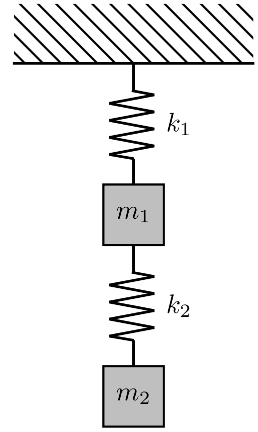
**Math 322** ASSIGNMENT 22, SPRING 2013 **Due at 3 PM Thursday, May 30th**

For several assignments you will be considering a coupled spring-mass system shown to the right. $k_1$ and $k_2$ are the spring constants of the upper and lower sprigs, respectively, $m_1$ and $m_2$ are the "masses of the masses," and we will represent the displacements of the two masses with $x_1$ and $x_2$. Using the fact that the sum of the forces acting on each mass must be zero we get the two differential equations




$$\begin{aligned} m_1x_1'' &= -(k_1+k_2)x_1 + k_2x_2 \\ m_2x_2'' &= k_2x_1 - k_2x_2 \end{aligned}$$

Dividing through by the masses then gives us

$$\begin{aligned} x_1'' &= -\frac{k_1+k_2}{m_1}x_1 + \frac{k_2}{m_1}x_2 \\ x_2'' &= \frac{k_2}{m_2}x_1 - \frac{k_2}{m_2}x_2 \end{aligned}$$

1. Suppose that the parameters have the following values, assuming correct units: $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, $k_2 = 2$. The initial position of the first mass is $-1$, the initial position of the second mass is $2$, and neither mass has any velocity initially. Give the matrix/vector form of the initial value problem. Write your answer down somewhere else as well, as you will need it for the next assignment.

2. For any $h > 0$, define the function $\delta_h$ by $\delta_h(t) = \begin{cases} \frac{1}{2h} & \text{for } -h \le x \le h \\ 0 & \text{elsewhere} \end{cases}$ **The notation $\delta_h$ will always refer to this function!**

   (a) Sketch the graphs of $\delta_{\frac{1}{2}}(t)$, $\delta_1(t-3)$ and $\delta_{\frac{1}{10}}(t-1)$ **on separate graphs. Label the important values on each axis.**

   (b) Using simple geometry, find the values of $\int_{-\infty}^{\infty} \delta_{\frac{1}{2}}(t)\,dt$, $\int_0^{\infty} \delta_1(t-3)\,dt$ and $\int_0^{\infty} \delta_{\frac{1}{10}}(t-1)\,dt$.

   (c) Let $f(t)$ be some function defined for $t \ge 0$ and let $t_0 > h$. Give a piecewise definition for the function $f(t)\,\delta_h(t-t_0)$ that does not use the function $\delta_h$.

3. Find **the exact value** of each of the following integrals, where $f(t) = t^2$. Then give each as an exact decimal, using the bar notation for repeating digits.

   (a) $\int_0^{\infty} f(t)\,\delta_1(t-2)\,dt$  (b) $\int_0^{\infty} f(t)\,\delta_{\frac{1}{2}}(t-2)\,dt$  (c) $\int_0^{\infty} f(t)\,\delta_{\frac{1}{4}}(t-2)\,dt$

4. Based on your results from Exercise 3, what do you think the value of $\lim_{h\to 0}\int_0^{\infty} f(t)\,\delta_h(t-2)\,dt$ is? (Give your answer by writing this full expression out and setting it equal to your guess.) Do you see how this relates to the function $f(t) = t^2$?

5. Give what you think the value of $\lim_{h\to 0}\int_0^{\infty} \sin t\,\delta_h(t-\frac{\pi}{2})\,dt$ is.

**Math 322** ASSIGNMENT 23, SPRING 2013 **Due at 3 PM Friday, May 31st**

1. Solve the initial value problem from Exercise 1 of Assignment 22 **using diagonalization**.

2. (a) Solve the IVP $y'' + 4y' + 13y = \delta(t-4)$, $y(0) = 0, y'(0) = 0$ using the Laplace transform. Use Wolfram Alpha to perform the needed inverse transform.

   (b) Change Wolfram's notation for the step function to ours, and then rewrite your answer to (a) so that every part of the solution is a function of $t-4$.

**Math 322** ASSIGNMENT 24, SPRING 2013 **Due at 3 PM Monday, June 3rd**

1. Solve the initial value problem from Exercise 1 of Assignment 22 by reducing it to a system of four first order ODEs, as follows.

   (a) Let $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_1'$, $y_4 = x_2'$. There are two first order ODE's included here, and the other two come from making the appropriate substitutions into your previously obtained system of two second order ODEs. Write the first order system you obtain in matrix form $\mathbf{y}' = A\mathbf{y}$. *Ten of the sixteen entries in your matrix should be zeros.*

   (b) Using the new variables, give the initial conditions as a single vector $\mathbf{y}(0)$.

   (c) Find the eigenvalues and associated eigenvectors for your matrix $A$ using your calculator or the last online tool listed at the class web page. The eigenvalues and some components of the eigenvectors will be imaginary.

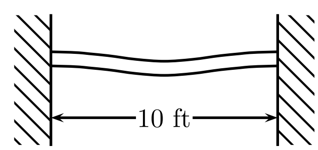
   (d) Write the solution in the same way that you always did for systems of two equations. Your exponentials will have imaginary exponents.

   (e) Apply the initial values to obtain a system of four equations in four unknowns, with complex number coefficients. Use the *rref* command with your calculator or Wolfram Alpha to solve the system.

   (f) Put your answers from (e) back into your solution from (d) and simplify it.

   (g) Give the displacement functions $x_1$ and $x_2$.

2. When a beam is suspended between two walls with its ends embedded in the walls the beam will deflect (sag) as shown in the diagram to the right. The amount of deflection at any point $x$ will be denoted by $y(x)$. The differential equation that this deflection satisfies is fourth order, so four *boundary conditions* are needed to determine the four unknown constants that arise. In this situation the full boundary value problem is




$$EI\frac{d^4y}{dx^4} = w(x), \quad y(0) = 0, \quad y'(0) = 0, \quad y(10) = 0, \quad y'(10) = 0,$$

   where $E$ is the modulus of elasticity of the material that the beam is made of, $I$ is the cross sectional moment of inertia of the beam, and $w(x)$ is the weight density of the beam at any point $x$ in the beam. For this exercise we will let $E = 10, I = 5$ and $w(x) = 100$, and we will solve the boundary value problem using the Laplace transform.

   (a) Substitute the numerical values and divide through by $EI$.

   (b) Take the Laplace transform of both sides. You will need to look up the Laplace transform of a higher derivative somewhere (it is in the Schaum's outline). The transform of the fourth derivative requires values for the function and the first three derivatives at zero. You don't know all of those, so let $y''(0) = A$ and $y'''(0) = B$.

   (c) Solve for $Y(s)$ and break the right hand side into three fractions with differing powers of $s$ in their denominators. Take the inverse Laplace transform using your table of transforms. Remember that the independent variable is $x$.

   (d) Your solution at this point has the two arbitrary constants $A$ and $B$ in it. Use the two conditions at $x = 10$ to solve for them, then give your final solution.

   (e) Graph your solution on your calculator from $x = 0$ to $x = 10$. Is its appearance what you expected?