your computations from Exercise 4, rather than re-doing all of them.)

6. Repeat parts (a) and (b) of Exercise 2 for a vertical column of length $L$ that is pinned at both ends, with modulus of elasticity $E$ and moment of inertia $I$. This will give a general form of the buckling modes and critical loads for pinned ends.

7. Repeat parts (a) and (b) of Exercise 5 for a vertical column of length $L$ that is embedded at both ends, with the ceiling allowed to drift. Again use a modulus of elasticity $E$ and moment of inertia $I$. This will give a general form of the buckling modes and critical loads for embedded ends.

## 5.4 The Heat Equation in One Dimension

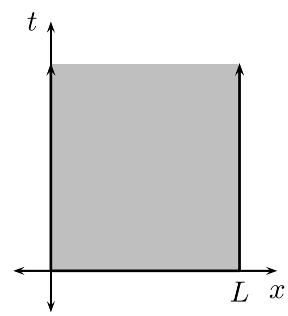
In this section we will consider the physically impossible but mathematically convenient situation: We have a metal rod of length $L$ (see picture below) that is perfectly insulated along its length, so that no heat can enter or escape along its length, but for which heat can enter or leave the ends. At any time $t$ greater than zero and any position $x$ along the rod, the function $u(x,t)$ gives the temperature at that point $x$ and time $t$. (We will think of the rod as being "infinitely thin," so that the rod has only one point at each $x$ position. If you are not happy with this, an alternative is to think that if the rod had some thickness the temperature at every point in a cross-sectional slice at some $x$ is the same, so we need not consider the other two space dimensions.) Suppose that at time zero there is some distribution of temperatures along the rod, given by a the function $f(x)$ for $0 \le x \le L$, and suppose also that the ends of the rod are held at temperature zero at all times $t \ge 0$. The function $f$ gives an initial condition for each point $x$ along the length of the rod, and the conditions that the ends are held at temperature zero are boundary conditions.




What we would like to know is whether, and how, we can determine the temperature at any point $x$ with $0 < x < L$ (we know the temperatures at $x = 0$ and $x = L$ are always zero), at any time $t > 0$. Here the dependent variable $u$ depends on the *two* independent variables $x$ and $t$. Some physical principles concerning heat give us a differential equation for this situation and, due to there being two independent variables, it is a *partial differential equation*. The equation (called the **heat equation**), and the conditions given above can all be stated as

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(L,t) = 0, \qquad u(x,0) = f(x) \tag{1}$$

Here the conditions $u(0,t) = u(L,t) = 0$ are boundary conditions and $u(x,0) = f(x)$ is essentially an initial condition (for every point along the rod). Thus we have a problem that is a sort of combination initial value/boundary value problem. But we can really think of it as a boundary value problem for this reason: If we were to think of the Cartesian plane as representing position $x$ along the horizontal axis and time $t$ along the vertical axis we get a picture like the one to the right, where each point in the shaded region represents a point $x$ in the rod and some time $t$. Our goal is then to find the temperature at each of those points; in this way we can think of trying to find function values in a region that is bounded by the line from zero to $L$ on the $x$-axis and the two "half-infinite" lines from zero to infinity in the $t$ direction at $x = 0$ and $x = L$. The condition $u(x,0) = f(x)$ can be thought of as a boundary condition along the bottom, and the conditions $u(0,t) = u(L,t) = 0$ are boundary conditions along the two sides.




Recall that if we have the function $f(x,y) = x^2y^3$, to get the partial derivative $\dfrac{\partial f}{\partial x}$ we simply take the derivative of $x^2y^3$, *treating $y$ (and therefore $y^3$) as a constant*. Similarly, we get $\dfrac{\partial f}{\partial y}$ by treating $x$ as a constant, so we have

$$\frac{\partial f}{\partial x} = 2xy^3 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 3x^2y^2$$

Let's try another.

◇ **Example 5.4(a):** Find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial t}$ for $u(x,t) = e^{-2t}\sin 3x$.

**Solution:** When finding $\frac{\partial u}{\partial x}$ we consider $t$ to be a constant, so $e^{-2t}$ is as well. The derivative is then

$$\frac{\partial u}{\partial x} = 3e^{-2t}\cos 3x.$$

When finding $\frac{\partial u}{\partial t}$, $\sin 3t$ is essentially a constant, so

$$\frac{\partial u}{\partial t} = -2e^{-2t}\sin 3x.$$

---

Now we'll see that we have a solution to the heat equation!

◇ **Example 5.4(b):** Show that $u(x,t) = e^{-2t}\sin 3x$ satisfies the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$.

**Solution:** We already have

$$\frac{\partial u}{\partial x} = 3e^{-2t}\cos 3x \qquad \text{and} \qquad \frac{\partial u}{\partial t} = -2e^{-2t}\sin 3x.$$

from the previous example. Taking the partial derivative of the first of these with respect to $x$ again gives us

$$\frac{\partial^2 u}{\partial x^2} = -9e^{-2t}\sin 3x$$

so

$$\frac{\partial u}{\partial t} = -2e^{-2t}\sin 3x = \frac{2}{9}\left(-9e^{-2t}\sin 3x\right) = k\frac{\partial^2 u}{\partial x^2}, \qquad k = \tfrac{2}{9}$$

---

We will soon see that $u(x,t) = e^{-2t}\sin 3x$ is not the most general solution to the equation.

Those who've had a multivariable calculus course will perhaps recall that the computation of partial derivatives can be significantly more complicated (and therefore difficult) than the ones done above, but if we understand the two examples just given we are ready to understand how

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(L,t) = 0, \qquad u(x,0) = f(x) \tag{1}$$

is solved. The method for doing it is called **separation of variables**, which is similar in execution, up to a point, to the method of the same name we used to solve separable first order ODEs up to a point. After that point we must proceed differently.

To begin, we assume that the function $u(x,t)$ is actually *a product of a function of* $x$ *alone and another function of* $t$ *alone*. There is no practical reason to think this might be the case, but the method works, so we'll use it! (This method was invented/discovered in the 1700s by Daniel Bernoulli, part of a family of a number of accomplished mathematicians and scientists. Daniel was also involved in the derivation of the ODE we used for horizontal beams.) If we let $X$ be the function of $x$ and $T$ be the function of $t$, then $u(x,t) = X(x)T(t)$. (This use of a capital letter for a function and the lower case of the same letter for the independent variable is common practice in the study partial differential equation solution methods.) Now remember that if we are taking the derivative of $X(x)T(t)$ with respect to $x$, $T(t)$ is treated as a constant, and when taking the derivative with respect to $t$, $X(x)$ is treated as a constant, so

$$\frac{\partial u}{\partial t} = X(x)T'(t) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t).$$

The differential equation in (1) then becomes $X(x)T'(t) = kX''(x)T(t)$. If we divide both sides by $kX(x)T(t)$ we get

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}. \tag{2}$$

Here is where we deviate from the procedure for solving first order separable equations. (2) needs to be true for all values of $x$ and $t$, and this is likely only the case if both sides of (2) are equal to some constant (again, we'll see that it works!) that we will call $-\lambda^2$. For reasons we won't go into here, $\lambda$ is positive. Setting each side equal to $-\lambda^2$ and multiplying by the denominators we get

$$\frac{dT}{dt}(t) = -k\lambda^2 T(t) \qquad \text{and} \qquad \frac{d^2X}{dx^2}(x) = -\lambda^2 X(x). \tag{3}$$

In addition to this, we also have the boundary conditions $X(0) = X(L) = 0$ for the second equation. The first equation in (3) tells us that $T(t)$ is an eigenfunction for the first derivative operator, with eigenvalue $-k\lambda^2$, and we know that $T(t)$ is then any constant multiple of $e^{-k\lambda^2 t}$. That is,

$$T(t) = C_1 e^{-k\lambda^2 t}.$$

The second equation says that $X(x)$ is an eigenfunction of the second derivative operator with eigenvalue $-\lambda^2$. Because $\lambda$ is positive, the eigenfunctions are constant multiples of $\sin\lambda x$ and $\cos\lambda x$, as determined in the previous section, and we have

$$X(x) = C_2 \sin\lambda x + C_2 \cos\lambda x.$$

The general solution to the heat equation then looks like

$$u(x,t) = X(x)T(t) = e^{-k\lambda^2 t}(A\sin\lambda x + B\cos\lambda x) \tag{4}$$

where $A = C_1C_2$ and $B = C_1C_3$.

Let's focus a bit more on the second ODE in (3) and its boundary values $X(0) = X(L) = 0$. The general solution to the ODE is $X(x) = A\sin\lambda x + B\cos\lambda x$. Applying the condition $X(0) = 0$ gives us $B = 0$, so the solution is $X(x) = A\sin\lambda t$. (At this point this story should be starting to feel familiar!) We now consider the boundary condition $X(L) = 0$, which gives us $0 = A\sin\lambda L$. As before, when considering vertical columns, we don't want to let $A = 0$, so we must have $\sin\lambda L = 0$. This implies that

$$\lambda L = 0,\ \pi,\ 2\pi,\ 3\pi, ... \qquad \Longrightarrow \qquad \lambda = 0,\ \frac{\pi}{L},\ \frac{2\pi}{L},\ \frac{3\pi}{L}, ...$$

and the solutions to the boundary value problem (disregarding constants and the zero solution arising from $\lambda = 0$) are

$$\sin\frac{\pi}{L}x,\ \sin\frac{2\pi}{L}x,\ \sin\frac{3\pi}{Lx}, ...$$

The solution $T$ then becomes $T(t) = e^{-\frac{k\pi^2}{L^2}t},\ e^{-\frac{4k\pi^2}{L^2}t},\ e^{-\frac{9k\pi^2}{L^2}t}, ...$ depending on $\lambda$, so we get a sequence of solutions $u(x,t) = X(x)T(t)$:

$$u(x,t) = e^{-\frac{k\pi^2}{L^2}t}\sin\tfrac{\pi}{L}x,\ e^{-\frac{4k\pi^2}{L^2}t}\sin\tfrac{2\pi}{L}x,\ e^{-\frac{9k\pi^2}{L^2}t}\sin\tfrac{3\pi}{L}x, ... \tag{5}$$

Recall that when solving an ODE like $y'' + 3y' + 2y = 0$ we assumed $y = e^{rt}$ for some constant $r$. From this we obtain $y = e^{-t}$ or $y = e^{-2t}$, but we saw that the sum of constant multiples of these two, $y = C_1e^{-t} + C_2e^{-2t}$ is the most general solution. By the same reasoning, the most general solution to the PDE we're looking at is an infinite sum of the solutions in (5):

$$u(x,t) = A_1 e^{-\frac{k\pi^2}{L^2}t}\sin\tfrac{\pi}{L}x + A_2 e^{-\frac{4k\pi^2}{L^2}t}\sin\tfrac{2\pi}{L}x + A_3 e^{-\frac{9k\pi^2}{L^2}t}\sin\tfrac{3\pi}{L}x + \cdots + A_n e^{-\frac{n^2k\pi^2}{L^2}t}\sin\tfrac{n\pi}{L}x + \cdots \tag{6}$$

This story goes on quite a bit longer, but let's end it with the following. In order to try to meet the condition $u(x,0) = f(x)$ we must have

$$f(x) = A_1 \sin \tfrac{\pi}{L} x + A_2 \sin \tfrac{2\pi}{L} x + A_3 \sin \tfrac{3\pi}{L} x + \cdots A_n \sin \tfrac{n\pi}{L} x + \cdots \qquad (7)$$

The right hand side of (7) is something called a **Fourier series**. This brings up the question

> In what way (or ways) do we interpret the equal sign in (7), and for what functions $f$ can such interpretation(s) be made?

Attempts to answer this question gave birth to a large amount of mathematics over many years, starting with Joseph Fourier's work in the early 1800s, and with a major result proved as late as 1966. Perhaps some of you will investigate this subject more in later coursework.

## 5.5 Chapter 5 Summary

- Boundary value problems arise when the independent variable of an ODE is length. Applications include the deflection of horizontal beams and vertical columns along their lengths.
- The differential equation for a horizontal beam is fourth order, and the solution is a fourth order polynomial with four arbitrary constants.
- There are two boundary conditions at each of the two ends of a horizontal beam, giving four conditions used to determine the values of the constants.
- There are three possible end conditions for each end of a horizontal beam:
  - **Embedded:** This is when the end of the beam is "clamped" horizontally. The mathematical conditions for such an end are $y = 0$ and $y' = 0$.
  - **Simply Supported (Pinned):** This is when the beam is held up but allowed to pivot. Mathematically, $y = 0$ and $y'' = 0$.
  - **Free:** This is when an end is completely unsupported, and the other end *must* be embedded. The mathematical conditions for such an end are $y'' = 0$ and $y''' = 0$.
- Let $A$ be an operator that operates on functions and $y$ a *nonzero* function. If there is a constant $\lambda$ such that

$$Ay = \lambda y$$

  then $y$ is an **eigenfunction** of the operator $A$, with corresponding **eigenvalue** $\lambda$.
- The ODE for a vertical column is second order, and the solution is either a sine function or a cosine function, depending on the end conditions:
  - When the ends are pinned (hinged) the solution is a sine function.
  - When the ends are embedded the solution is a cosine function.
- Mathematically, there are infinitely many solutions for a vertical column that is pinned at its ends.
  - Each is some multiple of a half period of a sine function beginning at $x = 0$.
  - The first solution, called the **first buckling mode**, is a single half-period of the sine function. This occurs physically when the column is allowed to deflect over its entire length.
  - Each additional solution (buckling mode) consists of $\frac{n}{2}$ periods of the sine function for $n = 2, 3, 4, 5, ....$ Physically, the solution consisting of $\frac{n}{2}$ periods of the sine function occurs when the the column is pinned along its length at $n - 1$ equally spaced points.
- Mathematically, there are infinitely many solutions for a vertical column that is embedded at its ends.
  - Each is some multiple of a half period of a cosine function beginning at $x = 0$.
  - In the case that the ends of the column are embedded, it is physically possible that the ceiling can **float** (move laterally).
  - If the ceiling is allowed to float the first buckling mode is a single half-period of the cosine function. Each additional buckling mode consists of $\frac{n}{2}$ periods of the cosine function for $n = 2, 3, 4, 5, ....$

- If the ceiling is *NOT* allowed to float the first buckling mode is a single period of the cosine function. Each additional buckling mode consists of $n$ periods of the cosine function for $n = 2, 3, 4, 5, ....$

- The load that causes the first buckling mode is called the **first buckling load**, and the $n$th buckling load leads to the $n$th buckling load.
- The $n$th buckling load is $n^2$ times the first buckling load.

# A Formula Sheet

**Exponents and Logarithms:**

$$x^a x^b = x^{a+b}, \qquad (x^a)^b = x^{ab}, \qquad x^{-n} = \frac{1}{x^n}, \qquad x^{\frac{1}{n}} = \sqrt[n]{x}$$

$$\log_a(uw) = \log_a u + \log_a w \qquad \log_a\left(\frac{u}{w}\right) = \log_a u - \log_a w \qquad \log(u^c) = c\log u$$

$$e^{\ln u} = u, \qquad \ln e^u = u$$

**Trigonometric Functions:**

$$\tan u = \frac{\sin u}{\cos u} \qquad \cot u = \frac{\cos u}{\sin u} \qquad \sec u = \frac{1}{\cos u} \qquad \csc u = \frac{1}{\sin u}$$

**Trigonometric Identities:**

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

**Useful Trigonometric Factoid:** $A\sin\omega t + B\cos\omega t = C\sin(\omega t + \phi)$, where

$$C = \sqrt{A^2 + B^2} \text{ and } \phi = \tan^{-1}\frac{B}{A} \text{ if } A > 0, \quad \phi = \pi + \tan^{-1}\frac{B}{A} \text{ if } A < 0$$

If $A = 0$, then $B\cos\omega t = B\sin\left(\omega t + \frac{\pi}{2}\right)$

**Euler's Relations:** $e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$

**Product Rule:** $[uv]' = uv' + vu'$ or $\dfrac{d[uv]}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

**Derivatives of Trig, Exponential and Log Functions:**

$$[\sin(u)]' = [\cos(u)](u)', \quad (\sin at)' = a\cos at \qquad [\cos(u)]' = [-\sin(u)](u)', \quad (\cos at)' = -a\sin at$$

$$[e^u]' = [e^u](u)', \quad (e^{at})' = ae^{at} \qquad [\ln(u)]' = \frac{1}{u}(u)'$$

**A Few Integration Formulas:** All formulas should include an arbitrary constant, which I have left off here to keep things a little cleaner.

$$\int dx = x \qquad \int cf(x)\,dx = c\int f(x)\,dx \qquad \int k\,dx = kx$$

$$\int [f(x) \pm g(x)]\ dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int u^n\ du = \frac{1}{n+1}\ u^{n+1} \text{ as long as } n \neq -1 \qquad \int u^{-1}\ du = \int \frac{1}{u}\ du = \ln|u|$$

$$\int \frac{1}{ax+b}\ dx = \frac{1}{a}\ln\left|\,ax+b\,\right| \qquad \int e^u\ du = e^u \qquad \int e^{at}\ dt = \frac{1}{a}\,e^{at}$$

$$\int t\,e^{at}\ dt = \frac{e^{at}(at-1)}{a^2} \qquad \int t^2 e^{at}\ dt = \frac{e^{at}(a^2t^2 - 2at + 2)}{a^3}$$

$$\int \sin u\ du = -\cos u \qquad \int \sin at\,dt = -\frac{1}{a}\cos at$$

$$\int \cos u\ du = \sin u \qquad \int \cos at\,dt = \frac{1}{a}\sin at$$

$$\int e^{at}\sin bt\ dt = \frac{e^{at}}{a^2+b^2}\ (a\sin bt - b\cos bt) \qquad \int e^{at}\cos bt\ dt = \frac{e^{at}}{a^2+b^2}\ (a\cos bt + b\sin bt)$$

**Solving** $\dfrac{dy}{dx} + p(x)y = q(x)$ **Using an Integrating Factor:**

- Compute $u = \int p(x)\ dx$, multiply both sides of the equation by $e^u$
- The left side becomes $\dfrac{d[e^u y]}{dx}$. Multiply both sides by $dx$ and integrate.

