Homogeneous Systems of Ordinary ODEs

- If the $n \times n$ matrix A has real eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of multiplicity one, with corresponding eigenvectors $\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n$, then the solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = c_1\mathbf{k}_1 e^{\lambda_1 t} + c_2\mathbf{k}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{k}_n e^{\lambda_n t} \tag{1}$$

- If the $n \times n$ matrix A has an eigenvalue λ_i of algebraic multiplicity m greater than one with m corresponding eigenvectors $\mathbf{k}_{i,1}, \mathbf{k}_{i,2}, ..., \mathbf{k}_{i,m}$, then the solution to $\mathbf{x}' = A\mathbf{x}$ is as (1) above, with the portion of the solution corresponding to the eigenvalue λ_i given by

$$c_{i,1}\mathbf{k}_{i,1}e^{\lambda_i t} + c_{i,2}\mathbf{k}_{i,2}e^{\lambda_i t} + \cdots + c_{i,m}\mathbf{k}_{i,m}e^{\lambda_i t}$$

- If the $n \times n$ matrix A has an eigenvalue λ_i of algebraic multiplicity two with only one corresponding eigenvector $\mathbf{k}_i$, then the solution to $\mathbf{x}' = A\mathbf{x}$ is as (1) above, with the portion of the solution corresponding to the eigenvalue λ_i given by

$$c_{i,1}\mathbf{k}_i e^{\lambda_i t} + c_{i,2}[\mathbf{k}_i t + \mathbf{p}_1]e^{\lambda_i t},$$

  where $\mathbf{p}$ is any solution to $(A - \lambda_i I)\,\mathbf{p} = \mathbf{k}$.

Variation of Parameters for Non-homogeneous Systems of ODEs

- Let $\mathbf{X}$ be the **fundamental matrix**, which is the matrix whose columns are the solutions $\mathbf{k}_i e^{\lambda_i t}$ to the homogeneous equation $\mathbf{x}' = A\mathbf{x}$. Then $\mathbf{x}_p$ is given by

$$\mathbf{x}_p = \mathbf{X}(t)\int \mathbf{X}^{-1}(t)\,\mathbf{f}(t)\,dt.$$

  When doing the indefinite integral of $\mathbf{X}^{-1}\,\mathbf{f}$ you need not include constants of integration.

- The inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Some Common Power Series

- $e^x = 1 + x + \dfrac{1}{2!}x^2 + \dfrac{1}{3!}x^3 + \dfrac{1}{4!}x^4 + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n$

- $\sin x = x - \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 - \dfrac{1}{7!}x^7 + \cdots + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$

- $\cos x = 1 - \dfrac{1}{2!}x^2 + \dfrac{1}{4!}x^4 - \dfrac{1}{6!}x^6 + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$

Ordinary and Singular Points of $a(x)y'' + b(x)y' + c(x)y = 0$

- An **ordinary point** of $a(x)y'' + b(x)y' + c(x)y = 0$ is any value of x_0 for which $a(x_0) \neq 0$.

- The radius of convergence R *about an ordinary point* for a power series solution to $a(x)y''+b(x)y'+c(x)y = 0$ is the distance from the origin to the closest zero of $a(x)$ in the complex plane. The series will converge in the interval $(-R, R)$. If $a(x)$ has no zeros, the series converges for all real numbers.

- A **singular point** of $a(x)y'' + b(x)y' + c(x)y = 0$ is any *real number* value of x_0 for which $a(x_0) = 0$.
- Divide both sides of $a(x)y''+b(x)y'+c(x)y = 0$ by $a(x)$ to get $y''+\frac{b(x)}{a(x)}y'+\frac{c(x)}{a(x)}y = y''+P(x)y'+Q(x)y = 0$. For x_0 a real number, suppose that a factor of the form $(x-x_0)^n$ appears in the denominator of $P(x)$ with n no larger than one and in the denominator of $Q(x)$ with n no larger than two. Then x_0 is called a **regular singular point** of $a(x)y'' + b(x)y' + c(x)y = 0$.

Solving About Regular Singular Points: The Method of Frobenius

- $y = x^\lambda \sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} a_n\, x^{\lambda+n}$ $\qquad y' = \sum_{n=0}^{\infty}(\lambda+n)\, a_n\, x^{\lambda+n-1} = x^\lambda \sum_{n=0}^{\infty}(\lambda+n)\, a_n\, x^{n-1}$

  $y'' = \sum_{n=0}^{\infty}(\lambda+n)(\lambda+n-1)\, a_n\, x^{\lambda+n-2} = x^\lambda \sum_{n=0}^{\infty}(\lambda+n)(\lambda+n-1)\, a_n\, x^{n-2}$

Laplace Transform: $\mathscr{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$

Table of Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}[f(t)]$	$f(t)$	$F(s) = \mathscr{L}[f(t)]$
δ	1	$f'(t)$	$sF(s) - f(0)$
1	$\frac{1}{s} \quad (s > 0)$	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
t^n	$\frac{n!}{s^{n+1}} \quad (s > 0)$	$u(t-c)$	$\frac{e^{-cs}}{s}$
e^{at}	$\frac{1}{s-a} \quad (s > a)$	$f(t-c)u(t-c)$	$e^{-cs}F(S)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}} \quad (s > a)$	$(g * f)(t)$	$G(s)F(s)$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2} \quad (s > 0)$	$e^{at} f(t)$	$F(s-a)$
$\cos \omega t$	$\frac{s}{s^2+\omega^2} \quad (s > 0)$	$f(kt)$	$\frac{1}{k}F\left(\frac{s}{k}\right)$
$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2} \quad (s > a)$	$tf(t)$	$-F'(s)$
$e^{at}\cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2} \quad (s > a)$	$\int_0^t f(u)\, du$	$\frac{F(s)}{s}$

Euler's Formulas: $e^{i\theta} = \cos\theta + i\sin\theta \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$