Series Solutions About Ordinary Points

Recall that a linear second order ordinary differential equation for a function y of an independent variable x is of the form

$$a(x)y'' + b(x)y' + c(x)y = f(x). \tag{1}$$

In the case that $a(x)$, $b(x)$ and $c(x)$ are all constants, we have a constant coefficient equation. Such an equation can generally be solved using the methods you learned in Math 321, or perhaps by Laplace transform methods. When any of $a(x)$, $b(x)$ and $c(x)$ are truly functions of the independent variable x, we must employ other methods to solve (1).

We will restrict ourselves to the homogeneous case, where $f(x) = 0$:

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{2}$$

Such equations often arise when solving partial differential equations. We will proceed by assuming that a solution of the form

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{n=0}^{\infty} a_n\, x^n \tag{3}$$

exists.

> An **ordinary point** of (2) is any value of x_0 for which $a(x_0) \neq 0$.

We will focus our attention on $x = 0$. *When zero is an ordinary point*, the process for finding a series solution "about zero" goes like this:

1) Take the first and second derivatives of (3), getting

$$y' = a_1 + 2\,a_2x + 3\,a_3x^2 + 4\,a_4x^3 + \cdots = \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} \tag{4}$$

and

$$y'' = 2\,a_2 + 3\cdot 2\,a_3x + 4\cdot 3\,a_4x^2 + 5\cdot 4\,a_5x^3 \cdots = \sum_{n=2}^{\infty} n(n-1)\, a_n\, x^{n-2} \tag{5}$$

2) Substitute (3), (4) and (5) into the left side of (2). Change indices to get all series so that they have the same power of x and remove terms from summations as needed so that all sums start at the same value of the index.

3) Combine the sums and factor out the power of x. This series can only be equal to zero for all allowable values of x (more on that later) if every coefficient is zero.

4) Set the common coefficient from the series equal to zero, and solve for the a_k with the highest index. This gives the **recurrence relation** used to calculate coefficients.

5) Begin with a_0 and a_1 equal to themselves, then find $a_2, a_3, a_4, ...,$ each in terms of either a_0 or a_1. Substitute your results into (3), group terms containing a_0 together, and the ones containing a_1 together, then factor those constants out. The result is the series solution to (2).

Sometimes the above steps will need some modification, especially at step five. Of concern with any power series is where it converges. The series obtained by the above process is centered at zero, and we can find its radius of convergence as follows:

> The radius of convergence R for a power series solution to (2) is the distance from the origin to the closest zero of $a(x)$ in the complex plane. The series will converge in the interval $(-R, R)$. If $a(x)$ has no zeros, the series converges for all real numbers.

Singular Points

We continue to work with equations of the form

$$a(x)y'' + b(x)y' + c(x)y = 0 \tag{2}$$

and we will further restrict ourselves to cases where $a(x)$, $b(x)$ and $c(x)$ are all polynomial functions. We first make this definition:

> An **singular point** of (2) is any *real number* value of x_0 for which $a(x_0) = 0$.

It is reasonable to think about dividing both sides of (2) by $a(x)$, resulting in

$$y'' + \frac{b(x)}{a(x)}y' + \frac{c(x)}{a(x)}y = y'' + P(x)y' + Q(x)y = 0. \tag{6}$$

With the restriction that $a(x)$, $b(x)$ and $c(x)$ are all polynomial functions, $P(x)$ and $Q(x)$ are rational functions, and their denominators can be factored. Any factor of the form $(x - x_0)^n$ in the denominator of either $P(x)$ or $Q(x)$, with x_0 a real number, indicates a singular point at x_0. The following definition implies that maybe some singular points are better than others!

> For x_0 a real number, suppose that a factor of the form $(x - x_0)^n$ appears in the denominator of $P(x)$ with n no larger than one and in the denominator of $Q(x)$ with n no larger than two. Then x_0 is called a **regular singular point** of (2).

Math 322 ASSIGNMENT 12, SPRING 2013 **Due at 3 PM Thursday, May 2nd**

1. For each of the following,
   - classify $x = 0$ as either an ordinary point (O), a regular singular point (RS) or a singular point that is not regular (SNR)
   - if $x = 0$ is an ordinary point, give the radius of convergence R and the interval of convergence for a series solution

   (a) $(2 + x^2)y'' - xy' + 4y = 0$ (b) $x^2y'' + (1 + x)y' + (1 + x^2)y = 0$ (c) $2y'' + xy' + 3y = 0$

   (d) $(4 - x^2)y'' + 2y = 0$ (e) $4x^2y'' - 4x^2y' + (1 - 2x)y = 0$

2. For each of the following, list each singular point and tell whether it is regular (RS) or singular but not regular (SNR).