# B Review of Calculus and Algebra

## B.1 Review of Differentiation

**Performance Criteria:**

B. 1. Apply the rules of differentiation, along with provided formulas, to find the derivatives of functions.

The purpose of this appendix is to remind you of how to find the derivatives of some functions of the sorts that we will encounter as we go on. For this we will usually take the independent variable to be $x$ and the dependent variable $y$; the statement $y = f(x)$ indicates that $y$ is a function of $x$; recall also the notations $y' = f'(x) = \dfrac{dy}{dx}$ for the derivative. Let's begin by giving the derivatives of some common functions, and some basic rules of taking derivatives. The notation $(\quad)'$ means the derivative of the quantity in the parentheses, and $k$ represents an arbitrary constant.

**Derivatives of Some Functions**

$$(k)' = 0 \qquad (x)' = 1 \qquad (x^n)' = nx^{n-1} \qquad (e^x)' = e^x$$

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x \qquad (\ln x)' = \frac{1}{x}$$

For the following, $k$ again represents an arbitrary constant, and $f(x)$, $g(x)$, $u = u(x)$ and $v = v(x)$ are functions of the independent variable $x$.

**Derivative Rules**

$$[kf(x)]' = kf'(x) \qquad [u \pm v]' = u' \pm v'$$

$$(uv)' = uv' + vu' \qquad \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

$$f[g(x)]' = f'[g(x)]g'(x)$$

You'll recall the third and fourth items above as the **product rule** and **quotient rule**, and the last item is the **chain rule**. The first two rules above tell us that the derivative is something called a **linear operator**. This is the same idea as a linear transformation, for those of you who have had linear algebra. Now we'll see how to use some of the rules and derivatives of common functions to find derivatives of some combinations of those functions.

### Derivatives of Polynomials

You'll recall that to find the derivative of a polynomial like $f(x) = 5x^3 - \frac{3}{4}x^2 - 7x + 2$ we simply use the various basic derivatives and rules as follows: Multiply the coefficient of each term by the corresponding power of $x$, and decrease the power by one, remember that the derivative of a constant is zero.

◇ **Example B.1(a):** Find the derivative of $f(x) = 5x^3 - \frac{3}{4}x^2 - 7x + 2$.

$$f(x) = 5x^3 - \frac{3}{4}x^2 - 7x + 2 \qquad \Longrightarrow \qquad f'(x) = 15x^2 - \frac{3}{2}x - 7$$

### Second Derivatives

For certain applications we will need to take the derivative of the derivative of a function. You'll recall that such a derivative is called the **second derivative** (so the kind of derivative you did above is called a **first derivative**). The notation is this: If the first derivative is $y'$ or $f'(x)$, then the second derivative is $y''$ or $f''(x)$. If the first derivative is $\frac{dy}{dx}$, then the second derivative is $\frac{d^2y}{dx^2}$. Note the different placement of the "exponents" in the numerator and denominator of this expression - there is a reason for this, but it's not too exciting, so I won't go into that here.

### Other Applications of the Power Rule

Since we can write things like $\frac{1}{x^2}$ and $\sqrt{x}$ using negative and fractional exponents, we can use the power rule to find their derivatives. Here are some examples:

◇ **Example B.1(b):** Find the derivative of $y = \dfrac{5}{x^3}$

$$y = \frac{5}{x^3} = 5x^{-3} \quad \Rightarrow \quad \frac{dy}{dx} = -15x^{-4} = \frac{15}{x^4}$$

◇ **Example B.1(c):** Find the derivative of $f(x) = \dfrac{\sqrt[3]{x}}{4}$

$$f(x) = \frac{\sqrt[3]{x}}{4} = \frac{1}{4}x^{\frac{1}{3}} \quad \Rightarrow \quad f'(x) = \frac{1}{12}x^{-\frac{2}{3}} = \frac{1}{12\sqrt[3]{x^2}}$$

Note that the four in the denominator is a constant, but it is really $\frac{1}{4}$, not 4.

### The Chain Rule

This is perhaps the most confusing (but also most important!) of the derivative rules. Let's use an example to try to explain it; suppose that $y = (x^5 + 2x - 4)^8$. Now if you knew $x = 3$ and wanted to compute $y$ it would be sort of a two step process. First you would compute $3^5 + 2(3) - 4$, then you would take the result to the eighth power. To do the derivative, you do the derivative of the last part first

$$[(\ stuff\ )^8]' = 8(\ stuff\ )^7,$$

and multiply by the derivative $(x^5 + 2x - 4)' = 5x^4 + 2$:

◇ **Example B.1(d):** Find the derivative of $y = (x^5 + 2x - 4)^8$.

$$y' = 8(x^5 + 2x - 4)^7 \cdot (x^5 + 2x - 4)' = 8(x^5 + 2x - 4)^7(5x^4 + 2)$$

In the notation of the rule in the box on the previous page, $g(x) = x^5 + 2x - 4$ and $f(u) = u^8$.

◇ **Example B.1(e):** Find the derivative of $f(t) = 5.6\sin(4.2t - 1.3)$.

$$f'(t) = 5.6[\cos(4.2t - 1.3)] \cdot (4.2t - 1.3)' = 5.6[\cos(4.2t - 1.3)] \cdot (4.2) = 23.6\cos(4.2t - 1.3)$$

---

In this last example there is the additional constant 5.6 that is just multiplied by the result of the derivative of the rest of the function. Note that the result of the derivative of $4.2t - 1.3$ is brought to the front and multiplied by the original constant at the end of the process. More on this in a bit.

In light of what you have seen, we can revise some of our basic derivatives some. Again, $u$ is a function of $x$: $u = u(x)$.

**Derivatives of Some Functions**

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx} \qquad \frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx} \qquad \frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$$

### Proper Manners(!) for Writing Functions

You will often have functions that are products of constants, powers of $x$ and exponential, trig or log functions. In the case of such a function, the correct order to write the factors is

constant, power of $x$, exponential function, trigonometric function

Occasionally you will have a logarithmic function, but these will not generally occur with trigonometric functions; a logarithmic function goes where the trigonometric function is listed above.

### The Product Rule

Since $(u + v)' = u' + v'$, one might hope that $(uv)' = u'v'$. A simple example shows that this is not true:

◇ **Example B.1(f):** Let $u(x) = x^2$ and $v(x) = x^3$, and find $[u(x)v(x)]'$ and $u'(x)v'(x)$.

We see that

$$[u(x)v(x)]' = [x^2x^3]' = (x^5)' = 5x^4 \quad \text{and} \quad u'(x)v'(x) = (2x)(3x^2) = 6x^3.$$

Therefore $[u(x)v(x)]'$ is NOT equal to $u'(x)v'(x)$!

---

So how do we find the derivative of a product of two functions? Well, we use the product rule, which says that if $u = u(x)$ and $v = v(x)$ are two functions of $x$, then

$$(uv)' = uv' + vu' \qquad \text{or} \qquad \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

◇ **Example B.1(g):** Find the derivative of $y = 3x^2 \sin 5x$.

$$\begin{aligned} y' &= (3x^2)(\sin 5x)' + (\sin 5x)(3x^2)' \\ &= (3x^2)(5\cos 5x) + (\sin x)(6x) \\ &= 15x^2 \cos 5x + 6x \sin x \end{aligned}$$

---

**The Quotient Rule**

The quotient rule is similar to the product rule. For two functions $u$ and $v$ of $x$,

$$\left(\frac{u}{v}\right)' = \frac{v\,u' - u\,v'}{v^2}$$

◇ **Example B.1(h):** Find the derivative of $f(x) = \dfrac{e^{2x}}{x^5}$

$$\begin{aligned} f'(x) &= \frac{(x^5)(e^{2x})' - (e^{2x})(x^5)'}{(x^5)^2} \\ &= \frac{(x^5)(2e^{2x}) - (e^{2x})(5x^4)}{x^{10}} \\ &= \frac{2x^5 e^{2x} - 5x^4 e^{2x}}{x^{10}} \end{aligned}$$

---

**Section B.1 Exercises** To Solutions

1. Find the derivative of each polynomial:

   (a) $y = 3x^5 - 7x^4 + x^2 - 3x + 2$ (b) $h(t) = -16t^2 + 23.7t + 3.5$

   (c) $f(x) = \frac{2}{3}x^4 + 5x^3 - \frac{1}{8}x^2 + 3$ (d) $s(t) = t^3 - 2t^2 + 3t - 5$

2. Find the second derivative of each of the functions in Exercise 1.

3. Find the derivative of each of the following. Give your final answers without negative or fractional exponents.

   (a) $f(x) = \dfrac{3}{x^2}$ (b) $g(t) = \dfrac{t^2}{6} - \dfrac{6}{t^2}$ (c) $y = \frac{1}{5}\sqrt{x}$ (d) $h(x) = \dfrac{4}{\sqrt[3]{x}}$

4. Find the second derivatives of the functions from 3(a) and (b).

5. Find the derivative of each of the following, utilizing the suggestions provided.

   (a) $g(x) = \sqrt{16 - x^2}$ - If you write the square root as an exponent you will have something very much like the first example above.

   (b) $y = 3\cos\left(\frac{\pi}{2}t\right)$

   (c) $A(t) = 500e^{-0.3t}$ - This is a two-step process (again, ignoring the constant of $500$), with the first step being $-0.3t$ and the second step being the exponential. Remember that the derivative of $e^x$ is $e^x$.

6. Find the derivative of each of the following.

   (a) $y = 5e^{2x}$ (b) $x = 4\sin(3t)$ (c) $g(x) = \dfrac{2}{(x^2 - 4x)^7}$

   (d) $s(t) = \cos(\frac{2}{5}t)$ (e) $y = e^{x^2}$

7. For each of the following, put the product in the correct order.

   (a) $e^{5t^3} \cdot 15t^3$ (b) $3[-\sin(5x - 2)] \cdot 5$ (c) $4\sin(5t + 3) \cdot (e^{-2t})$

8. For each of the following, multiply and put the final product in the correct order.

   (a) $4\sin(5t + 3) \cdot (-2e^{-2t})$ (b) $7e^{-t}\cos(3t - 1) \cdot 2t$ (c) $(3\ln x)(4x^3)$

9. Use the product rule to find the derivative of each of the following.

   (a) $f(t) = t^2 e^{7t}$ (b) $y = 3x\cos 2x$ (c) $h(t) = 2e^{-3t}\sin \pi t$

10. Find the derivative of each of the following.

   (a) $f(x) = \dfrac{3\sin 2t}{e^{7t}}$ (b) $y = \dfrac{4e^{-5t}}{t^2}$ (c) $g(x) = \dfrac{\cos 6x}{2x^3}$

11. Use the quotient rule and the fact that $\tan x = \dfrac{\sin x}{\cos x}$ to determine the derivative of $\tan x$.

## B.2 Review of Integration

**Performance Criteria:**

B. 2. Apply rules of integration, along with formulas, to find indefinite integrals of functions.

In this course we need on occasion to find anti-derivatives, or indefinite integrals, of functions. As you learned in integral calculus, doing so is often a challenging endeavor! Fortunately we only need to be able to find indefinite integrals of a handful of types of functions in our study of differential equations. We can use some simple formulas, found on the formula sheet (Appendix A), that are derived from methods you learned before, like substutution and integration by parts. We begin with linearity of the indefinite integral and the most basic antiderivative:

**Linearity of the Indefinite Integral**

Let $c$ be any constant, and let $f$ and $g$ be functions.

$$\int c\,f(x)\,dx = c\int f(x)\,dx, \qquad \int \big[f(x) \pm g(x)\big]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

**Indefinite Integral of $x^n$**

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \;\text{ if }\; n \neq -1, \qquad \int \frac{1}{x}\,dx = \ln|x| + C,$$

where $C$ is an arbitrary constant.

We note at this point that any indefinite integral includes the addition of an arbitrary constant, as shown above, but *we will not alwways include the constant in future formulas, even though it belongs in all of them.*

Here are a couple of special examples of two of the above things:

◇ **Example B.2(a):** Find the integrals $\int dx$ and $\int k\,dx$, where $k \neq 0$ is a constant.

**Solution:** $\int dx = \int x^0\,dx = \frac{1}{0+1}x^{0+1} + C = x + C, \quad \int k\,dx = k\int dx = k(x+C) = kx + C$

Note that when we write $k(x+C) = kx + C$, the two constants $C$ are actually different, but it is common to abuse notation this way.

The results of Example B.2(b) and the integral of $x^n$, $n \neq -1$ are commonly used when finding the integral of a polynomial. You probably don't think of it quite like this, but here is what happens:

◇ **Example B.2(b):** Find $\displaystyle\int (5x^2 - 6x + 4)\,dx$.

**Solution:**

$$\begin{aligned}\int (5x^2 - 6x + 4)\,dx &= \int 5x^2\,dx - \int 6x\,dx + \int 4\,dx \\ &= 5\int x^2\,dx - 6\int x\,dx + \int 4\,dx \\ &= 5\left(\tfrac{1}{3}x^3\right) - 6\left(\tfrac{1}{2}x^2\right) + 4x + C \\ &= \tfrac{5}{3}x^3 - 3x^2 + 4x + C\end{aligned}$$

---

It is not necessary that you show all of the above steps when doing such an integral - anything is pretty much alright as long as you arrive at the correct result!

We will occasionally see integral slike the following, which are quite easy *if we use the results from the previous page and negative exponents*.

◇ **Example B.2(c):** Find $\displaystyle\int \frac{7}{x^4}\,dx$.

**Solution:** Using the fact that $\dfrac{1}{x^4} = x^{-4}$ and noting that when we add one to $-4$ we get $-3$, we have the following:

$$\int \frac{7}{x^4}\,dx = 7\int \frac{1}{x^4}\,dx = 7\int x^{-4}\,dx = 7 \cdot \frac{1}{-3}\,x^{-3} + C = -\frac{7}{3x^3} + C$$

---

Exponential functions are extremely important in applications, and a large part of their importance is the result of the fact that they are essentially their own derivatives and integrals:

> **Indefinite Integral of** $e^{ax}$
>
> $$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax} + C$$

◇ **Example B.2(d):** Find $\displaystyle\int e^{-5x}\,dx$.

**Solution:** $\displaystyle\int e^{-5x}\,dx = \tfrac{1}{-5}\,e^{-5x} + C = -\tfrac{1}{5}\,e^{-5x} + C$.

---

Note that, unlike the integral of $x^{-5}$, *the exponent does not change when we integrate an exponential function*. This holds true for the integrals of trigonometric functions as well. When working with applications of differential equations, we usually only need the following.

**Indefinite Integrals of Sine and Cosine**

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

◇ **Example B.2(e):** Find $\displaystyle\int 3\sin\tfrac{\pi}{2}x\,dx$.

**Solution:** $\displaystyle\int 3\sin\tfrac{\pi}{2}x\,dx = 3\int \sin\tfrac{\pi}{2}x\,dx = 3\left(-\frac{1}{\frac{\pi}{2}}\cos\tfrac{\pi}{2}x + C\right) = -\tfrac{6}{\pi}\cos\tfrac{\pi}{2}x + C.$

---

Note that the second $C$ above is really three times the first $C$!

There are a number of somewhat specialized formulas on the formula sheet that we will use fairly often. We'll give an example of the use of one of them here, and the others will be addressed in the exercises. Here is the formula we'll use in the next example:

$$\int e^{at}\cos bt\;dt = \frac{e^{at}}{a^2+b^2}\,(a\cos bt + b\sin bt) + C \tag{1}$$

Note that the variable is $t$, rather than $x$. This will be common for us.

◇ **Example B.2(f):** Find and simplify $\displaystyle\int 3e^{-5t}\cos 2t\,dt$.

