Math 322, Assignment 21

(1) $y^{(5)}-2y^{(4)}-6y^{(3)}+3y''-y'+2y=3\sin 2t$

$x_1=y$
$x_2=y'=x_1'$
$x_3=y''=x_2'$
$x_4=y^{(3)}=x_3'$
$x_5=y^{(4)}=x_4'$
$x_5'=y^{(5)}$

$x_5'=2x_5+6x_4-3x_3+x_2-2x_1+3\sin 2t$

$x_1'=x_2$
$x_2'=x_3$
$x_3'=x_4$
$x_4'=x_5$
$x_5'=-2x_1+x_2-3x_3+6x_4+2x_5+3\sin 2t$

$$\boxed{\vec{x}'=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\-2&1&-3&6&2\end{bmatrix}\vec{x}+\begin{bmatrix}0\\0\\0\\0\\3\sin 2t\end{bmatrix}}$$

(2) a) $x_1'=x_1+x_2+2e^{2t}$
$x_2'=4x_1+x_2+e^{2t}$ $\Longrightarrow$ $\boxed{\vec{x}'=\begin{bmatrix}1&1\\4&1\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}e^{2t},\ \vec{x}(0)=\begin{bmatrix}0\\1\end{bmatrix}}$

b) $A=\begin{bmatrix}1&1\\4&1\end{bmatrix}\Longrightarrow \lambda=3,-1,\ \vec{k}=\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}-1\\2\end{bmatrix}$

Solution to $\vec{x}'=A\vec{x}$ is $\boxed{\vec{x}_h=c_1\begin{bmatrix}1\\2\end{bmatrix}e^{3t}+c_2\begin{bmatrix}-1\\2\end{bmatrix}e^{-t}}$

c) $\vec{x}_p=\begin{bmatrix}a\\b\end{bmatrix}e^{2t}\Longrightarrow \vec{x}_p'=2\begin{bmatrix}a\\b\end{bmatrix}e^{2t}=\begin{bmatrix}2a\\2b\end{bmatrix}e^{2t}$

$A\vec{x}_p=\begin{bmatrix}1&1\\4&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}e^{2t}=\begin{bmatrix}a+b\\4a+b\end{bmatrix}e^{2t}$

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②, continued

Substituting these into $\vec{x}' = A\vec{x} + \vec{f}$ gives

$$\begin{bmatrix} 2a \\ 2b \end{bmatrix} e^{2t} = \begin{bmatrix} a+b \\ 4a+b \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} \Longrightarrow \begin{matrix} 2a = a+b+2 \\ 2b = 4a+b+1 \end{matrix} \Longrightarrow \begin{matrix} a-b=2 \\ -4a+b=1 \\ \hline -3a = 3 \end{matrix}$$

$$a=-1$$
$$b=-3$$

$$\boxed{\vec{x}_p = \begin{bmatrix} -1 \\ -3 \end{bmatrix} e^{2t}}$$

d) $\boxed{\vec{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t}}$

e) $$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Longrightarrow \begin{matrix} c_1 - c_2 = 1 \\ 2c_1 + 2c_2 = 4 \end{matrix} \quad \begin{matrix} 2c_1 - 2c_2 = 2 \\ 2c_1 + 2c_2 = 4 \\ \hline 4c_1 = 6 \end{matrix}$$

$$c_1 = \tfrac{3}{2}, \; c_2 = \tfrac{1}{2}$$

$$\boxed{\vec{x} = \frac{3}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + \frac{1}{2}\begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t}}$$

Any equivalent form is acceptable

③ $y'' + 3y' + 2y = 5\cos 2t \quad y(0) = 1, \; y'(0) = -2$

$x_1 = y$
$x_2 = y' = x_1'$
$x_2' = y''$

$$x_2' = -2x_1 - 3x_2 + 5\cos 2t$$
$$x_1' = x_2$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_2 \\ -2x_1 - 3x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 5\cos 2t \end{bmatrix} \Longrightarrow \vec{x}' = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 5\cos 2t \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \Longrightarrow \lambda = -2, -1, \; \vec{k} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \vec{x}_h = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

$y(0) = x_1(0) = 1$
$y'(0) = x_2(0) = -2$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

③, continued

Variation of Parameters:

$$\underline{X} = \begin{bmatrix} -e^{-2t} & -e^{-t} \\ 2e^{-2t} & e^{-t} \end{bmatrix} \Longrightarrow \underline{X}^{-1} = \frac{1}{-e^{-3t}+2e^{-3t}} \begin{bmatrix} e^{-t} & e^{-t} \\ -2e^{-2t} & -e^{-2t} \end{bmatrix}$$

$$= e^{3t} \begin{bmatrix} e^{-t} & e^{-t} \\ -2e^{-2t} & -e^{-2t} \end{bmatrix} = \begin{bmatrix} e^{2t} & e^{2t} \\ -2e^{t} & -e^{t} \end{bmatrix}$$

$$\int \underline{X}^{-1} f \, dt = \int \begin{bmatrix} e^{2t} & e^{2t} \\ -2e^{t} & -e^{t} \end{bmatrix} \begin{bmatrix} 0 \\ 5\cos 2t \end{bmatrix} dt$$

$$= \int \begin{bmatrix} 5e^{2t}\cos 2t \\ -5e^{t}\cos 2t \end{bmatrix} dt = \begin{bmatrix} \frac{5e^{2t}}{8}(2\cos 2t + 2\sin 2t) \\ -\frac{5e^{t}}{5}(\cos 2t + 2\sin 2t) \end{bmatrix}$$

$$\underline{X} \int \underline{X}^{-1} f \, dt = \begin{bmatrix} -e^{-2t} & -e^{-t} \\ 2e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{5e^{2t}}{8}(2\cos 2t + 2\sin 2t) \\ -\frac{5e^{t}}{5}(\cos 2t + 2\sin 2t) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{5}{8}(2\cos 2t + 2\sin 2t) + (\cos 2t + 2\sin 2t) \\ \frac{10}{8}(2\cos 2t + 2\sin 2t) - (\cos 2t + 2\sin 2t) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{4}\cos 2t + \frac{3}{4}\sin 2t \\ \frac{3}{2}\cos 2t + \frac{1}{2}\sin 2t \end{bmatrix} = \vec{x}_p$$

$$\vec{x} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} -\frac{1}{4}\cos 2t + \frac{3}{4}\sin 2t \\ \frac{3}{2}\cos 2t + \frac{1}{2}\sin 2t \end{bmatrix}$$

③, still continued!

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} \\ \frac{3}{2} \end{bmatrix} \Longrightarrow \begin{aligned} -c_1 - c_2 &= \frac{5}{4} \\ 2c_1 + c_2 &= -\frac{7}{2} \end{aligned}$$

$$c_1 = -\frac{9}{4} \longrightarrow -\frac{9}{2} + c_2 = -\frac{7}{2}$$

$$c_2 = 1$$

$$\boxed{\vec{x} = -\frac{9}{4} \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} -\frac{1}{4}\cos 2t + \frac{3}{4}\sin 2t \\ \frac{3}{2}\cos 2t + \frac{1}{2}\sin 2t \end{bmatrix}}$$