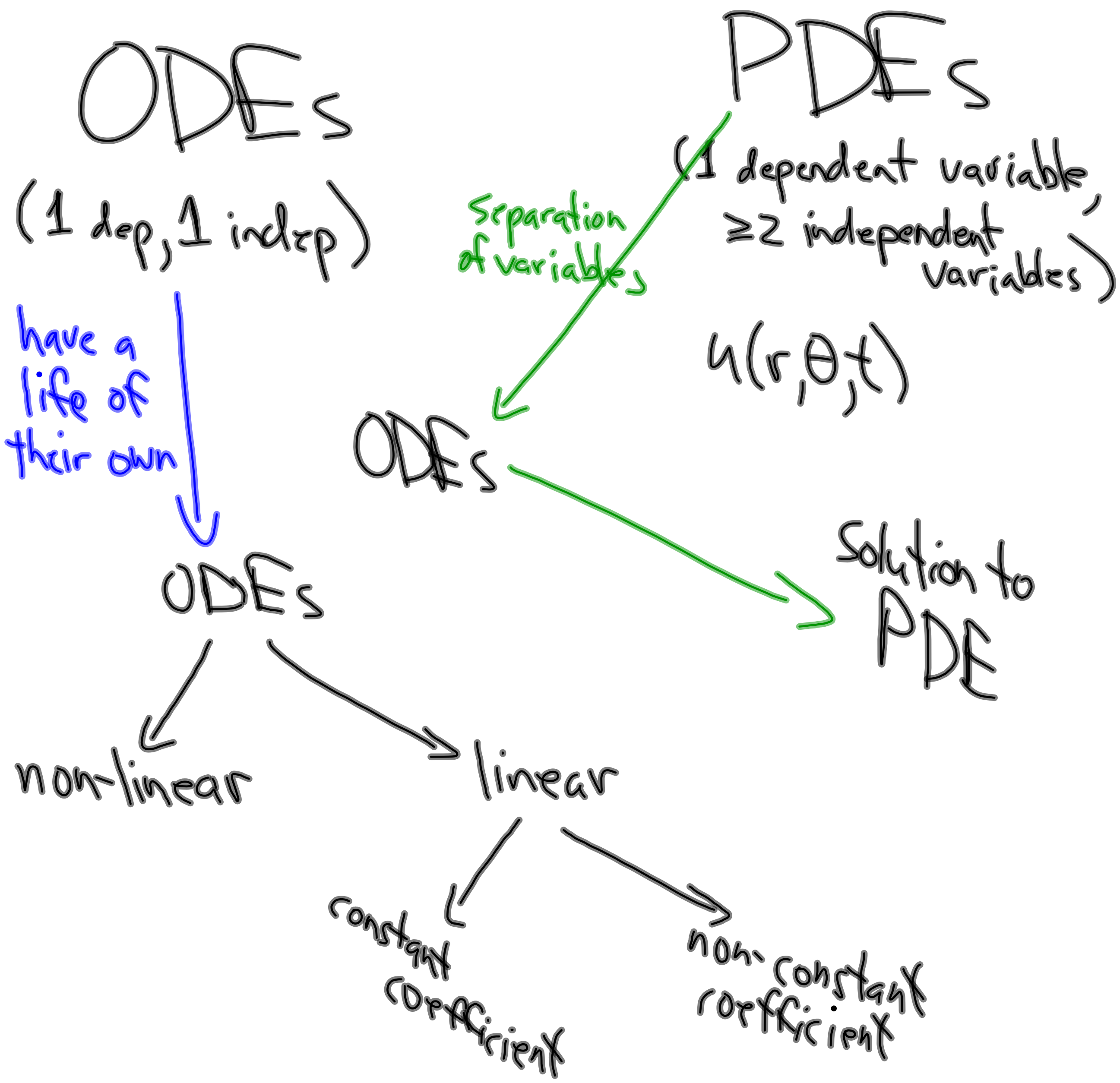
Differential Equations



Linear ODE $\quad y = y(x)$

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = f(x)$$

nth order linear ODE

When $a_n(x), \dots, a_1(x)$ are constants, we have a constant coefficient ODE.

It can be turned into a system of n first-order, constant coefficient ODEs.

If $f(x) = 0$, the ODE is homogeneous.

2nd order, linear, constant coeff.

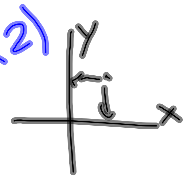
$$ay'' + by' + cy = f(t)$$

If $f(t)$ is a polynomial, exp or trig function, use Math 321 methods.

When $f(t)$ is discontinuous (or other...), Laplace methods become nice.

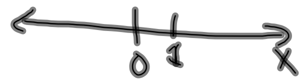
2nd order, linear, homogeneous

$$a(x)y''+b(x)y'+c(x)y=0 \quad (2)$$



There are ordinary points and singular points. They are values of x.

$y=y(x)$



Values of x where $a(x)\neq 0$ are called ordinary points. Values where $a(x)=0$ are singular points.

Examples:

① $(1-x^2)y''-xy'+p^2y=0$ $\quad a(x)=1-x^2$

$x=1,-1$ are singular points

All other x are ordinary points

② $y''+2x^3y=0$ $\quad a(x)=1$

All x are ordinary points.

③ $x^2(1-x)y''+(x-2)y'-3xy=0$ $\quad a(x)=x^2(1-x)$

$x=0,1$ are singular points

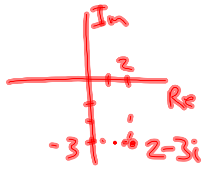
All other x are ordinary points

If x_0 is an ordinary point, we look for a solution to (2) of the form $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

We'll look at $y = \sum_{n=0}^{\infty} a_n x^n$

What is the radius of convergence R of a series solution $y = \sum_{n=0}^{\infty} a_n x^n$ "about zero".

Answer: R is distance from zero to the closest zero (root) of $a(x)$ in the complex plane.

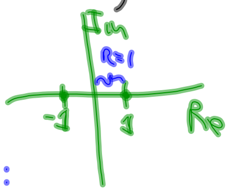


Examples:

① $(1-x^2)y'' + b(x)y' + c(x)y = 0$

$a(x) = 1-x^2 \implies$ zeros are $1, -1$

radius of convergence is $R=1$



Interval of convergence:

$$(0-R, 0+R)$$

$$(-R, R)$$

② $(5+x^2)y'' + xy' - 3y = 0$

$a(x) = 5+x^2 = 0$

$x^2 = -5$

$x = \pm i\sqrt{5}$

$R = \sqrt{5}$

interval of convergence is

$0 + i\sqrt{5}$, $0 - i\sqrt{5}$