**Solution:** First we note that, in the context of formula (1), $a = -5$ and $b = 2$. Pulling the constant $3$ out of the integral and applying the formula we get

$$\begin{aligned}\int 3e^{-5t}\cos 2t\,dt &= 3\left[\frac{e^{-5t}}{(-5)^2+2^2}\big(-5\cos 2t + 2\sin 2t\big)\right] + C\\ &= \frac{3e^{-5t}}{29}\big(-5\cos 2t + 2\sin 2t\big) + C\\ &= e^{-5t}\big(-\tfrac{15}{29}\cos 2t + \tfrac{6}{29}\sin 2t\big) + C\end{aligned}$$

---

## Section B.2 Exercises

To Solutions

1. We will commonly encounter integrals for which this formula is useful:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\left|\,ax+b\,\right| + C$$

Use the formula to compute the following integrals.

(a) $\displaystyle\int \frac{5}{2x+3}\,dx$ (b) $\displaystyle\int \frac{2}{3-5x}\,dx$ (c) $\displaystyle\int \frac{1}{1.6-0.08A}\,dA$

2. Evaluate each of the following indefinite integrals, using the formula sheet as you need. For exercises involving decimals, round to two significant figures. **Be sure to note and use the correct variable, in the case (upper or lower) given**.

(a) $\displaystyle\int (x^2 - 7x + 3)\,dx$

(b) $\displaystyle\int 7\sin 3t\,dt$

(c) $\displaystyle\int 3te^{-2t}\,dt$

(d) $\displaystyle\int \frac{3}{x^2}\,dx$

(e) $\displaystyle\int \frac{dA}{2.0 - 0.1A}$

(f) $\displaystyle\int 3e^{-t}\sin 5t\,dt$

(g) $\displaystyle\int \frac{3\,dx}{5x - 1}$

(h) $\displaystyle\int 5t^2e^{-3t}\,dt$

(i) $\displaystyle\int e^{-4t}\cos 3t\,dt$

(j) $\displaystyle\int 5\cos \tfrac{\pi}{2}t\,dt$

## B.3 Solving Systems of Equations

In this course we will often need to solve systems of two equations in two unknowns, and it is important to be able to do this quickly *and correctly*. There are two methods, the **addition method** and the **substitution method**. Each has its own advantages and disadvantages; both methods will be demonstrated here.

### The Addition Method

We'll begin with the addition method, going from the easiest scenario to the most difficult (which still isn't too hard).

◇ **Example B.3(a):** Solve the system $\begin{array}{rcl} 3x - y & = & 5 \\ 2x + y & = & 15 \end{array}$.

The basic idea of the addition method is to add the two equations together so that one of the unknowns goes away. In this case, as shown below and to the left, nothing fancy need be done. The remaining unknown is then solved for and placed back into *either* equation to find the other unknown as shown below and to the right.

$$\begin{array}{rcl} 3x - y & = & 5 \\ 2x + y & = & 15 \\ \hline 5x & = & 20 \\ x & = & 4 \end{array} \qquad \longrightarrow \qquad \begin{array}{rcl} 3(4) - y & = & 5 \\ 12 - y & = & 5 \\ 12 & = & y + 5 \\ 7 & = & y \end{array}$$

The solution to the system is $(4, 7)$.

What made this work so smoothly is the $-y$ in the first equation and the $+y$ in the second; when we add the two equations, the sum of these is zero and $y$ "has gone away." In the next two examples we see what to do in slightly more difficult situations.

◇ **Example B.3(b):** Solve the system $\begin{array}{rcl} 3x + 4y & = & 13 \\ x + 2y & = & 7 \end{array}$.

We can see that if we just add the two equations together we get $4x + 6y = 20$, which doesn't help us find either of $x$ or $y$. The trick here is to multiply the second equation by $-3$ so that the the first term of that equation becomes $-3x$, the opposite of the first term of the first equation. When we then add the two equations the $x$ terms go away and we can solve for $y$:

$$\begin{array}{rcl} 3x + 4y & = & 13 \\ x + 2y & = & 7 \end{array} \quad \begin{array}{c} \Longrightarrow \\ \underset{\text{times } -3}{\Longrightarrow} \end{array} \quad \begin{array}{rcl} 3x + 4y & = & 13 \\ -3x - 6y & = & -21 \\ \hline -2y & = & -8 \\ y & = & 4 \end{array} \quad \longrightarrow \quad \begin{array}{rcl} x + 2(4) & = & 7 \\ x + 8 & = & 7 \\ x & = & -1 \end{array}$$

The solution to the system of equations is $(-1, 4)$. Note that we could have eliminated $y$ first instead of $x$:

$$\begin{array}{rcl} 3x + 4y & = & 13 \\ x + 2y & = & 7 \end{array} \quad \begin{array}{c} \Longrightarrow \\ \underset{\text{times } -2}{\Longrightarrow} \end{array} \quad \begin{array}{rcl} 3x + 4y & = & 13 \\ -2x - 4y & = & -14 \\ \hline x & = & -1 \end{array} \quad \longrightarrow \quad \begin{array}{rcl} -1 + 2y & = & 7 \\ 2y & = & 8 \\ y & = & 4 \end{array}$$

Three things need to be pointed out at this time:

- There is no need for the equations to have the forms

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}.$$

  All that is necessary is that the unknown to be eliminated is on the same side of both equations.

- The unknowns in this course will usually be denoted by letters other than $x$ and $y$, like $C_1$, $C_2$, $A$ and $B$.

- Often *both* equations must be multiplied by a value in order to eliminate an unknown.

The following example illustrates both of these points.

◇ **Example B.3(c):** Solve the system $\begin{aligned} 4 &= 3A - 5B + 1 \\ 0 &= -5A - 3B - 2 \end{aligned}$.

In this case we will chose to eliminate $A$ because its coefficients already have opposite signs in the two equations. We'll multiply the first equation by $5$ and the second by $3$ so that the coefficients of $A$ will be the same, but with opposite signs. When we then add the two equations the $A$ terms go away and we can solve for $B$:

$$\begin{array}{rcl c rcl} 4 &=& 3A - 5B + 1 & \stackrel{\text{times } 5}{\Longrightarrow} & 20 &=& 15A - 25B + 5 \\ 0 &=& -5A - 3B - 2 & \underset{\text{times } 3}{\Longrightarrow} & 0 &=& -15A - 9B - 6 \\ \hline &&&& 20 &=& -34B - 1 \\ &&&& 21 &=& -34B \\ &&&& -\frac{21}{34} &=& B \end{array}$$

Ordinarily we would substitute this value into one of the original equations and solve for $A$ at this point. However, when one of the unknowns is a messy fraction it is often easier to repeat the same procedure, but eliminate the other unknown. Let's multiply the first equation by $3$ and the second by $-5$:

$$\begin{array}{rcl c rcl} 4 &=& 3A - 5B + 1 & \stackrel{\text{times } 3}{\Longrightarrow} & 12 &=& 9A - 15B + 3 \\ 0 &=& -5A - 3B - 2 & \underset{\text{times } -5}{\Longrightarrow} & 0 &=& 25A + 15B + 10 \\ \hline &&&& 12 &=& 34A + 13 \\ &&&& -1 &=& 34A \\ &&&& -\frac{1}{34} &=& A \end{array}$$

The solution to the system is $A = -\frac{1}{34}$, $B = -\frac{21}{34}$.

---

Let's summarize the steps for the addition method, which you've seen in the above examples.

**The Addition Method**

To solve a system of two linear equations by the addition method,

(1) Multiply each equation by something as needed in order to make the coefficients of one unknown the same but opposite in sign in the two equations.

(2) Add the two equations and solve the resulting equation for whichever unknown remains.

(3) Substitute that value into either original equation and solve for the other unknown *OR* repeat steps (1) and (2) for the other unknown.

### The Substitution Method

We will now describe the substitution method, then give an example of how it works.

**The Substitution Method**

To solve a system of two linear equations by the substitution method,

1) Pick one of the equations in which the coefficient of one of the unknowns is either one or negative one. Solve that equation for that unknown.

2) Substitute the expression for that unknown into *the other* equation and solve for the unknown.

3) Substitute that value into the equation from (1), or into either original equation, and solve for the other unknown.

◇ **Example B.3(d):** Solve the system of equations $\begin{aligned} x - 3y &= 6 \\ -2x + 5y &= -5 \end{aligned}$ using the substitution method.

Solving the first equation for $x$, we get $x = 3y + 6$. We now replace $x$ in the second equation with $3y + 6$ and solve for $y$. Finally, that result for $y$ can be substituted into $x = 3y + 6$ to find $x$:

$$\begin{aligned} -2(3y+6) + 5y &= -5 \\ -6y - 12 + 5y &= -5 \\ -y - 12 &= -5 \\ -y &= 7 \\ y &= -7 \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} x - 3(-7) &= 6 \\ x + 21 &= 6 \\ x &= -15 \end{aligned}$$

The solution to the system of equations is $(-15, -7)$.

When solving a system of three equations in three unknowns, matrix methods are usually employed. However, we will encounter certain systems of three equations in three unknowns for which the substitution method yields a solution rather easily. The next example illustrates this.

◇ **Example B.3(e):** Solve the system of equations $\begin{aligned} 9A \quad\quad\quad\quad &= 3 \\ 12A + 8B \quad\quad &= 0 \\ 3A + 6B + 12C &= 8 \end{aligned}$ using the substitution method.

We divide both sides of the first equation by $9$ to obtain $A = \frac{1}{3}$. Substituting this into the second equation we get

$$\begin{aligned} 12(\tfrac{1}{3}) + 8B &= 0 \\ 4 + 8B &= 0 \\ 8B &= -4 \\ B &= -\tfrac{1}{2} \end{aligned}$$

We can now substitute $A = \frac{1}{3}$ and $B = -\frac{1}{2}$ into the third equation to get

$$\begin{aligned} 3(\tfrac{1}{3}) + 6(-\tfrac{1}{2}) + 12C &= 8 \\ 1 - 3 + 12C &= 8 \\ 12C &= 10 \\ C &= \tfrac{5}{6} \end{aligned}$$

The solution to the system is $A = \frac{1}{3}$, $B = -\frac{1}{2}$, $C = \frac{5}{6}$.

---

## Section B.3 Exercises

To Solutions

1. Solve each of the following systems by both the addition method and the substitution method.

   (a) $\begin{aligned} 2x + y &= 13 \\ -5x + 3y &= 6 \end{aligned}$ (b) $\begin{aligned} 2x - 3y &= -6 \\ 3x - y &= 5 \end{aligned}$ (c) $\begin{aligned} x + y &= 3 \\ 2x + 3y &= -4 \end{aligned}$

2. Solve each of the following systems by the addition method.

   (a) $\begin{aligned} 7x - 6y &= 13 \\ 6x - 5y &= 11 \end{aligned}$ (b) $\begin{aligned} 5x + 3y &= 7 \\ 3x - 5y &= -23 \end{aligned}$ (c) $\begin{aligned} 5x - 3y &= -11 \\ 7x + 6y &= -12 \end{aligned}$

3. Consider the system of equations $\begin{aligned} 2x - 3y &= 4 \\ 4x + 5y &= 3 \end{aligned}$.

   (a) Solve for $x$ by using the addition method to eliminate $y$. Your answer should be a fraction.

   (b) Ordinarily you would substitute your answer to (a) into either equation to find the other unknown. However, dealing with the fraction that you got for part (a) could be difficult and annoying. Instead, use the addition method again, but eliminate $x$ to find $y$.

4. Consider the system of equations $\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{3}y &= 2 \\ \tfrac{1}{4}x + \tfrac{2}{3}y &= 6 \end{aligned}$. The steps below indicate how to solve such a system of equations.

(a) Multiply both sides of the first equation by the least common denominator to “kill off” all fractions.

(b) Repeat for the second equation.

(c) You now have a new system of equations without fractional coefficients. Solve that system by the addition method.

5. Solve each of the following systems of equations.Each is of the sort that arise in solving various initial value and boundary value problems.

(a)
$$\begin{aligned} 4 &= C_1 + C_2 \\ -3 &= -2C_1 - 5C_2 + 7 \end{aligned}$$

(b)
$$\begin{aligned} 7A - 2B &= 4 \\ -2A - 7A &= 0 \end{aligned}$$

(c)
$$\begin{aligned} 8A \qquad\qquad &= 4 \\ 10A + 8B \qquad &= -3 \\ 4A + 5B + 8C &= 10 \end{aligned}$$

(d)
$$\begin{aligned} -3 &= C_1 + C_2 - 3 \\ 2 &= -4C_1 - C_2 + 1 \end{aligned}$$

(e)
$$\begin{aligned} 3A - 8B &= -2 \\ -8A - 3B &= 1 \end{aligned}$$

(f)
$$\begin{aligned} 0 &= \tfrac{800}{12} + \tfrac{800}{6}C_1 + \tfrac{80}{2}C_2 \\ 0 &= \tfrac{800}{3} + \tfrac{80}{2}C_1 + 8C_2 \end{aligned}$$

## B.4 Partial Fraction Decomposition

There are times when we wish to take an expression of the form $\dfrac{Ax+B}{(x-x_1)(x-x_2)}$, where either (but not both) of $A$ or $B$ might be zero, and find two expressions

$$\frac{C}{x-x_1} \qquad \text{and} \qquad \frac{D}{x-x_2} \tag{1}$$

such that

$$\frac{Ax+B}{(x-x_1)(x-x_2)} = \frac{C}{x-x_1} + \frac{D}{x-x_2}. \tag{2}$$

The sum to the right of the equal sign in (2) is called the **partial fraction decomposition** of the expression to the left of the equal sign there. The process of obtaining the right hand side is also called partial fraction decomposition, and we illustrate it in the following example.

◇ **Example B.4(a):** Find two expressions of the form (2) whose sum is $\dfrac{x+17}{x^2+4x-5}$.

First we note that

$$\frac{x+17}{x^2+4x-5} = \frac{x+17}{(x-1)(x+5)},$$

so we are looking for $\dfrac{C}{x-1}$ and $\dfrac{D}{x+5}$ such that

$$\frac{C}{x-1} + \frac{D}{x+5} = \frac{x+17}{x^2+4x-5}.$$

But

$$\frac{C}{x-1} + \frac{D}{x+5} = \frac{C(x+5)}{(x-1)(x+5)} + \frac{D(x-1)}{(x-1)(x+5)} = \frac{Cx+5C+Dx-D}{(x-1)(x+5)}$$

Now if this last expression is to equal $\dfrac{x+17}{(x-1)(x+5)}$, then the numerators of both fractions must be equal (because they both have the same denominator). Note that both fractions have the same denominator, so the two fractions will be equal only if their numerators are equal:

$$Cx + 5C + Dx - D = x + 17$$

By "grouping like terms," this can be rewritten (be sure you see how) as

$$(C+D)x + (5C-D) = 1x + 17,$$

and these will be equal only if $C+D=1$ and $5C-D=17$. Now we have two equations in two unknowns, which we know how to solve (see Appendix B.3). If we add the two equations together we get $6C=18$, so $C=3$. Substituting this into the first equations gives $D=-2$. Thus

$$\frac{x+17}{x^2+4x-5} = \frac{3}{x-1} + \frac{-2}{x+5} = \frac{3}{x-1} - \frac{2}{x+5}.$$

The method of partial fractions has many complications that can arise when the expression to be decomposed has other forms. Those complications are dealt with by varying the above process slightly, but for our needs the above method is sufficient.

**Section B.4 Exercises** To Solutions

1. Add $\dfrac{3}{x-1}$ and $\dfrac{-2}{x+5}$. Check your answer with the original expression from Example B.3(a).

2. Find the partial fraction decomposition of each expression.

   (a) $\dfrac{4x+7}{x^2+5x+6}$

   (b) $\dfrac{-14}{x^2-3x-10}$

   (c) $\dfrac{11-x}{x^2-x-2}$

   (d) $\dfrac{4x-10}{x^2-1}$

## B.5 Series and Euler's Formula

Try the following: Set your calculator for radians (as you always should in this course) and find $\sin(0.05)$, $\sin(0.1)$ and $\sin(0.5)$. You should get (to five places past the decimal) 0.04998, 0.09983 and 0.47943, respectively. These numbers are fairly close to the numbers that you were finding the sine of, with the one closest to zero being the best approximation. This seems to indicate that

$$\sin x \approx x$$

for values of $x$ near zero. Since of course $\sin(5)$ must be a number between $-1$ and $1$ this will certainly not hold in that case!

Now consider the function $f(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$. Here we find that if we round to five places past the decimal we get $f(0.5) = 0.47943$, the value of $\sin(0.5)$ when rounded to the same number of decimal places. Let's try something bigger:

$$f(1.3) = 0.96477, \quad \sin(1.3) = 0.96356 \qquad\qquad f(2.0) = 0.93333, \quad \sin(2.0) = 0.90930$$

It appears that this function $f$ comes pretty close to approximating the sine function, especially for values of $x$ nearer to zero.

The fraction coefficients in the equation for the function $f$ appear to be a bit mysterious, but it turns out that $6 = 3\cdot2\cdot1$ and $120 = 5\cdot4\cdot3\cdot2\cdot1$. We call the quantity $n(n-1)(n-2)\cdots3\cdot2\cdot1$ "$n$ factorial," denoted by $n!$ So $6 = 3!$, $120 = 5!$ and

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

It turns out that (in a sense that you really must take a course in sequences and series to understand)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

This is called the (infinite) **series representation** of the sine function. As you have seen, for values of $x$ near zero, very good approximations of $\sin x$ can be obtained by using just the first term or few terms of this series. For values farther from zero, more terms must be used to get a good approximation.

