Math 322 SOLVING NON-HOMOGENEOUS SYSTEMS **Spring 2013**

To solve $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$, $\mathbf{x}(0) = \mathbf{c}$:

1) Solve the homogeneous equation $\mathbf{x}' = A\mathbf{x}$ to get the homogeneous solution $\mathbf{x}_h$.
2) Use either undetermined coefficients or variation of parameters to find the particular solution $\mathbf{x}_p$.
   - *Undetermined Coefficients:* Assume that $\mathbf{x}_p$ has a particular form that has certain unknown coefficients in it. Substitute the guess into $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ and equate like terms to obtain a system of equations that can be solved to find the coefficients. Put them back into the guess to get the particular solution.
   - *Variation of Parameters:* Let $\mathbf{X}$ be the **fundamental matrix**, which is the matrix whose columns are the solutions $\mathbf{k}_i e^{\lambda_i t}$ to the homogeneous equation $\mathbf{x}' = A\mathbf{x}$. Then $\mathbf{x}_p$ is given by

$$\mathbf{x}_p = \mathbf{X}(t) \int \mathbf{X}^{-1}(t)\,\mathbf{f}(t)\,dt.$$

   When doing the indefinite integral of $\mathbf{X}^{-1}\mathbf{f}$ you need not include constants of integration. Remember that the inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
3) Add the homogeneous and particular solutions to get the general solution $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$.
4) Apply the initial conditions to obtain values of arbitrary constants.

Here are some other things that will be useful for this assignment. First,

$$\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}\,(a\sin bt - b\cos bt) \qquad \int e^{at}\cos bt\,dt = \frac{e^{at}}{a^2+b^2}\,(a\cos bt + b\sin bt)$$

Next, there is a very useful way to think of the product of a matrix and a vector:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} a \\ c \end{bmatrix} + x_1\begin{bmatrix} b \\ d \end{bmatrix}$$

Math 322 ASSIGNMENT 20, SPRING 2013 **Due at 3 PM Monday May 20th**

For this assignment you will be considering the initial value problem

$$\mathbf{x}' = \begin{bmatrix} -4 & 1 \\ -6 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 3\cos t \\ -\sin t \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The eigenvalues and eigenvectors for $A = \begin{bmatrix} -4 & 1 \\ -6 & 1 \end{bmatrix}$ are $\lambda = -2, -1$ and $\mathbf{k} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

1. Give the homogeneous solution to the differential equation.
2. Find the particular solution using variation of parameters.
3. Find the particular solution using undetermined coefficients, assuming a solution of the form

$$\mathbf{x}_p = \begin{bmatrix} a\cos t + b\sin t \\ c\cos t + d\sin t \end{bmatrix}.$$

4. Give the general solution.
5. Apply the initial condition to find the values of the unknown constants, and give your final solution. Whew!
6. Use the Method of Frobenius to find the series solution to $x^2y'' + xy' + (x^2 - 1)y = 0$ that corresponds to the larger root of the indicial equation. **Include a BRIEF explanation of why** a_1 **must be zero.** Find enough terms of the series that you could tell the next term without computing it (or stop after four nonzero terms if you don't see it by then!).