$$a(x)y''+b(x)y'+c(x)y=0$$

If $a(x_0)=0$, then x_0 is a singular point of the ODE.

Divide by $a(x)$

$$y''+\frac{b(x)}{a(x)}y'+\frac{c(x)}{a(x)}y=0$$

$$y''+P(x)y'+Q(x)y=0$$

Assume $a(x)$, $b(x)$, $c(x)$ are polynomials. Then $P(x)$ + $Q(x)$ are rational functions.

Example $x^2y''+xy'+(x^2-p^2)y=0$

$x=0$ is singular.

$x-x_0$ is x

$$y''+\frac{x}{x^2}y'+\frac{(x^2-p^2)}{x^2}y=0$$

$\frac{1}{x}$

Because the powers here and here are ≤ 1 and 2, respectively, $x=0$ is a <u>regular singular point</u>.

$$x^2(1-x)y''+(x-2)y'-3xy=0$$

$$y''+\frac{(x-2)}{x^2(1-x)}y'-\frac{3x}{x^2(1-x)}y=0$$

$x=0$ >1, so $x=0$ is singular, but not regular singular.

$x-0$

$x=1$ ≤ 1 and 2 respectively, so $x=1$ is a regular singular point.

Method of Frobenius

$3xy'' + y' - y = 0$ $\quad$ $x=0$ is singular, but regular

We want a series solution about $x=0$.

Guess $y = x^{\lambda} \sum_{n=0}^{\infty} a_n x^n$

$y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$

$y = \sum_{n=0}^{\infty} a_n x^{\lambda+n}$

$y' = \sum_{n=0}^{\infty} a_n (\lambda+n) x^{\lambda+n-1} = x^{\lambda} \sum_{n=0}^{\infty} a_n (\lambda+n) x^{n-1}$

$y'' = \sum_{n=0}^{\infty} a_n (\lambda+n)(\lambda+n-1) x^{\lambda+n-2}$

$= x^{\lambda} \sum_{n=0}^{\infty} a_n (\lambda+n)(\lambda+n-1) x^{n-2}$

$$3xy'' + y' - y = 0$$

$$3xx^{\lambda}\sum_{n=0}^{\infty} a_n(\lambda+n)(\lambda+n-1)x^{n-2}$$

$$+ x^{\lambda}\sum_{n=0}^{\infty} a_n(\lambda+n)x^{n-1} - x^{\lambda}\sum_{n=0}^{\infty} a_n x^n$$

$$x^{\lambda}\left\{\sum_{n=0}^{\infty} 3a_n(\lambda+n)(\lambda+n-1)x^{n-1} + \sum_{n=0}^{\infty} a_n(\lambda+n)x^{n-1} - \sum_{n=0}^{\infty} a_n x^n\right\}$$

$$x^{\lambda}\left[3a_0\lambda(\lambda-1)x^{-1} + a_0\lambda x^{-1} + \sum_{n=1}^{\infty} 3a_n(\lambda+n)(\lambda+n-1)x^{n-1}\right.$$

$$\left. + \sum_{n=1}^{\infty} a_n(\lambda+n)x^{n-1} - \sum_{n=0}^{\infty} a_n x^n\right]$$

$$x^{\lambda}\left\{a_0x^{-1}\left(3\lambda(\lambda-1)+\lambda\right) + \sum_{n=1}^{\infty} 3a_n(\lambda+n)(\lambda+n-1)x^{n-1}\right.$$

$$\left. + \sum_{n=1}^{\infty} a_n(\lambda+n)x^{n-1} - \sum_{n=0}^{\infty} a_n x^n\right\}$$

let $\theta = n-1$ $\quad n = \theta+1 \quad$ $n=1 \Rightarrow \theta=0$

$n=\infty \Rightarrow \theta=\infty$

$$x^{\lambda}\left\{a_0x^{-1}\left(3\lambda(\lambda-1)+\lambda\right)+\sum_{n=1}^{\infty}3a_n(\lambda+n)(\lambda+n-1)x^{n-1}\right.$$
$$\left.+\sum_{n=1}^{\infty}a_n(\lambda+n)x^{n-1}-\sum_{n=0}^{\infty}a_nx^n\right\}$$

let $\theta=n-1$ $\quad n=\theta+1$ $\quad n=1\Rightarrow\theta=0$, $n=\infty\Rightarrow\theta=\infty$

$$x^{\lambda}\left\{a_0x^{-1}\left[3\lambda^2-2\lambda\right]+\sum_{\theta=0}^{\infty}3a_{\theta+1}(\lambda+\theta+1)(\lambda+\theta)x^{\theta}\right.$$
$$\left.+\sum_{\theta=0}^{\infty}a_{\theta+1}(\lambda+\theta+1)x^{\theta}-\sum_{n=0}^{\infty}a_nx^n\right\}$$

let $\theta=n$

$$x^{\lambda}\left\{a_0x^{-1}\underbrace{(3\lambda^2-2\lambda)}_{\text{set}=0}+\sum_{n=0}^{\infty}\left[3a_{n+1}(\lambda+n+1)(\lambda+n)+a_{n+1}(\lambda+n+1)-a_n\right]x^n\right\}$$

$$3a_{n+1}(\lambda+n+1)(\lambda+n)+a_{n+1}(\lambda+n+1)-a_n=0$$

$$a_{n+1}\left[3(\lambda+n+1)(\lambda+n)+(\lambda+n+1)\right]=a_n$$

$$a_{n+1}=\frac{a_n}{(\lambda+n+1)\left[3\lambda+3n+1\right]}$$

* recurrence relation

$3\lambda^2-2\lambda=0$ indicial equation

Continued on 5/3

May 2-9:36 AM