Cosine and the exponential function have series representations as well:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

and

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

Recall now $i$, the imaginary unit ($j$ for you electrical engineering students), for which $i^2 = -1$. We can also compute things like

$$i^1 = i, \qquad i^3 = i^2\cdot i = -i, \qquad i^4 = i^3\cdot i = -i\cdot i = -i^2 = 1, \qquad i^5 = i^4\cdot i = 1i = i$$

and so on. Because we can compute these, we can now find something like $e^{i\theta}$ by using the series representation of $e^x$:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots \\ &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} + \cdots\right) \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

A similar computation can be done for $e^{-i\theta}$. The result of the two of these is

**Euler's Relations:**

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

# C Numerical Solutions to ODEs

**Performance Criteria:**

C. (1) Given a first order IVP, a recursion formula, and a step size, generate a numerical sequence of solution values.

(2) Given the direction field for a first order ODE and an initial value, sketch the solution generated by Euler's method.

*The point of* **numerical methods** *is that they can be used to solve IVPs that cannot be solved analytically.* (We will sometimes use numerical methods with IVPs that we *CAN* solve analytically, so that we can compare numerical solutions with analytical solutions.) So numerical methods are needed when we want a function $y = y(t)$ that we can't use algebra and calculus methods to find an equation for. The use of numerical methods (both for solving IVPs and for other things) is a huge part of mathematics, science and engineering. Here you will be introduced to some methods for finding solutions to first order initial value problems. The simplest method is called **Euler's method**; the name comes from Leonhard Euler (pronounced "oiler"), a very prolific Swiss mathematician who lived in the 1700s. You will see that this is not a very good method, but we will study it because it is simple to implement, it illustrates what are called iterative techniques, and the ideas behind it are the same ones that make some better methods work.

Before actually finding a numerical solution to an IVP, let's go over what a solution will consist of, and the special notation we'll be using in this section. Suppose that we have an IVP

$$\frac{dy}{dt} = f(t,y), \qquad y(t_0) = y_0;$$

the solution to the IVP is some unknown function $y = y(t)$. Consider the sequence of times $t = 0, 0.2, 0.4, 0.6, 0.8, ...$ seconds, which we'll denote as $t_0, t_1, t_2, t_3, t_4, ...$. Note that these times are spaced apart by the increment of $0.2$ seconds - we could just as well have used an increment of $0.1$ seconds, which would give us instead

$$t_0 = 0, \quad t_1 = 0.1, \quad t_2 = 0.2, \quad t_3 = 0.3, ...$$

The increment used is denoted by the letter $h$, so for the first sequence of times listed above $h = 0.2$ and for the second sequence $h = 0.1$. We will refer to the value of $h$ as the **step size**, meaning that time advances in steps of size $h$. Regardless of the value of $h$, we always start with a known value of $t_0$ (usually zero) and $h$, then we use the relations

$$t_1 = t_0 + h, \qquad t_2 = t_1 + h = t_0 + 2h, \qquad t_3 = t_2 + h = t_0 + 3h, \qquad t_4 = t_3 + h = t_0 + 4h, ...$$

to determine the other time values.

Let's continue this discussion for the first sequence of times, $t_0 = 0$, $t_1 = 0.2$, $t_2 = 0.4$, $t_3 = 0.6$, and so on. If we knew the exact solution $y = y(t)$, we could compute the values of $y(0.2)$, $y(0.4)$, $y(0.6)$, ... (we already know the value of $y(0) = y_0$). Because we don't know the exact solution, we instead come up with a sequence $y_1, y_2, y_3, ...$ where

$$y_1 \approx y(t_1) = y(0.2), \qquad y_2 \approx y(t_2) = y(0.4), \qquad y_3 \approx y(t_3) = y(0.6), ...$$

Each value in the sequence must be determined before the following term can be found, so we use the known value of $y_0$ to find $y_1$, then use that value of $y_1$ to find $y_2$, use that to find $y_3$, and so on, as far as we wish to have values for $y$. It should be noted that to find any value in the $y$ sequence we always use the previous value of $y$, but might also use other prior $y$ values and perhaps some time values as well.

Any method that proceeds sequentially like this is called an **iterative method**. To carry it out we need an iteration formula that tells us how to obtain a $y$ value from previous $y$ values and perhaps previous time values. The simplest such formula, as mentioned before, is the one for Euler's method:

$$y_{n+1} = y_n + h \left.\frac{dy}{dt}\right|_{(t_n, y_n)} = y_n + hf(t_n, y_n) \tag{1}$$

Let's think about what this tells us. It says that to get the next $y$ value $y_{n+1}$ we take the "current" $y$ value $y_n$ and add the product of the increment $h$ and the derivative evaluated at our current $t$ and $y$ values. But because the ODE can be put in the form $\frac{dy}{dt} = f(t, y)$, we use the final expression above to compute each new $y$ value.

You may (understandably) be totally confused by now! Let's look at a specific example.

◇ **Example C(1):** Use the Euler's method equation (1) to find $y_0, y_1, y_2, y_3$ and $y_4$ for the initial value problem

$$\frac{dy}{dt} = t - y, \qquad y(0) = 2,$$

using a step size of $h = 0.2$.

First we identify $t_0 = 0$ and $y_0 = 2$ (from the initial value), $t_1 = 0.2$, $t_2 = 0.4$, ... from $t_0$ and $h$, and $f(t, y) = t - y$ (from the ODE). By (1),

$$y_1 = y_0 + hf(t_0, y_0) = 2 + (0.2)(0 - 2) = 1.6$$

We then use this value and (1) again to obtain

$$y_2 = y_1 + hf(t_1, y_1) = 1.6 + (0.2)(0.2 - 1.6) = 1.32$$

Continuing, we get

$$y_3 = y_2 + hf(t_2, y_2) = 1.32 + (0.2)(0.4 - 1.32) = 1.136,$$

$$y_4 = y_3 + hf(t_3, y_3) = 1.136 + (0.2)(0.6 - 1.136) = 1.0288.$$

More $y$ values, if desired, could be obtained by continuing in the same way.

---

You might be wondering why the formula $y_{n+1} = y_n + h \left.\frac{dy}{dt}\right|_{(t_n, y_n)}$ works. Recall the definition of the derivative

$$\frac{dy}{dt} = \lim_{h \to 0} \frac{y(t+h) - y(t)}{h}.$$

This says that as $h$ approaches zero, the quantity on the right approaches the derivative. Thus, for small values of $h$,

$$\frac{dy}{dt} \approx \frac{y(t+h) - y(t)}{h}.$$

Multiplying both sides by $h$ and solving for $y(t+h)$ gives us

$$y(t+h) \approx y(t) + h\frac{dy}{dt},$$

and if $t = t_n$, then $t + h = t_{n+1}$; substituting these into the above gives us

$$y(t_{n+1}) \approx y(t_n) + h \left.\frac{dy}{dy}\right|_{t=t_n}.$$

But $y(t_{n+1}) \approx y_{n+1}$ and $y(t_n) \approx y_n$, and our expression $f(t, y)$ for the derivative may contain both $t$ and $y$, so we must evaluate the derivative at $(t_n, y_n)$, giving us the final result

$$y_{n+1} = y_n + h \left.\frac{dy}{dt}\right|_{(t_n,y_n)} = y_n + hf(t_n, y_n). \tag{1}$$

This formula can also be derived using something called a **Taylor polynomial**, which many of you have not yet seen. For this reason we will not go into it here but, if you are interested, you can see the derivation of Euler's method from a Taylor polynomial in most any introductory differential equations book. Using Taylor polynomials has a couple advantages:

- Their use can lead to better formulas than the one for Euler's method.
- Their use can give us a *bound* for the error that occurs when using a numerical method.

The distinction is a bit subtle and often blurred, but when we just use an iterative formula, as we will, we are performing a **numerical method**. When we try to analyze our method for how accurate it is, we are doing something called **numerical analysis**.

There is also a geometric explanation for how and why Euler's method works. Suppose that we have the initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

with solution $y = y(t)$. Figure C(1) on the next page shows the solution curve (which is really unknown) through $(t_0, y_0)$ and the points on that curve corresponding to two times $t_1$ and $t_2$ as described previously. We can think of generating the approximate numerical solution consisting of a sequence $y_0, y_1, y_2, y_3, \ldots$ as follows:

- We are given the initial value $y_0$.
- We construct a line through $(t_0, y_0)$ that is tangent to the solution curve - its slope is the derivative at $(t_0, y_0)$. See Figure C(2). The point on that line with $t$-coordinate $t_1$ is $y_1$ (Figure C(3)).
- We construct a line through $(t_1, y_1)$ that is tangent *to the solution curve passing through that point*. Its slope is the derivative at $(t_1, y_1)$, and the point on the line with $t$-coordinate $t_2$ is $y_2$. See Figure C(4).
- Continue in this manner to obtain the values $y_3, y_4, \ldots$.

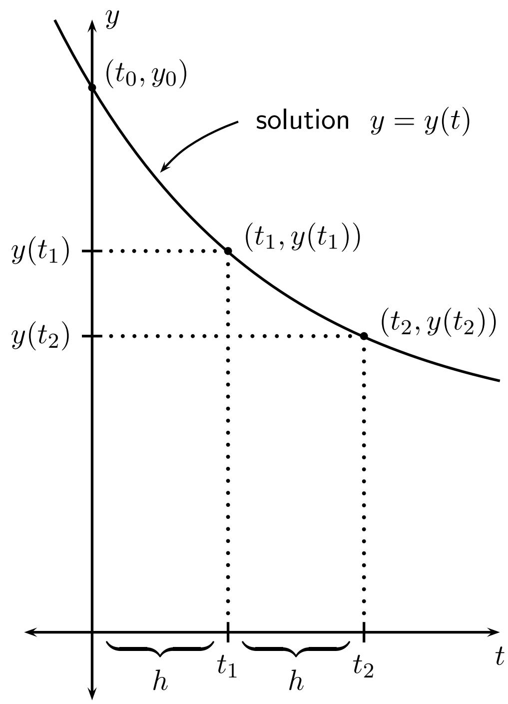



Figure C(1)

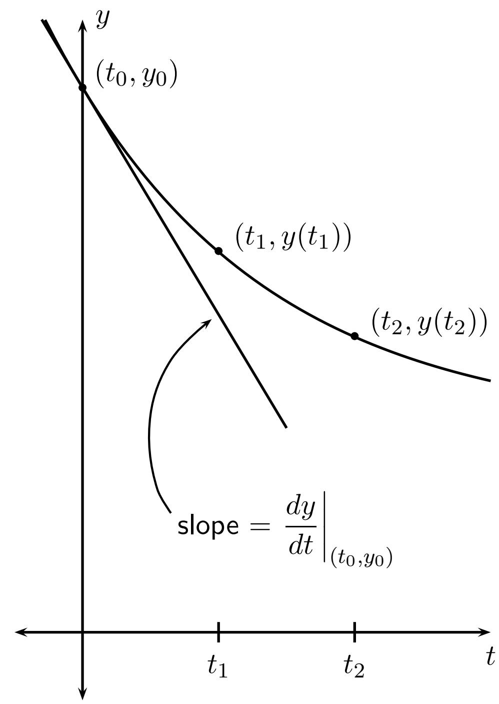



Figure C(2)

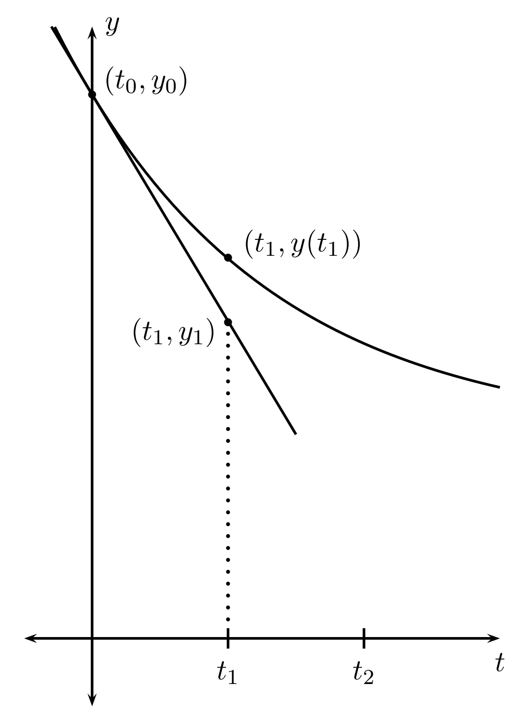



Figure C(3)

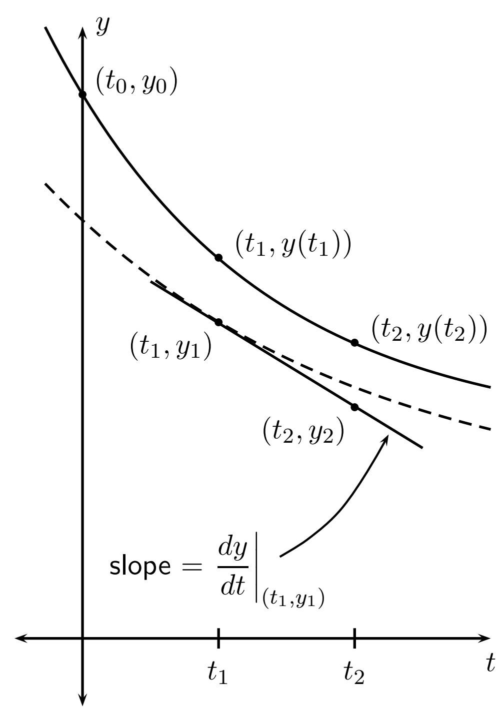



Figure C(4)

We should make a couple observations/comments:

- It should be clear from Figure C(2) that a smaller value of $h$ would give us a value of $y_1$ that is closer to the true value $y(t_1)$.
- The error in the approximate solution does not necessarily grow as $t$ gets larger.

The following summarizes the idea of how Euler's method works.

**Euler's Method**

To solve the first order IVP $\dfrac{dy}{dt} = f(t,y),\quad y(t_0) = y_0$ for some $x$ increment $h$, use the recursive relation

$$y_{n+1} = y_n + h\left.\frac{dy}{dt}\right|_{(t_n,y_n)} = y_n + hf(t_n, y_n) \tag{1}$$

as follows:

1) $y_0 = y(t_0)$
2) Use the above relation with $n = 0$ to find $y_1$.
3) Use the relation with $n = 1$, remembering that $t_1 = t_0 + h$, to find $y_2$.
4) Repeat as long as desired, remembering that $t_n = t_0 + nh$.

There other methods besides Euler's method for generating approximate solutions to IVPs. The general method for applying them is the same as Euler's method, but the iteration formula (1) is different for them. You will see some of those other methods in the exercises.

**Appendix C Exercises** To Solutions

For Exercises 1 and 2 you will be considering the ODE $\dfrac{dy}{dt} = y - t$.

1. In this exercise you will use Euler's method to construct a numerical solution for an initial value of $y(0) = 2$ and a step size of $h = 0.2$

   (a) Build a table with three columns, the first for $n = 0, 1, 2, ...,$ the second for $t_n$, and the third for $y_n$. Complete the second column for $t = 0$ to $t = 1$, and finish the top row by putting in the initial value $y_0 = y(0)$.

   (b) Use the recursion formula $y_{n+1} = y_n + hf(t_n, y_n)$ and the values in the first row of your table to find $y_1$. Put the result in your table. Then continue computing $y_n$ values up to time one. **Do not round any of your answers and use all digits of each $y_n$ to find $y_{n+1}$.**

   (c) Solve the IVP, using the integrating factor method to solve the ODE.

   (d) Add a fourth column to your table for the exact solution values $y(t_0)$, $y(t_1)$, $y(t_2)$, ... Compute those values, rounding to four places past the decimal using your answer to (c), and add them to your table.

(e) The percent error is $percent\ error = \dfrac{100 \times |\,approximate\ value - actual\ value\,|}{|\,actual\ value\,|}$. Find the percent error for each of your approximations, and add them to the table in a fifth column. **Round to the nearest hundredth of a percent**.

2. (a) Find a recursion formula for the given ODE and a step size $h = 0.05$ as follows: For the formula given in 1(b), insert 0.05 for $h$ and apply the function $f(t, y) = y - t$ to $t_n$ and $y_n$. Distribute the 0.05 and combine like terms. Your final result should be something of the form $y_{n+1} = ay_n + bt_n$, for some constants $a$ and $b$.

   (b) Repeat Exercise 1, but for an initial value of $y(0) = 1.4$, a step size of $h = 0.05$, and from $t = 0$ to $t = 0.2$, using your formula from part (a) to find each successive $y_n$. **Round both the $y_n$ and $y(t_n)$ values to four places past the decimal, and the percent error to two places past the decimal.**

3. Consider the IVP $\dfrac{dy}{dt} + ty = 0,\ \ y(0) = 2.$

   (a) Using Euler's method, which is the recursion formula given in Exercise 1(b), find the first four approximate solutions (including $y_0$, which is really exact) to the IVP using a step size of $h = 0.1$. Compile your results in a table, like you have been doing.

