Math 322 Assignment 10, Spring 2013 **Due at 3 PM Monday, April 29th**

◇ **Example 1:** When we take the second derivative of the series $y = \sum_{n=0}^{\infty} a_n x^n$ we get $y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2}$. (Make sure you see how this is obtained!) Make a change of the index variable n to get a series with terms involving x^n.

We will let $m = n-2$, so that the terms x^{n-2} become x^m. Note that if $m = n-2$, then $n = m+2$ and $n(n-1)a_n\, x^{n-2}$ becomes $(m+2)[(m+2)-1]a_{m+2}\, x^m = (m+2)(m+1)a_{m+2}\, x^m$. Also, when $n = 2$ we have $m = 0$, and $m = \infty$ when $n = \infty$ as well. Thus the summation $\sum_{n=2}^{\infty}$ becomes $\sum_{m=0}^{\infty}$. Putting all this together, we have

$$\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}\, x^m$$

The variable m in the second summation is just a "dummy" variable, so in the end we can change it back to n. ♠

1. Rewrite each series so that the general term involves x^n.

(a) $\sum_{n=1}^{\infty} n\, a_n\, x^{n+2}$ (b) $\sum_{n=0}^{\infty} a_n\, x^{n+2}$ (c) $\sum_{n=2}^{\infty} (n+1)n\, a_n\, x^{n-2}$

(d) $\sum_{n=0}^{\infty} a_n\, x^{n+1}$ (e) $\sum_{n=3}^{\infty} (2n-1)\, a_n\, x^{n-3}$

2. If we have two series that are indexed the same and whose corresponding terms have the same power of x, they can be added like this:

$$\sum_{n=a}^{\infty} a_n\, x^m + \sum_{n=a}^{\infty} b_n\, x^m = \sum_{n=a}^{\infty} [a_n\, x^m + b_n\, x^m] = \sum_{n=a}^{\infty} [a_n + b_n]\, x^m \qquad (1)$$

Now consider the sum $\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} + 4\sum_{n=0}^{\infty} a_n\, x^n$. Simplify the sum as follows:

- Distribute the 4 into the second series.
- Change the index of summation of the first series so that the terms involve x^m, then change the variable back to n.
- Use (1) to combine the two series into one.

Sometimes when we change the index variables of sums to get series with x^n terms, the summations don't start in the same place. Here's how we handle that:

◇ **Example 2:** Simplify the expression $\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} + x\sum_{n=0}^{\infty} a_n\, x^n$ by combining the two series.

When we change the index of summation in the first series to get terms involving x^n we get $\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}\, x^m$.

Distributing the x into the second series and changing its index gives us $\sum_{m=1}^{\infty} a_{m-1}\, x^m$. We can "pull out" the first term of the first series, *then* add the resulting series:

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} + x\sum_{n=0}^{\infty} a_n\, x^n &= \sum_{m=0}^{\infty} (m+2)(m+1)\, a_{m+2}\, x^m + \sum_{m=1}^{\infty} a_{m-1}\, x^m \\
&= 2a_2 + \sum_{m=1}^{\infty} (m+2)(m+1)\, a_{m+2}\, x^m + \sum_{m=1}^{\infty} a_{m-1}\, x^m \\
&= 2a_2 + \sum_{m=1}^{\infty} [(m+2)(m+1)\, a_{m+2} + a_{m-1}]\, x^m
\end{aligned}$$ ♠

3. Use the procedure of Example 2 to simplify each of the following. For the second there will be *two* extra terms added to a series. (α is just a constant.)

(a) $\displaystyle\sum_{n=1}^{\infty} n\, a_n\, x^{n-1} + x\sum_{n=0}^{\infty} a_n\, x^n$ (b) $\displaystyle\sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} - \alpha^2 \sum_{n=0}^{\infty} a_n\, x^{n+2}$

Math 322 ASSIGNMENT 11, SPRING 2013 **Due at 3 PM Wednesday, May 1st**

◇ **Example 3:** Obtain a recursion relation for the coefficients a_n of a power series solution $y = \displaystyle\sum_{n=0}^{\infty} a_n\, x^n$ to the differential equation $y'' - xy' + 2y = 0$.

We take the first and second derivative of the series, substitute it into the left side of the ODE, get all sums in terms of x^n, and combine in the manner of Example 2:

$$\begin{aligned}
y'' - xy' + 2y &= \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} - x\sum_{n=1}^{\infty} n\, a_n\, x^{n-1} + 2\sum_{n=0}^{\infty} a_n\, x^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2}\, x^n - \sum_{n=1}^{\infty} n\, a_n\, x^n + \sum_{n=0}^{\infty} 2\, a_n\, x^n \\
&= 2a_2 + 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)\, a_{n+2} - n\, a_n + 2\, a_n]\, x^n \\
&= 2a_2 + 2a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1)\, a_{n+2} - (n-2)\, a_n]\, x^n
\end{aligned}$$

The only way that we can have $y'' - xy' + 2y$ equal to zero is if every coefficient of the final series above is zero, giving us $(n+2)(n+1)\, a_{n+2} - (n-2)\, a_n = 0$. If we solve this for a_{n+2} we get

$$a_{n+2} = \frac{n-2}{(n+2)(n+1)} a_n,$$

which is the desired recursion relation. ♠

1. Obtain a recurrence relation for each of the following ODEs, assuming a solution of the form $y = \displaystyle\sum_{n=0}^{\infty} a_n\, x^n$.

   (a) $y'' - xy = 0$ (b) $(1-x^2)y'' - 6xy' - 4y = 0$ (c) $y'' - (x+1)y = 0$

   (d) $y'' - \alpha^2 x^2 y = 0$ (e) $(x^2+1)y'' + xy' - y = 0$

2. Find the series solution to each of the ODEs in Exercise 1, in terms of a_0 and a^1 (or two other coefficients when either or both of those are zero).