   (a) $x^2(1 - x)^2y'' + 2xy' + 4y = 0$ (b) $x^2(1 - x^2)y'' + \frac{2}{x}y' + 4y = 0$

Solving About Regular Singular Points: The Method of Frobenius

When attempting to solve (2) when zero is a regular singular point, we assume a solution of the form

$$y = x^\lambda \sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} a_n\, x^{\lambda+n} = a_0\, x^\lambda + a_1\, x^{\lambda+1} + a_2\, x^{\lambda+2} + \cdots . \tag{7}$$

Here λ could perhaps be a complex number, but we'll stick to situations where it is real. We now need to take the derivative of (7), using any one of the forms given there. If we were to use $y = x^\lambda \sum_{n=0}^{\infty} a_n\, x^n$ we would need to apply the product rule, taking x^λ as our first function and $\sum_{n=0}^{\infty} a_n\, x^n$ as our second. It is easier to use the second or third for above while taking the derivative, then taking the x^λ back out of the sum when we're done:

$$\begin{aligned} y' &= \lambda a_0\, x^{\lambda-1} + (\lambda+1)\, a_1\, x^{\lambda} + (\lambda+2)\, a_2\, x^{\lambda+1} + (\lambda+3)\, a_3\, x^{\lambda+2} + \cdots \\ &= \sum_{n=0}^{\infty} (\lambda+n)\, a_n\, x^{\lambda+n-1} \\ &= x^\lambda \sum_{n=0}^{\infty} (\lambda+n)\, a_n\, x^{n-1} \end{aligned} \tag{8}$$

and

$$\begin{aligned} y'' &= \lambda(\lambda-1)\, a_0\, x^{\lambda-2} + (\lambda+1)\lambda\, a_1\, x^{\lambda-1} + (\lambda+2)(\lambda+1)\, a_2\, x^{\lambda} + (\lambda+3)(\lambda+2)\, a_3\, x^{\lambda+1} + \cdots \\ &= \sum_{n=0}^{\infty} (\lambda+n)(\lambda+n-1)\, a_n\, x^{\lambda+n-2} \\ &= x^\lambda \sum_{n=0}^{\infty} (\lambda+n)(\lambda+n-1)\, a_n\, x^{n-2} \end{aligned} \tag{9}$$

We are now ready to examine how the method of Frobenius is executed, which is quite similar to how solutions about ordinary points are found:

1) Substitute the versions of y'', y' and y in (7), (8) and (9) that have the x^λ outside the summation into the left side of the differential equation, then factor the x^λ out right away. *DO NOT* factor out other powers of x. Multiply other factors outside of sums *into the sums.*

2) Evaluate the terms of the sums for the indices that give terms with x^{-1} in them, to "pull them out" of their sums.

3) Take the terms that were pulled out of the series and factor out $a_0 x^{-1}$. Change indices of the series so that they all have the same power of x, then combine the sums and factor out the power of x.

4) You should now have something of the form $x^\lambda\{[\text{ stuff }]a_0 x^{-1} + \sum[\text{ stuff }]x^n\}$. We deal with the two stuffs separately:

   - The second stuff is just set equal to zero and solved for the a_k with the highest index to get the recurrence relation, as done previously. The resulting formula will contain λ.
   - The first stuff is called the **indicial equation** (pronounced "in-dish-ull"), and will always be a second degree polynomial. Set it equal to zero and solve to get two values of λ.

At this point there are several ways to proceed, depending on the relationship between the values of λ. We will do only the case in which there are two distinct real values λ_1 and λ_2 that *do not differ by an integer.*

5) Substitute λ_1 into the recurrence relation, and simplify it to eliminate fractions.

6) Set $a_0 = 1$ and use your recurrence relation for λ_1 to find $a_1, a_2, a_3, \ldots$. One solution to the ODE is then $y_1 = x^{\lambda_1} \sum_{n=0}^{\infty} a_n\, x^n$. Remember - we assumed the solutio had this form.

7) Repeat steps 5 and 6 for λ_2 to get a second solution y_2.

8) The general solution is $y = c_1 y_1 + c_2 y_2$.

Math 322 ASSIGNMENT 13, SPRING 2013 **Due at 3 PM Monday, May 6th**

1. The equation $2xy'' + (1+x)y' + y = 0$ has a regular singular point at $x = 0$. Use the method of Frobenius to find two linearly independent solutions to the equation.