   (b) Use separation of variables to solve the IVP. Use your answer to add a column to your table for the exact values $y(t_n)$.

So far you have approximated solutions to an ODE using an iterative method called **Euler's method**. Now you will use two other methods that give better approximations. For *any* first order equation, written in the form $\dfrac{dy}{dt} = f(t, y)$, Euler's method uses the recursion formula given in Exercise 1(b) to find each successive solution approximation from the previous approximation.bEuler's method is the simplest of a variety of methods called **Runge-Kutta methods**. The derivation of these methods is beyond the scope of our course, but we'll use two of them without necessarily understanding their derivations. The first method we'll look at is called the **midpoint method**, which has a fairly straightforward geometric interpretation that we'll go over in class. Here is the formula for the method:

$$y_{n+1} = y_n + hf\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t_n, y_n)\right) \tag{1}$$

4. Let's use the midpoint method to obtain an approximate solution to the initial value problem

$$\frac{dy}{dt} = y - t,\quad y(0) = 2,$$

   with a step size of $h = 0.2$, from $t = 0$ to $t = 1$. As in Exercise 2, you will first derive a recursion formula specific to this problem.

   (a) Simplify the expression $y_n + \dfrac{h}{2}f(t_n, y_n)$ for our particular ODE and step size, in the manner you did for Exercise 2(a).

   (b) Insert your answer from (a) into the appropriate place in (1) and simplify (1) for our ODE. Go until you obtain something of the form $y_{n+1} = ay_n + bt_n + c$ for some constants $a$, $b$ and $c$.

(c) Use your answer to (b) to fill out a table of values like you did for Exercise 1, up to time $t = 1$. As before, include a column at the right for exact values of the solution, rounded to four places past the decimal. (Remember that the exact solution for this IVP is $y = t + 1 + e^t$.)

(d) How do the percent errors for this method compare with those for Euler's method?

The most commonly used Runge-Kutta method uses a set of equations to obtain each successive approximation. These could all be combined into one equation like (1), but it would be very cumbersome. Instead we use

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\,,$$

where

$$k_1 = hf(t_n, y_n), \quad k_2 = hf\left(t_n + \frac{h}{2}, y_n + \frac{1}{2}k_1\right), \quad k_3 = hf\left(t_n + \frac{h}{2}, y_n + \frac{1}{2}k_2\right), \quad k_4 = hf(t_{n+1}, y_n + k_3)$$

The values $k_1$, $k_2$, $k_3$ and $k_4$ must be computed each time you iterate to find a new $y_{n+1}$.

5. Now we'll use the above method to approximate a solution to the same IVP $\dfrac{dy}{dt} = y - t$, $y(0) = 2$.

(a) Find simplified forms of $k_1$ through $k_4$ for our particular ODE, again with a step size of $h = 0.2$.

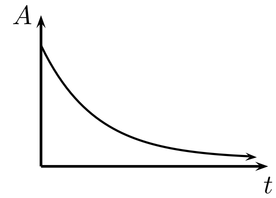
(b) Build a table of values with columns for $n$, $t_n$, $y_n$, $k_1$, ... , $k_4$ and fill it out up through time $t = 1$. As before, round to four places past the decimal when necessary.

(c) Compare your approximate values with the exact values in your table for Exercises 1 and 4.

# D Solutions to Exercises

## D.1 Chapter 1 Solutions

**Section 1.1 Solutions** Back to 1.1 Exercises




1. (a) The independent variable is time, and the dependent variable is the amount of radioactive material.

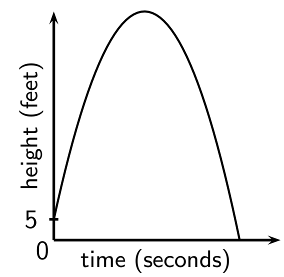
   (b) See graph to the right.

2. (a) The independent variables are time $t$ and the distance $x$ out from the edge of the table, and the dependent variable is the deflection of the ruler at any point and time.

   (b) $0 \le t$, $0 \le x \le 6$, where $x$ is measured in inches. (Substitute $0.5$ for $6$ if measuring in feet instead of inches.)

3. $0 \le r \le 5$ inches, $0 \le \theta \le 2\pi$, $0 \le t$ Note that, unless we are doing right triangle trigonometry, angles will be measured in radians.

4. If measuring in feet, $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.



Exercise 5(a)

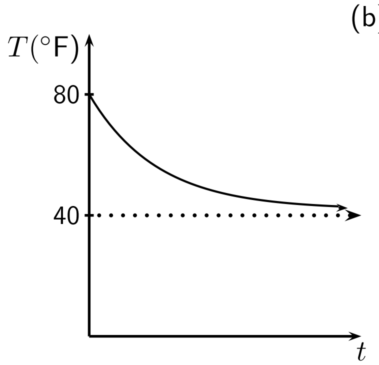
5. (a) See graph to the right.

   (b) The shape is a parabola, opening downward.

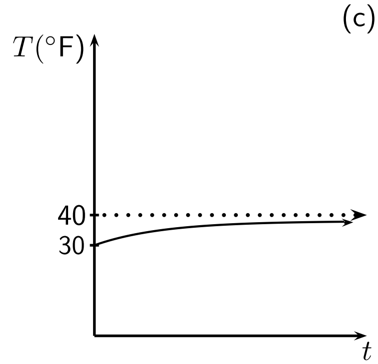
   (c) We would model the height with a quadratic function.

   (d) In this case the domain is finite, of the form $0 \le t \le a$ for the time $a$ when the rock hits the ground, whereas in Exercise 1(b) the domain was infinite, from time zero "to infinity."

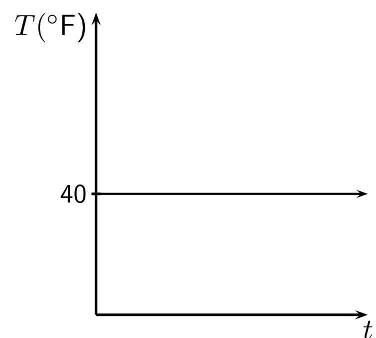
6. (a)



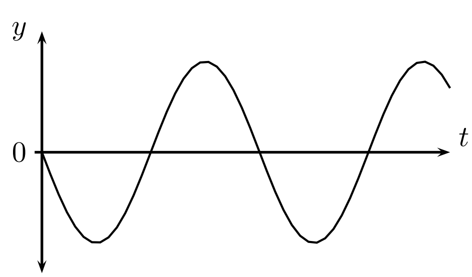

(b)



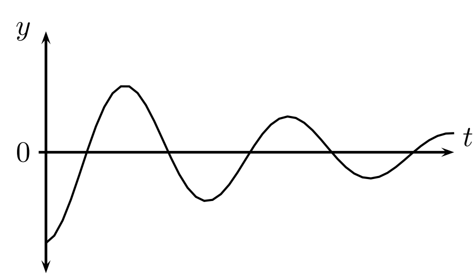

(c)



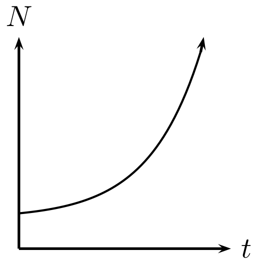

7. (a)

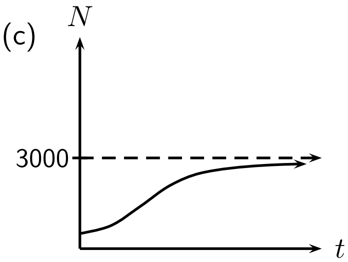


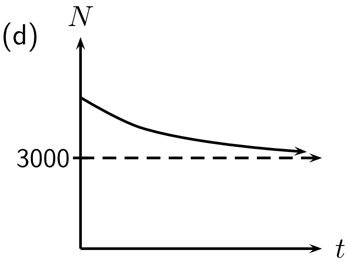

(b)



8. (a) Time is independent, number of individuals is dependent.

(b) 


(c) 


(d) 


9. (c) Changing $A$ affects the amplitude, changing $w$ affects the period. The phase is not affected by changing $A$ or $w$.
   (d) Changing $A$ affects both the amplitude and the phase, and changing $w$ affects only the period. Changing $B$ also affects both the amplitude and phase.

11. (b) $a$ is the value of the horizontal asymptote, and it also affects the $y$-intercept. $b$ affects the $y$-intercept and (sort of) affects the rate at which the function approaches the asymptote. $r$ affects only the rate at which the function approaches the asymptote.

12. (a) $a = 30$, $b = -20$, $r$ cannot be determined.

13. (b) $a = 70$, $b = 90$, $r$ cannot be determined.

14. (a) $t = 0$ at the $y$-intercept, the $y$-intercept is $a + b$.
    (b) The limit of $y$ as $t$ goes to infinity is $a$ , so the graph has a horizontal aymptote of $y = a$.

16. (a) III (b) I (c) II

**Section 1.2 Solutions**

Back to 1.2 Exercises

1. (a) $\dfrac{dy}{dx} = 6\cos 3x$ (b) $y' = -2e^{-0.5t}$ (c) $x' = 2t + 5$
   (d) $y' = -4.42\sin(1.3t - 0.9)$ (e) $\dfrac{dy}{dt} = -3te^{-3t} + e^{-3t}$
   (f) $x' = 12e^{-2t}\cos(3t + 5) - 8e^{-2t}\sin(3t + 5)$

2. (a) $\dfrac{d^2y}{dx^2} = -18\sin 3x$ (b) $y'' = e^{-0.5t}$ (c) $x' = 2$

3. At seven minutes, the temperature is increasing at 2.7°F per minute.

4. At 12.5 minutes, the amount of salt in the tank is decreasing at 1.3 pounds per minute.

5. (a) At 2 seconds the mass is moving downward at 5 inches per second.
   (b) At 2 seconds the mass is accelerating *upward* at 3 inches per second per second (in/sec$^2$).
   (c) At 2 seconds the mass is slowing down because the acceleration is in the direction opposite the velocity.

6. At 5.4 hours, the number of bacteria in the dish is increasing at a rate of 430 bacteria per hour.

7. (a) For $x > 0$ the derivative will be positive, because the deflection increases as $x$ increases.
   (b) The absolute value of the derivative at $x_2$ will be greater than the absolute value of the derivative at $x_1$.

8. (a) $y = e^{-3x}$, $y = Ce^{-3x}$ for any constant $C$.
   (b) $y = \sin 3t$ or $y = \cos 3t$. Any function of the form $y = C_1 \sin 3t + C_2 \cos 3t$ will do it, for any constants $C_1$ and $C_2$.
   (c) $y = e^{3t}$, $y = Ce^{3t}$ for any constant $C$.
   (d) $y = \sin \sqrt{5}\,x$ or $y = \cos \sqrt{5}\,x$. Any function $y = C_1 \sin \sqrt{5}\,x + C_2 \cos \sqrt{5}\,x$ will do it, for any constants $C_1$ and $C_2$.

9. (a) $\dfrac{dy}{dx} = -3y$ (b) $\dfrac{d^2y}{dt^2} = -9y$ (c) $\dfrac{d^2x}{dt^2} = 9x$ (d) $\dfrac{d^2y}{dx^2} = -5y$

## Section 1.3 Solutions

Back to 1.3 Exercises

1. (a) The parameters are the the initial temperature $T_0$, the temperature $T_m$ of the medium, and the constant $k$.
   (b) The independent variable is time $t$.
   (c) The dependent variable is temperature $T$.

2. The parameters are the amount of the mass, the stiffness of the spring (the spring constant), the amount the mass is pulled downward before letting it go, and the viscosity of the oil in the oil bath. The shape of the mass is probably a parameter as well.

3. The independent variable is time $t$ and the dependent variable is current $i$. The parameters are the inductance $L$, the resistance $R$, the voltage $E$ and the initial current $i_0$.

4. (a) The independent variables are time and the distance along the string.
   (b) Some parameters would be physical properties of the material the string is made of, the thickness and cross-section of the string (which is probably circular, but could be different) and the tension in the string.

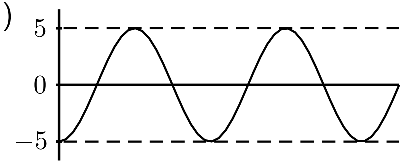
5. (a) The independent variable is time, and the dependent variable is the amount of salt in the tank.
   (b) The parameters are the rate at which fluid is entering and leaving the tank, and the concentration of the incoming fluid. We might think that the amount of salt in the tank to begin with is a parameter, but it is instead what we call an **initial value**. It is a value of the dependent variable when the independent variable is zero.

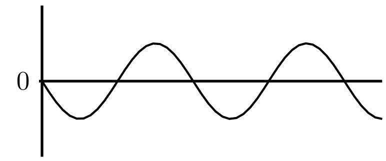
## Section 1.4 Solutions

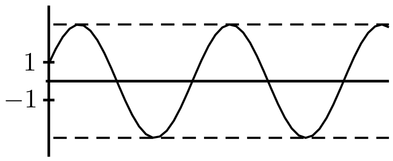
Back to 1.4 Exercises

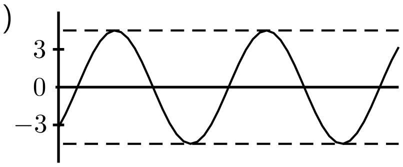
1. (a) $y = 7$ when $x = 0$, $y' = -3$ when $x = 0$
   (b) $x = 1$ when $t = 0$, $x' = 5$ when $t = 0$
   (c) $y = 0$ when $x = 0$, $y'' = 0$ when $x = 0$, $y = 0$ when $x = 15$, $y' = 0$ when $x = 0$

2. (a) $y(0) = -5,\ y'(0) = 0$ (b) $y(0) = 0,\ y'(0) = -2$
   (c) $y(0) = 1,\ y'(0) = 2$ (d) $y(0) = -3,\ y'(0) = 1$

3. (a)



(b)



(c)



(d)



4. (a) $y(0) = y(20) = 0,\ y''(0) = 0,\ y'(20) = 0$ (b) $y(0) = y(12) = 0,\ y''(0) = y''(12) = 0$

5. $u(0) = 32, \quad u(70) = 115$

## Section 1.5 Solutions

Back to 1.5 Exercises

1. (a) Independent variable: $x$ Dependent variable: $y$
   (b) Independent variable: can't determine Dependent variable: $y$
   (c) Independent variables: $x_1$, $x_2$, $x_3$, $t$ Dependent variable: $u$
   (d) Independent variable: $t$ Dependent variable: $y$
   (e) Independent variable: $x$, $t$ Dependent variable: $u$
   (f) Independent variable: $x$ Dependent variable: $u$
   (g) Independent variable: $x$ Dependent variable: $y$
   (h) Independent variable: $x$ Dependent variable: $y$
   (i) Independent variable: $r$, $t$ Dependent variable: $u$

2. (a) Independent variable: $x$ Dependent variable: $y$
   (b) Independent variable: $t$ Dependent variable: $y$

3. no 4. yes 7. $A = \frac{36}{65}, \ B = \frac{28}{65}$.

9. (a) no (b) $y = ce^{3x}$ is only a solution if $c = 2$ (c) $y = 2e^{3x}$
   (d) $y = ce^x$ is a solution for any value of $c$

## Section 1.6 Solutions

Back to 1.6 Exercises

1. (a), (b), (d), (f), (g), (h)

2. (a), (c), (e) and (i) are first order. (b), (d), (f) and (g) are second order, and (h) is fourth order.

3. (a) $F(x, y) = 2y$ (c) $F(x, y) = y - y^2$ (e) $F(x, y) = x - xy$ (i) $F(x, y) = 1 - xy$

4. (a) $f(x) = 0,\ a_0(x) = -2,\ a_1(x) = 1$
   (b) $f(x) = 0,\ a_0(x) = -1,\ a_1(x) = 0,\ a_2(x) = 1$

(c) not linear

(d) $f(t) = 26e^{-2t},\ a_0(t) = 9,\ a_1(t) = 0, a_2(t) = 1$

(e) $f(x) = 1,\ a_0(x) = 1,\ a_1(x) = \frac{1}{x}$

(f) $f(t) = 10\sin t,\ a_0(t) = 6,\ a_1(t) = -5,\ a_2(t) = 1$

(g) not linear

(h) $f(x) = w,\ a_0(x) = a_1(x) = a_2(x) = a_3(x) = 0,\ a_4(x) = 1$

(i) $f(x) = 1,\ a_0(x) = x,\ a_1(x) = 1$

5. (a) $g(x) = 1,\ h(y) = 2y$

   (c) $g(x) = 1,\ h(y) = y - y^2$

   (e) $g(x) = x,\ h(y) = 1 - y$

   (i) not separable

6. (a) and (c)

**Section 1.7 Solutions** Back to 1.7 Exercises

2. Yes, it is a solution.

3. (a) Not a solution, $y'(0) \neq 1$ and $y$ is not a solution to the ODE.

   (b) Not a solution, $y$ is not a solution to the ODE.

   (c) Solution.

   (d) Solution.

4. (a) $y = 2e^{-2t} + 3\cos 2t$

   (b) $x = -\frac{14}{3}e^{-t} + \frac{5}{3}e^{-4t} + 3t + 1$

   (c) $y = \frac{2}{3\sqrt{5}}\sin\sqrt{5}t - 3\cos\sqrt{5}t$

   (d) $y = -\frac{1}{2}\sin 2t + 6\cos 2t + e^{-3t}$

5. (a) $C_1 = 2,\ C_2 = -4$

   (b) $C_1 = 6,\ C_2 = 5$

6. (a) The function is not a solution to the BVP, it doesn't satisfy the boundary condition $y'(5) = 0$.
   (b) The function is a solution to the BVP.
   (c) The function is a solution to the BVP.
   (d) The function is not a solution to the BVP, it doesn't satisfy the ODE.
   (e) The function is a solution to the BVP.
   (f) The function is not a solution to the BVP, it doesn't satisfy the boundary condition $y(0) = 0$.

## D.2 Chapter 2 Solutions

**Section 2.1 Solutions** Back to 2.1 Exercises

1. (a) $\sin y = -\frac{1}{2}x^2 + C$
   (b) $\dfrac{1}{2}y^2 = \dfrac{1}{2x^2} + C$ or $y^2 = \dfrac{1}{x^2} + C$
   (c) $\frac{1}{5}y^5 = -\frac{1}{3}x^3 + C$ or $3y^5 = -5x^3 + C$
   (d) $\ln|y| = \frac{1}{2}\ln|2x+3| + C$
   (e) $-e^{-y} = -\frac{1}{x} + C$ or $e^{-y} = \frac{1}{x} + C$
   (f) $\frac{1}{2}y^2 = \frac{5}{2}x^2 + 3x + C$ or $y^2 = 5x^2 + 6x + C$

2. (a) $\ln|y| = \frac{1}{2}x^2 + \ln 3 - \frac{1}{2}$
   (b) $\frac{1}{2}x^2 = -\frac{5}{2}t^2 + 3t + 12$ or $x^2 = -5t^2 + 6t + 24$
   (c) $e^y = \frac{3}{2}x^2 + e^2$
   (d) $-\dfrac{1}{3y^3} = \sin t - \dfrac{1}{24}$

3. (a) $y = 4e^{3x}$
   (b) not separable
   (c) $\ln|y| = 2x^2 + \ln 2$
   (d) $y = \frac{7}{e^2}e^{\frac{1}{2}x^2} - 2$
   (e) not separable
   (f) $y = \frac{1}{2}x + \frac{5}{2}$

4. (a) $y = \ln(x^2 + x + 1)$
   (b) $y = \ln(x^2 + x + e^3 - 2)$

5. (b) $y = Ce^{-t^2}$
   (c) $y = 7e^{-t^2}$

6. The final solution is $v = \dfrac{C}{x}$.

7. (a) $y = \ln|x(x+3)| + C$
   (b) $y = \ln\left|\dfrac{x}{x+3}\right| + C$
   (c) $y = \ln\left(\dfrac{x+2}{x-5}\right)^2 + C$

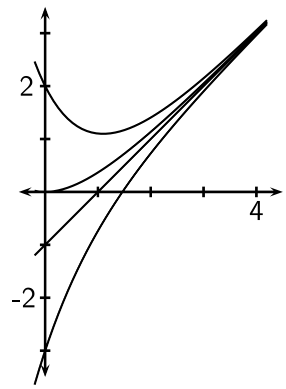
8. (a) $\dfrac{|y|}{|y-3|} = x + C$ or $\left|\dfrac{y}{y-3}\right| = x + C$
   (b) $\dfrac{y}{y-3} = Ce^x$
   (c) $y = \dfrac{3C}{C - e^{-x}}$
   (d) In order, the constants are $C = \frac{1}{7},\ 0,\ -\frac{1}{2},\ 4$ and the solutions are $y = \dfrac{3}{1 - 7e^{-x}}$, $y = 0$, $y = \dfrac{3}{1 + 2e^{-x}}$, $y = \dfrac{12}{4 - e^{-x}}$
   (f) $y \to 3$ as $x \to \infty$ in all cases
   (g) The value of the constant cannot be determined - the equation to be solved has no solution.

## Section 2.2 Solutions

Back to 2.2 Exercises

1. Constants for each initial value are given below, graph is to the right.

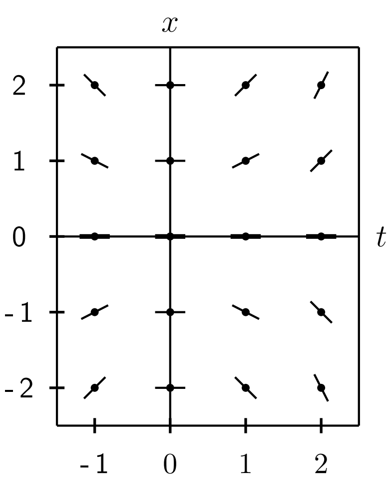
   (a) $C = 3$
   (b) $C = 1$
   (c) $C = 0$
   (d) $C = -2$



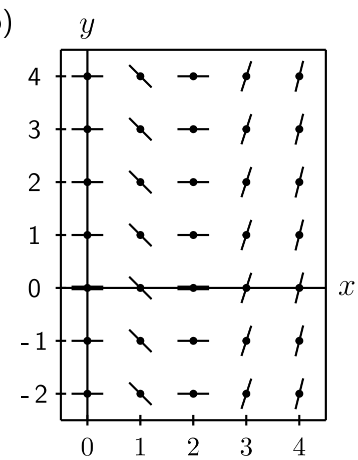

3. (a) II (b) I (c) III (d) IV

5. (a) The top, U-shaped curve is for initial value $y(0) = 1$. The next curve down has initial value $y(0) = -1$, and the one below that has initial value $y(0) = -2$.
   (b) Using the point $(2, -1)$, the value obtained for $C$ is $-2$. If instead one uses the point $(-2, -1)$, the same value of $C$ is obtained!
   (c) When the solution is graphed for $C = -2$, the graph includes the U-shaped curve as well as the two curves in the lower left and lower right. They are all parts of the same graph, which has vertical asymptotes at $x = -\sqrt{2}$ and $x = \sqrt{2}$.

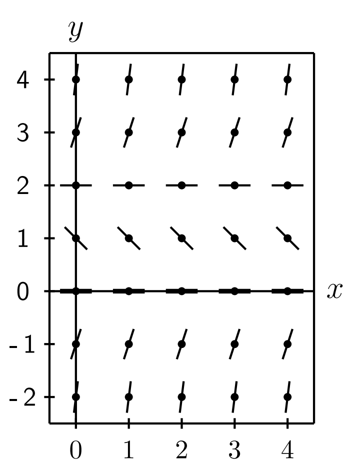
8. (a)



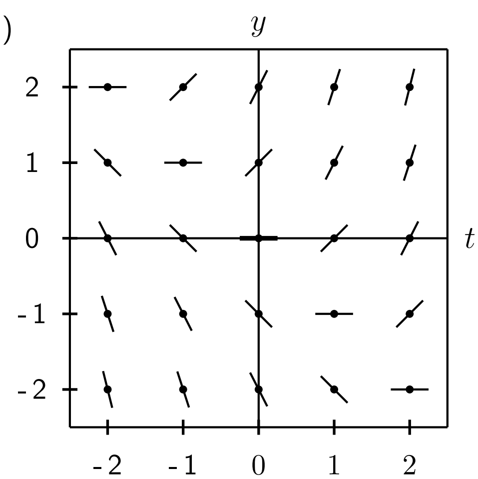

(b)



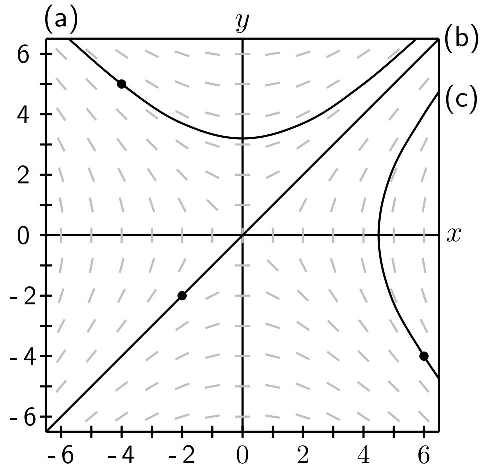

(c)



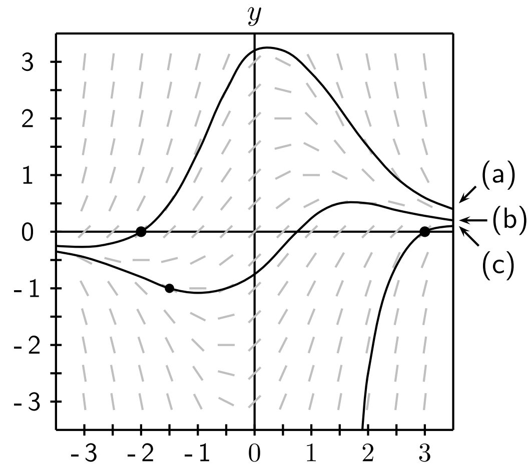

(d)



9.




10.




## Section 2.3 Solutions

Back to 2.3 Exercises

1. $y=-\frac{3}{2}e^{3x}+\frac{1}{2}e^{5x}$

2. (a) $y=0.4te^{-2t}+Ce^{-2t}$ (b) $y=0.4te^{-2t}+3e^{-2t}$

3. (a) and (b) $y=Ce^{\frac{1}{2}x}$ (c) $y=\frac{3}{2}e^{\frac{1}{2}x}$

4. (a) $y=\frac{6}{29}\sin 2t-\frac{15}{29}\cos 2t+Ce^{5t}$ (b) $y=\frac{6}{29}\sin 2t-\frac{15}{29}\cos 2t-\frac{101}{29}e^{5t}$

5. (a) $y=Ce^{-3t}+\frac{1}{3}t^2+\frac{13}{9}t-\frac{22}{27}$ (b) $y=\frac{76}{27}e^{-3t}+\frac{1}{3}t^2+\frac{13}{9}t-\frac{22}{27}$

6. (a) $y=\frac{1}{2}e^{3x}+\frac{7}{2}e^{x}$ (b) $y=x\ln x+2x$

7. $y=\frac{7}{e^2}e^{\frac{1}{2}x^2}-2$

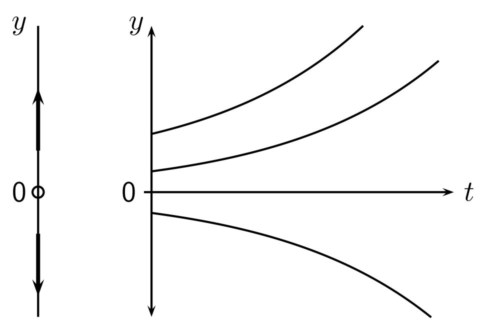
8. (a) $r=5$ (b) $y=\frac{6}{29}\sin 2t-\frac{15}{29}\cos 2t$ (c) $y=\frac{6}{29}\sin 2t-\frac{15}{29}\cos 2t+Ce^{5t}$

## Section 2.4 Solutions

Back to 2.4 Exercises

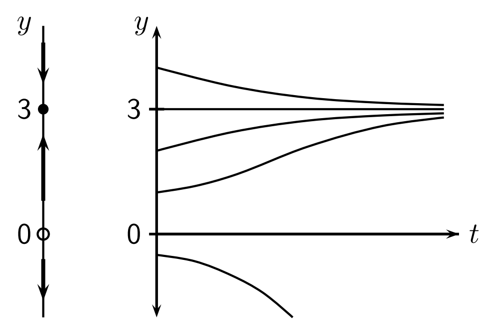
1. The ODEs in parts (a), (c), (d) and (f) are autonomous.

2. (a)



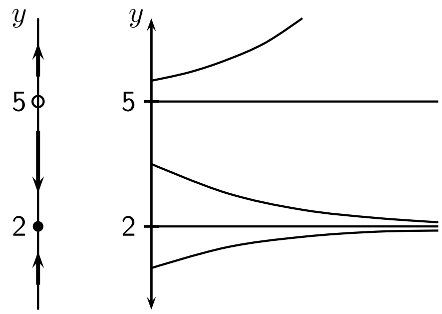

$y=0$ is an unstable equilibrium

(c)



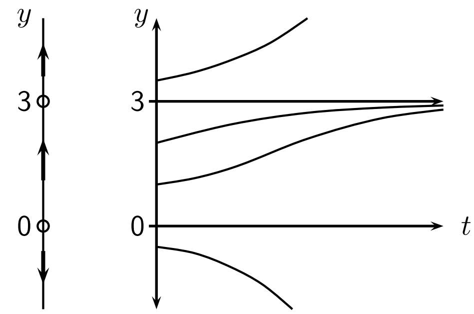

$y=0$ is an unstable equilibrium
$y=3$ is a stable equilibrium

(d)




$y = 2$ is a stable equilibrium
$y = 5$ is an unstable equilibrium

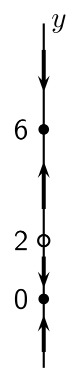
(f)



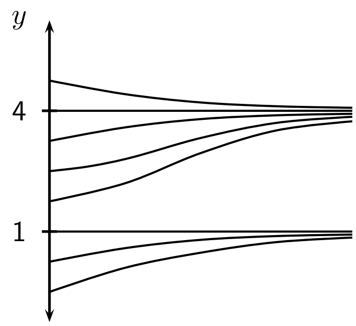

$y = 0$ is an unstable equilibrium
$y = 3$ is a semi-stable equilibrium

3. (a) $y = 0$ is a stable equilibrium, $y = 2$ is an unstable equilibrium, $y = 6$ is a stable equilibrium
   (b) $y = 1$ is a semi-stable equilibrium, $y = 4$ is a stable equilibrium
   (c) $y = 0$ is an unstable equilibrium, $y = 2$ is a stable equilibrium, $y = 5$ is an unstable equilibrium

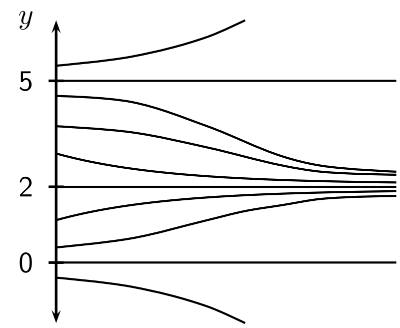
4. (a)




(b)

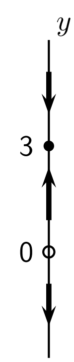


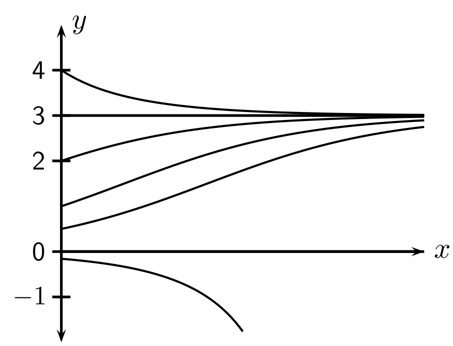

(c)




5.
   (a) $\dfrac{dy}{dt} = -y(y-2)(y-6)$
   (b) $\dfrac{dy}{dt} = -(y-4)(y-1)^2$
   (c) $\dfrac{dy}{dt} = y(y-2)(y-5)$

6. (c)




6. We can factor the right side of the ODE to get $\dfrac{dy}{dx} = -\frac{1}{3}y(y-3)$.
   (a) $y = 3$ is a critical value, so there is an equilibrium solution of $y = 3$.
   (b) $y = 3$ is a stable equilibrium solution, $y = 0$ is an unstable equilibrium solution.
   (c) See above and to the right.

**Section 2.5 Solutions** Back to 2.5 Exercises

1. (a) $A(0) = 80$ (b) $A = 0$ is a stable equilibrium solution

   (c) $k = -0.104, \quad A = 80e^{-0.104t}$

2. (a) $\dfrac{dA}{dt} = -\dfrac{3}{10}A, \quad A(0) = 87$ (b) $A = 87e^{-\frac{3}{10}t}$ (c) $t = 9.52$ hours

   (d) The transient solution is $87e^{-\frac{3}{10}t}$ and the steady state solution is zero. (Saying there is no steady-state solution could be considered correct as well.)

3. (a) $T = 70 + Ce^{-kt}$ (b) $C = -38$, so $T = 70 - 38e^{-kt}$

   (c) $k = \ln \frac{19}{6} \approx 1.15$ (d) Steady-state: 70 Transient: $-38e^{-\ln \frac{19}{6} t} \approx -38e^{-1.15t}$

4. (a) $i = \frac{6}{5} - \frac{6}{5}e^{-20t}$ or $i = \frac{6}{5}(1 - e^{-20t})$

   (b) $i = \frac{5}{101}(20 \sin 2t - 2 \cos 2t) - \frac{10}{101}e^{-20t}$ or $i = \frac{100}{101} \sin 2t - \frac{10}{101} \cos 2t - \frac{10}{101}e^{-20t}$

   (c) The transient part is $-\frac{10}{101}e^{-20t}$ and the steady-state part is $\frac{5}{101}(20 \sin 2t - 2 \cos 2t)$ or $\frac{101}{101} \sin 2t - \frac{10}{101} \cos 2t$.

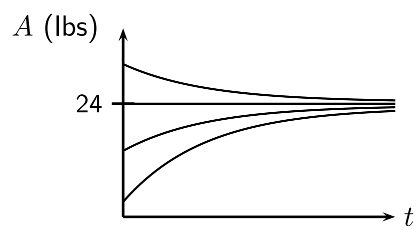
5. (a) If $T_0 = T_m$ the temperature will not change because the initial temperature of the object will be the same as the temperature of the medium; the solution will be completely steady-state.

   (b) The steady-state solution is $T_m$.

   (c) The transient solution is $(T_0 - T_m)e^{-kt}$

6. (a) $A = 24$ is a stable equilibrium solution

   (b) See to the right.




7. (a) $\dfrac{dT}{dt} = -k(T - 73), \quad T(0) = 29$

   (b) There will be a steady state solution of $T = 73$.

   (c) $T = 73 - 44e^{-kt}$ (d) $T = 73 - 44e^{-0.263t}$

8. (a) Initial value problem: $\frac{3}{4}\dfrac{di}{dt} + 15i = 6 \cos 2t, \quad i(0) = 2$

   Solution: $i = \frac{2}{101}(20 \cos 2t + 2 \sin 2t) + \frac{162}{101}e^{-20t}$

   (b) Steady-state: $\frac{2}{101}(20 \cos 2t + 2 \sin 2t)$ Transient: $\frac{162}{101}e^{-20t}$

9. (b) Initial value problem: $\dfrac{dA}{dt} = 7 - \dfrac{7A}{150}, \quad A(0) = 450$ Solution: $A = 150 + 300e^{-\frac{7}{150}t}$

   (a) Steady-state: 150 Transient: $300e^{-\frac{7}{150}t}$

10. (a) $i = 5e^{-54t}$

    (b) The transient part of the solution is $5e^{-54t}$ and there is no steady-state part.

## D.3 Chapter 3 Solutions

**Section 3.1 Solutions** Back to 3.1 Exercises

1. (a) $y = C_1x + C_2x^4$ (b) $y = C_1x^{-1} + C_2x^{-2}$ (c) $y = C_1x + C_2x^{\frac{1}{3}}$

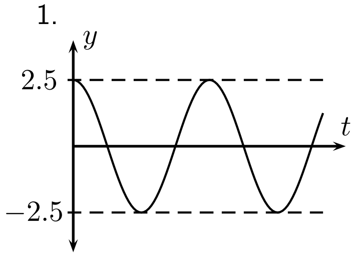
2. (a) $y = C_1e^{3t} + C_2e^{-t}$ (b) $y = e^{-t}(C_1 \sin 3t + C_2 \cos 3t)$
   (c) $y = C_1e^{-5t} + C_2te^{-5t}$ (d) $y = e^{-3t}(C_1 \sin 2\sqrt{2}\,t + C_2 \cos 2\sqrt{2}\,t)$
   (e) $y = C_1e^{-t} + C_2e^{-2t}$ (f) $y = C_1 \sin \sqrt{2}\,t + C_2 \cos \sqrt{2}\,t$
   (g) $y = C_1e^{-t} + C_2te^{-t}$ (h) $y = C_1 \sin 4t + C_2 \cos 4t$
   (i) $y = e^{-1.55t}(C_1 \sin 1.45t + C_2 \cos 1.45t)$

3. (a) $r^2 + 25 = 0$ and $r^2 + 25r = 0$ (b) $y = C_1 + C_2e^{-25t}$

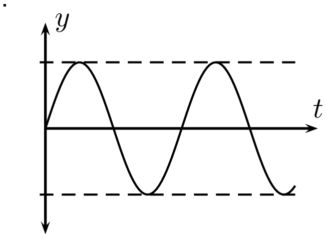
4. (a) $y = C_1 \sin \lambda t + C_2 \cos \lambda t$ (b) $y = C_1e^{r_1t} + C_2e^{r_2t}$
   (c) $y = e^{kt}(C_1 \sin \lambda t + C_2 \cos \lambda t)$ (d) $y = C_1e^{rt} + C_2te^{rt}$

5. (a) $y = Cx^2$

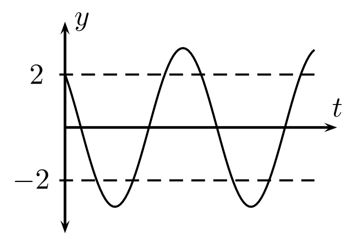
**Section 3.2 Solutions** Back to 3.2 Exercises

1.



$y(0) = 2.5,\ y'(0) = 0$

2.



$y(0) = 0,\ y'(0) = 3$

3.



$y(0) = 2,\ y'(0) = -8$

4. (a) $\frac{3}{4}y'' + 15y = 0, \quad y(0) = 2.5,\ y'(0) = 0$
   (b) $y = 2.5 \cos 4.47\ t$ (d) $y = 2.5 \sin(4.47\ t + 1.57)$
   (e) amplitude is 2.5, angular frequency is 4.47, period is 1.41, frequency is 0.71, phase shift is -0.35

5. (a) $y = -1.79 \sin(4.47\,t) + 2.00 \cos(4.47\,t)$
   (c) $y = 2.68 \sin(4.47\,t + 2.30)$
   (d) amplitude is 2.68, angular frequency is 4.47, period is 1.41, frequency is 0.71, phase shift is -0.51

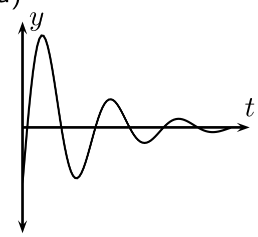
6. (a) $\frac{4}{10}y'' + 4y = 0, \quad y(0) = -4,\ y'(0) = 9$
   (b) $y = 2.85 \sin 3.16\ t - 4.00 \cos 3.16\ t \quad \Longrightarrow \quad y = 4.91 \sin(3.16\ t - 0.95)$
   (c) amplitude 4.91, angular frequency 3.16, period 1.99, frequency 0.50, phase shift 0.30

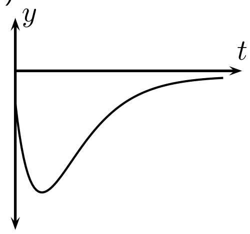
**Section 3.3 Solutions** Back to 3.3 Exercises

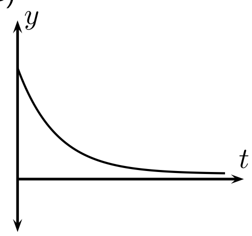
1. (a) $y(t) = e^{-0.6t}(-1.2 \sin 4.0t + 2.0 \cos 4.0t) \quad \Rightarrow \quad y(t) = 2.3e^{-0.6t} \sin(4.0t + 4.2)$

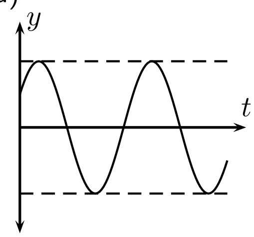
2. (a) $y(t) = \frac{11}{3}e^{-2t} - \frac{5}{3}e^{-8t}$

3. (a) $\beta = 40$ (b) $y(t) = 2e^{-4t} + 14te^{-4t}$

5. (a)


(b)


(c)


(d)



6. (a) The system is underdamped because $R^2 - 4L \cdot \frac{1}{C} < 0$.
   (b) $q(t) = e^{-1670t}(2.18 \times 10^{-5} \sin 1530t + 2.00 \times 10^{-5} \cos 1530t)$ coulombs
   (c) $q(t) = 2.96 \times 10^{-5} e^{-1670t} \sin(1530t + 0.742)$ coulombs
   (d) $i(t) = \dfrac{dq}{dt} = -4.94 \times 10^{-2} \cos(1530t + 0.742)$ amperes

**Section 3.4 Solutions** Back to 3.4 Exercises

1. (a) $y_p = At + B$ (b) $y_p = A \sin 3t + B \cos 3t$ (c) $y_p = Ae^{5t}$
   (d) $y_p = A \sin t + B \cos t$ (e) $y_p = A$ (f) $y_p = At + B + Ce^{-t}$
2. (a) $y_p = \frac{5}{2}t - \frac{17}{4}$ (b) $y_p = \frac{12}{169} \sin 2t + \frac{5}{169} \cos 2t$ (c) $y_p = \frac{1}{17}e^{5t}$
   (d) $y_p = -\frac{9}{13} \sin t + \frac{6}{13} \cos t$ (e) $y_p = 7$ (f) $y_p = \frac{2}{3}t + \frac{3}{4}e^{-t}$
5. (a) $y = e^{-2t}(C_1 \sin 5\,t + C_2 \cos 5\,t)$ (b) $y = C_1 e^{-5t} + C_2 e^{-\frac{1}{2}t}$
   (c) $y = C_1 e^{-3t} + C_2 t e^{-3t}$ (d) $y = C_1 \sin \sqrt{3}\,t + C_2 \cos \sqrt{3}\,t$

**Section 3.5 Solutions** Back to 3.5 Exercises

1. (a) $D(y) = 2t^2 + 20t + 23$ (b) $D(y) = 0$
   (c) $D(y) = -10 \cos 2t - 30 \sin 2t$ (d) $D(y) = 5t - 1$
2. $y = \frac{5}{2}t - \frac{17}{4}$ is a particular solution to the ODE $y'' + 3y' + 2y = 5t - 1$.
3. (a) $D(Ce^{-2t}) = 0$ for all values of $C$.
   (b) $D(e^{kt}) = 0$ only when $k = -2$ or $k = -1$.
4. $D(y) = 0$ in all three cases.
5. (a) $S(\cos t) = \cos t + 3$, $S(t^2 + 5t - 1) = t^2 + 5t + 2$
   (b) $S(4 \cos t) = 4 \cos t + 3$, $4S(\cos t) = 4(\cos t + 3) = 4 \cos t + 12$ These are not the same, so $S$ is not linear.
   (c) $S(\cos t + e^{2t}) = \cos t + e^{2t} + 3$, $S(\cos t) + S(e^{2t}) = \cos t + 3 + e^{2t} + 3 = \cos t + e^{2t} + 6$

**Section 3.6 Solutions** Back to 3.6 Exercises

1. (a) $y = C_1e^{-2t} + C_2e^{-t} + \frac{5}{2}t - \frac{17}{4}$
   (b) $y_h = C_1e^{-3t} + C_2te^{-3t} + \frac{5}{18}\sin 3t$
   (c) $y_h = C_1\sin 3t + C_2\cos 3t + \frac{1}{17}e^{5t}$
   (d) $y_h = C_1e^{-t} + C_2e^{5t} - \frac{9}{13}\sin t + \frac{6}{13}\cos t$
   (e) $y_h = C_1e^{-\frac{1}{2}t} + C_2e^{-t} + 7$
   (f) $y_h = C_1\sin\sqrt{3}t + C_2\cos\sqrt{3}t + \frac{2}{3}t + \frac{3}{4}e^{-t}$

2. (a) $y = \frac{7}{6}\sin 3t + 2\cos 3t + \frac{1}{2}\sin t$
   (b) $y = -\frac{1}{5}e^{-2t} - te^{-2t} + \frac{1}{5}e^{3t}$
   (c) $y = \frac{7}{5}e^{5t} - \frac{1}{5}e^{5t} + \frac{6}{5}t + \frac{3}{5}$

3. (a) $y = C_1x^{-2} + C_2x^3$
   (b) $y = C_1x^{\frac{1}{2}} + C_2x^{-\frac{1}{2}}$

4. (a) $y = e^{-t}(1.4\sin 3t + 5.2\cos 3t) + \sin t - 0.2\cos t$
   (c) Transient part: $e^{-t}(1.4\sin 3t + 5.2\cos 3t)$ Steady-state part: $\sin t - 0.2\cos t$

## D.4 Chapter 4 Solutions

**Section 4.1 Solutions** Back to 4.1 Exercises

1. (a) linearly independent (b) $c_1 = 1,\ c_2 = -2$
   (c) $c_1 = 2,\ c_3 = 3$ (d) linearly independent

2. (b) $c_1 = 1,\ c_2 = -1$ (c) $c_1 = 1,\ c_3 = 1$

3. (a) $W(x) = -6x^2 - 6x - 10$ Any value of $x$ will give $W(x) \neq 0.$
   (b) $W(x) = -9e^{7x}$ Any value of $x$ will give $W(x) \neq 0.$

4. $W(t) = e^{2kt}$ Any value of $t$ will give $W(t) \neq 0.$

5. $W(t) = -2$ Any value of $t$ will give $W(t) = -2 \neq 0.$

**Section 4.2 Solutions** Back to 4.2 Exercises

3. $y_2 = x^{\frac{3}{2}}$ 4. $y_2 = \dfrac{1}{x^2}$

**Section 4.3 Solutions** Back to 4.3 Exercises

1. (a) $y_p = -\frac{5}{6}t - \frac{13}{36}$ (b) $y_p = \frac{35}{986}\sin 5t - \frac{217}{986}\cos 5t$ (c) $y_p = \frac{4}{5}te^{2t}$

3. (a) $y = -\frac{1}{5}e^{-2t} - te^{-2t} + \frac{1}{5}e^{3t}$ (b) $y = -e^{-2t} + 3e^{-5t} + 2te^{-2t}$
   (c) $y = \frac{13}{8}\sin 2t + \frac{1}{2}\cos 2t - \frac{3}{4}t\cos 2t$

4. (a) Steady-state: $-\frac{2}{3}\sin 3t + \frac{5}{3}\cos 3t$ Transient: none
   (b) Steady-state: $\frac{3}{4}\cos 7t$ Transient: $e^{-3t}(4\sin t + 7\cos t)$
   (c) Steady-state: $\frac{3}{5}\sin 5t - \frac{6}{5}\cos 5t$ Transient: $\frac{7}{2}e^{-2t}$
   (d) Steady-state: none Transient: $3te^{-5t} - 7e^{-5t} + e^{-t}$

5. (a) solution (b) solution (c) solution (d) not a solution
   (e) not a solution (f) solution (g) solution (h) not a solution

**Section 4.4 Solutions** Back to 4.4 Exercises

1. (a) $x(t) = 0.4\cos 2.2t - 0.4\cos 5t$
   (c) We should expect no transient solution. There is no damping, so the homogeneous solution is periodic, hence steady-state. Because the forcing function is periodic with different frequency than the homogeneous solution, it results in a periodic particular solution, so the general solution is then periodic, so steady-state.

2. (a) $x(t) = 1.8t\sin 2.2t$
   (c) What causes the resonance is that the frequency of the forcing function $f(t) = 8\cos 2.2t$ is the same as the natural frequency of the system, which is seen in the homogeneous solution.

3. (a) $x(t) = 9.5\cos 2t - 9.5\cos 2.2t$

   (c) The beats are being caused by the fact that the frequency of the forcing function is close to, but not the same as, the natural frequency of the system.

   (d) $x(t) = 19\sin(0.1t)\sin(2.1t)$ The first sine function (and the factor of 19) acts as a sort of "variable amplitude" for the higher frequency second sine function.

**Chapter 4 Exercises Solutions** Back to Chapter 4 Exercises

1. (a) Undamped: ii, iv, v, viii Under-damped: vi, vii

   Critically damped: iii Over-damped: i

   (b) Transient: i, iii, vi, vii Steady-state: ii, iv, v, vi, viii

   (c) Resonance: iv Beats: viii

2. (a) (i) entire solution is transient (ii) entire solution is transient

   (iii) entire solution is steady-state (iv) entire solution is transient

   (v) Transient: $e^{-0.4t}[C_1\cos 2t + C_2\cos 2t]$ Steady-state: $-1.3\cos 7t$

   (vi) entire solution is steady-state (vii) Steady-state: $C_1\cos 3t + c_2\sin 3t$

   (viii) entire solution is steady-state

   (b) i, ii, iii, iv

   (c) Undamped: iii, vi, vii, viii Under-damped: iv, v

   Critically damped: ii Over-damped: i

3. (a) Undamped: ii, iii, v, vii Under-damped: iv, vi Critically or over-damped: i, vi

   (b) i, vi

   (c)

| Equation | Solution | Graph |
|---|---|---|
| i | i | i |
| ii | vi | ii |
| iii | ii | i |

| Equation | Solution | Graph |
|---|---|---|
| iv | vii | vii |
| v | iii | iii |
| vi | v | iv |
| vii | iv | vi |
| viii | viii | v |

## D.5 Chapter 5 Solutions

**Section 5.1 Solutions** Back to 5.1 Exercises

1. (a) $y(0) = 0,\ y''(0) = 0,\ y(12) = 0, y'(12) = 0$
   (b) $y(0) = 0,\ y'(0) = 0,\ y''(8) = 0, y'''(8) = 0$
   (c) $y(0) = 0,\ y'(0) = 0,\ y(20) = 0,\ y''(20) = 0$
   (d) $y(0) = 0,\ y''(0) = 0,\ y(15) = 0, y''(15) = 0$

2. Only (a) is possible, $y(0) = 0,\ y'(0) = 0,\ y(8) = 0, y''(8) = 0$

3. (a) $(30)(80)\dfrac{d^4y}{dx^4} = 150, \qquad y(0) = y'(0) = y(8) = y'(8) = 0$
   (b) $y = \frac{1}{384}x^4 - \frac{1}{24}x^3 + \frac{1}{6}x^2$
   (d) The maximum deflection should appear at the middle of the beam $(x = 4)$. The deflection there is $\frac{2}{3}$.

4. (b) $y = \frac{1}{12}x^4 - \frac{10}{3}x^3 + 50x^2$
   (d) The maximum deflection is $2500$ at $x = 10$, the right hand end of the beam.

5. (b) $y = \frac{1}{12}x^4 - \frac{5}{3}x^3 + \frac{250}{3}x$
   (d) The maximum deflection is $\frac{3125}{12}$ at $x = 5$

6. (a) $(30)(80)\dfrac{d^4y}{dx^4} = 150, \qquad y(0) = y'(0) = y(8) = y'' = 0$
   (b) $y = \frac{1}{384}x^4 - \frac{5}{96}x^3 + \frac{1}{4}x^2$
   (d) The maximum deflection is about 1.39 at $x = 4.6$

7. (a) $J$ and $L$ (b) $E$ and $H$ (c) $F$ and $G$

**Section 5.2 Solutions** Back to 5.2 Exercises

1. (a) The eigenvalue is $\lambda = 3$. (b) $y = e^{-5x}$
   (c) Eigenfunction: $y = e^{kx}$ Eigenvalue: $k$

2. (a) $y = 3x,\ y = 5,\ y = 2x - 1$, etc. (b) $y = Ax + B$
   (c) $y = \sin 2x,\ y = \cos 2x$ (d) $y = \sin\sqrt{3}x,\ y = \cos\sqrt{3}x$

3. (a) $D(e^{-2t}) = 4e^{-2t} - 4e^{-2t} - 3e^{-2t} = -3e^{-2t}$, the eigenvalue is $-3$
   (b) $D(e^{kt}) = k^2e^{kt} + 2ke^{kt} - 3e^{kt} = (k^2 + 2k - 3)e^{kt}$, the eigenvalue is $k^2 + 2k - 3$
   (c) $k = -3, 1$ (d) $k = -4, 2$

**Section 5.3 Solutions** Back to 5.3 Exercises

1. (a) $\lambda = \frac{\pi}{5},\ \frac{2\pi}{5},\ \frac{3\pi}{5},\ \frac{4\pi}{5},...,\qquad y = C\sin\frac{\pi}{5}x,\ C\sin\frac{2\pi}{5}x,\ C\sin\frac{3\pi}{5}x,\ C\sin\frac{4\pi}{5}x,...$
   (b) $\lambda = \frac{1}{3},\ \frac{2}{3},\ \frac{3}{3},\ \frac{4}{3},...,\qquad y = C\cos\frac{1}{3}x,\ C\cos\frac{2}{3}x,\ C\cos\frac{3}{3}x,\ C\cos\frac{4}{3}x,...$
   (c) $\lambda = \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ \frac{7}{2},...,\qquad y = C\sin\frac{\pi}{5}x,\ C\sin\frac{2\pi}{5}x,\ C\sin\frac{3\pi}{5}x,\ C\sin\frac{4\pi}{5}x,...$
   (d) $\lambda = \frac{\pi}{14},\ \frac{3\pi}{14},\ \frac{5\pi}{14},\ \frac{7\pi}{14},...,\qquad y = C\cos\frac{\pi}{14}x,\ C\cos\frac{3\pi}{14}x,\ C\cos\frac{5\pi}{14}x,\ C\cos\frac{7\pi}{14}x,...$
   (e) $\lambda = \frac{\pi}{10},\ \frac{2\pi}{10},\ \frac{3\pi}{10},\ \frac{4\pi}{10},...,\qquad y = C\cos\frac{\pi}{10}x,\ C\cos\frac{2\pi}{10}x,\ C\cos\frac{3\pi}{10}x,\ C\cos\frac{4\pi}{10}x,...$
   (f) $\lambda = \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},...,\qquad y = C\sin\frac{1}{5}x,\ C\sin\frac{2}{5}x,\ C\sin\frac{3}{5}x,\ C\sin\frac{4}{5}x,...$

2. (a) $y = C\sin\frac{\pi}{12}\,x,\ y = C\sin\frac{2\pi}{12}\,x,\ y = C\sin\frac{3\pi}{12}\,x,\ y = C\sin\frac{4\pi}{12}\,x,\ \ldots$
   (b) $P = \frac{6250}{9}\pi^2,\ 4(\frac{6250}{9}\pi^2),\ 9(\frac{6250}{9}\pi^2),\ 16(\frac{6250}{9}\pi^2),\ \ldots$
   (c) The third critical load is nine times the first critical load.

3. (a) $y = C\sin\frac{\pi}{6}\,x,\ y = C\sin\frac{2\pi}{6}\,x,\ y = C\sin\frac{3\pi}{6}\,x,\ y = C\sin\frac{4\pi}{6}\,x,\ \ldots$
   $P = \frac{25000}{9}\pi^2,\ 4(\frac{25000}{9}\pi^2),\ 9(\frac{25000}{9}\pi^2),\ 16(\frac{25000}{9}\pi^2),\ \ldots$
   (b) Each critical load is four times the corresponding critical load for the twelve foot column.
   (c) The first buckling mode for the six foot column is the same as the second buckling mode for the twelve foot column, but is half as high, so it only includes half a period of the sine function, whereas the twelve foot column's second buckling mode has a full period of the sine function.

4. (a) $y = C\cos\frac{2\pi}{12}\,x,\ y = C\cos\frac{4\pi}{12}\,x,\ y = C\cos\frac{6\pi}{12}\,x,\ \ldots$
   (b) $P = 4(\frac{6250}{9}\pi^2),\ 16(\frac{6250}{9}\pi^2),\ 36(\frac{6250}{9}\pi^2),\ \ldots$
   (c) The third critical load is nine times the first critical load.
   (d) Each critical load is four times the corresponding critical load for the column with pinned ends.

5. (a) $y = C\cos\frac{\pi}{12}\,x,\ y = C\cos\frac{2\pi}{12}\,x,\ y = C\cos\frac{3\pi}{12}\,x,\ y = C\cos\frac{4\pi}{12}\,x,\ \ldots$
   (b) $P = \frac{6250}{9}\pi^2,\ 4(\frac{6250}{9}\pi^2),\ 9(\frac{6250}{9}\pi^2),\ 16(\frac{6250}{9}\pi^2),\ \ldots$
   (c) The third critical load is (again!) nine times the first critical load.
   (d) The critical loads are the same as those for the pinned ends.

6. (a) $y = C\sin\frac{\pi}{L}\,x,\ y = C\sin\frac{2\pi}{L}\,x,\ y = C\sin\frac{3\pi}{L}\,x,\ y = C\sin\frac{4\pi}{L}\,x,\ \ldots$
   (b) $P = \frac{EI}{L^2}\pi^2,\ 4(\frac{EI}{L^2}\pi^2),\ 9(\frac{EI}{L^2}\pi^2),\ 16(\frac{EI}{L^2}\pi^2),\ \ldots$

7. (a) $y = C\cos\frac{\pi}{L}\,x,\ y = C\cos\frac{2\pi}{L}\,x,\ y = C\cos\frac{3\pi}{L}\,x,\ y = C\cos\frac{4\pi}{L}\,x,\ \ldots$
   (b) $P = \frac{EI}{L^2}\pi^2,\ 4(\frac{EI}{L^2}\pi^2),\ 9(\frac{EI}{L^2}\pi^2),\ 16(\frac{EI}{L^2}\pi^2),\ \ldots$

## D.6 Solutions for Appendices

**Section B.1 Solutions** Back to B.1 Exercises

1. (a) $y' = 15x^4 - 28x^3 + 2x - 3$ (b) $h'(t) = -32t + 23.7$
   (c) $f'(x) = \frac{8}{3}x^3 + 15x^2 - \frac{1}{4}x$ (d) $s'(t) = 3t^2 - 4t + 3$
2. (a) $y'' = 60x^3 - 84x^2 + 2$ (b) $h''(t) = -32$
   (c) $f''(x) = 8x^2 + 30x - \frac{1}{4}$ (d) $s''(t) = 6t - 4$
3. (a) $f'(x) = -\dfrac{6}{x^3}$ (b) $g'(t) = \dfrac{1}{3}t + \dfrac{12}{t^3}$
   (c) $y' = \dfrac{1}{10\sqrt{x}}$ (d) $h'(t) = -\dfrac{4}{3\sqrt[3]{x^4}}$
4. (a) $f''(x) = \dfrac{18}{x^4}$ (b) $g''(t) = \dfrac{1}{3} - \dfrac{36}{t^4}$
5. (a) $g(x) = \dfrac{x}{\sqrt{16 - x^2}}$ (b) $y' = -\frac{3\pi}{2}\sin\left(\frac{\pi}{2}t\right)$ (c) $A'(t) = -150e^{-0.3t}$
6. (a) $y' = 10e^{2x}$ (b) $x' = 12\cos 3t$ (c) $g'(x) = -\frac{14(2x-4)}{(x^2-4x)^7}$
   (d) $s'(t) = -\frac{2}{5}\sin\left(\frac{2}{5}t\right)$ (e) $y' = 2xe^{x^2}$
7. (a) $15t^3e^{5t^3}$ (b) $-15\sin(5x - 2)$ (c) $4e^{-2t}\sin(5t + 3)$
8. (a) $-8e^{-2t}\sin(5t + 3)$ (b) $14te^{-t}\cos(3t - 1)$ (c) $12x^3\ln x$
9. (a) $f'(t) = 7t^2e^{7t} + 2te^{7t}$ (b) $y' = -6x\sin 2x + 3\cos 2x$
   (c) $h'(t) = 2\pi e^{-3t}\cos \pi t - 6e^{-3t}\sin \pi t$
10. (a) $f'(x) = \dfrac{6e^{7t}\cos 2t - 21e^{7t}\sin 2t}{e^{14t}}$ (b) $y' = \dfrac{-20t^2e^{-5t} - 8te^{-5t}}{t^4}$
    (c) $g'(x) = \dfrac{-12x^3\sin 6x - 6x^2\cos 6x}{4x^6}$

**Section B.2 Solutions** Back to B.2 Exercises

1. (a) $\frac{5}{2}\ln\left|2x + 3\right| + C$ (b) $-\frac{2}{5}\ln\left|3 - 5x\right| + C$ (c) $-12.5\ln\left|1.6 - 0.08A\right| + C$
2. (a) $\frac{1}{3}x^3 - \frac{7}{2}x^2 + 3x + C$ (b) $-\frac{7}{3}\cos 3t + C$
   (c) $-\frac{3}{2}te^{-2t} - \frac{3}{4}e^{-2t} + C$ (d) $-\dfrac{3}{x} + C$
   (e) $-10\ln\left|2.0 - 0.1A\right| + C$ (f) $-e^{-t}\left(\frac{3}{26}\sin 5t + \frac{15}{26}\cos 5t\right) + C$
   (g) $\frac{3}{5}\ln\left|5x - 1\right| + C$ (h) $-\frac{5}{3}t^2e^{-3t} - \frac{10}{9}te^{-3t} - \frac{10}{27}e^{-3t} + C$
   (i) $e^{-4t}\left(\frac{3}{25}\sin 3t - \frac{4}{25}\cos 3t\right) + C$ (j) $\frac{10}{\pi}\sin\frac{\pi}{2}t + C$

**Section B.3 Solutions** Back to B.3 Exercises

1. (a) $(-15,-7)$ (b) $(3,4)$ (c) $(13,-10)$
2. (a) $(1,-1)$ (b) $(-1,4)$ (c) $(-2,\frac{1}{3})$
3. $(\frac{29}{22},-\frac{5}{11})$
4. $(8,6)$
5. (a) $C_1=\frac{10}{3},\ C_2=\frac{2}{3}$ (b) $A=\frac{28}{53},\ B=-\frac{8}{53}$
   (c) $A=\frac{1}{2},\ B=-1,\ C=\frac{13}{8}$ (d) $C_1=-\frac{1}{3},\ C_2=\frac{1}{3}$
   (e) $A=-\frac{14}{73},\ B=\frac{13}{73}$ (f) $C_1=-19,\ C_2=\frac{185}{3}$

**Section B.4 Solutions** Back to B.4 Exercises

1. (a) $\dfrac{4x+7}{x^2+5x+6}=\dfrac{5}{x+3}+\dfrac{-1}{x+2}=\dfrac{5}{x+3}-\dfrac{1}{x+2}$
   (b) $\dfrac{-14}{x^2-3x-10}=\dfrac{-2}{x-5}+\dfrac{2}{x+2}$
   (c) $\dfrac{11-x}{x^2-x-2}=\dfrac{-4}{x+1}+\dfrac{3}{x-2}$
   (d) $\dfrac{4x-10}{x^2-1}=\dfrac{7}{x+1}+\dfrac{-3}{x-1}=\dfrac{7}{x+1}-\dfrac{3}{x-1}$

**Appendix C Solutions** Back to Appendix C Exercises

1.

| $n$ | $t_n$ | $y_n$ | $y(t_n)$ | % error |
|---|---|---|---|---|
| 0 | 0.0 | 2 | 2 | 0 |
| 1 | 0.2 | 2.4 | 2.4214 | 0.88 |
| 2 | 0.4 | 2.84 | 2.8918 | 1.79 |
| 3 | 0.6 | 3.328 | 3.4221 | 2.75 |
| 4 | 0.8 | 3.8736 | 4.0255 | 3.77 |
| 5 | 1.0 | 4.48832 | 4.7183 | 4.87 |

2. (a) $y_{n+1}=1.05y_n-0.05t_n$

   (b)

| $n$ | $t_n$ | $y_n$ | $y(t_n)$ | % error |
|---|---|---|---|---|
| 0 | 0.00 | 1.4 | 1.4 | 0.00 |
| 1 | 0.05 | 1.47 | 1.4705 | 0.03 |
| 2 | 0.10 | 1.541 | 1.5421 | 0.07 |
| 3 | 0.15 | 1.6131 | 1.6147 | 0.10 |
| 4 | 0.20 | 1.6863 | 1.6886 | 0.14 |

3. (a)

| $n$ | $t_n$ | $y_n$ | $y(t_n)$ | % error |
|---|---|---|---|---|
| 0 | 0.0 | 2 | 2 | 0.00 |
| 1 | 0.1 | 2 | 2.01 | 0.50 |
| 2 | 0.2 | 2.02 | 2.0404 | 1.00 |
| 3 | 0.3 | 2.0604 | 2.0921 | 1.52 |

   (b) $y=2e^{-\frac{1}{2}t^2}$

4. (a) $1.1y_n - 0.1t_n$ (b) $y_{n+1} = 1.22y_n - 0.22t_n - 0.02$

(c)

| $n$ | $t_n$ | $y_n$ | $y(t_n)$ | % error |
|---|---|---|---|---|
| 0 | 0.0 | 2 | 2 | 0 |
| 1 | 0.2 | 2.42 | 2.4214 | 0.06 |
| 2 | 0.4 | 2.8884 | 2.8918 | 0.12 |
| 3 | 0.6 | 3.4158 | 3.4221 | 0.18 |
| 4 | 0.8 | 4.0153 | 4.0255 | 0.25 |
| 5 | 1.0 | 4.7027 | 4.7183 | 0.33 |

(d) The error using the midpoint method are much smaller than those obtained using Euler's method with the same step size.

5. (a) $k_1 = .2y_n - .2t_n$, $k_2 = 0.22y_n - 0.22t_n - 0.02$, $k_3 = 0.222y_n - 0.222t_n - 0.022$, $k_4 = 0.2444y_n - 0.0444t_n - 0.044 - 0.2t_{n+1}$
*Note the appearance of both* $t_n$ *and* $t_{n+1}$ *in* $k_4$.

(b)

| $n$ | $t_n$ | $y_n$ | $k_1$ | $k_2$ | $k_3$ | $k_4$ |
|---|---|---|---|---|---|---|
| 0 | 0.0 | 2 | 0.4 | 0.42 | 0.422 | 0.4444 |
| 1 | 0.2 | 2.4214 | 0.4443 | 0.4687 | 0.4712 | 0.4985 |
| 2 | 0.4 | 2.8918 | 0.4984 | 0.5282 | 0.5312 | 0.5646 |
| 3 | 0.6 | 3.4221 | 0.5644 | 0.6009 | 0.6045 | 0.6453 |
| 4 | 0.8 | 4.0255 | 0.6451 | 0.6896 | 0.6941 | 0.7439 |
| 5 | 1.0 | 4.7182 | | | | |

(c) The approximations obtained using this method are very close to the exact values.

# Index