Systems of Equations

$$A(a\vec{x}+b\vec{y}) = aA\vec{x}+bA\vec{y}$$

$$\frac{d}{dt}\left(ax_1(t)+bx_2(t)\right) = a\frac{dx_1}{dt}+b\frac{dx_2}{dt}$$

Solution to $\vec{x}' = A\vec{x}$

Guess $\vec{x} = \vec{k}e^{rt} \Longrightarrow \vec{x}' = r\vec{k}e^{rt}$

$$\begin{bmatrix} k_1 e^{rt} \\ k_2 e^{rt} \end{bmatrix}$$

$r\vec{k}\cancel{e^{rt}} = A\vec{k}\cancel{e^{rt}}$ divide be e^{rt}

$$\vec{0} = A\vec{k} - r\vec{k}$$

$$\vec{0} = \underbrace{(A - \lambda I)}_{B}\vec{k}$$

We want $\det(A - \lambda I) = 0$

λ is an eigenvalue & $\vec{k}$ is the corresponding eigenvector.

Solution: $\vec{x} = \vec{k}e^{\lambda t}$

Suppose $\vec{x}_1, \vec{x}_2$ are both solutions $\vec{x}_1' = A\vec{x}_1$, $\vec{x}_2' = A\vec{x}_2$

$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 \longrightarrow \vec{x}' = c_1\vec{x}_1' + c_2 x_2'$

$\longrightarrow A(c_1\vec{x}_1 + c_2\vec{x}_2)$

$= c_1 A x_1 + c_2 A\vec{x}_2$

$\vec{x}' \stackrel{?}{=} A\vec{x}$

$c_1\vec{x}_1' + c_2\vec{x}_2' \stackrel{?}{=} c_1 A_1 x_1 + c_2 A\vec{x}_2$

This is only better than $\vec{x}_1$ or x_2 alone if they are linearly independent.

May 20-9:08 AM

$\vec{x}_1$ and $\vec{x}_2$ linearly independent?

Set $c_1\vec{x}_1 + c_2\vec{x}_2 = \vec{0}$

If the only values of c_1 + c_2 that make this true are $c_1 = c_2 = 0$, then $\vec{x}_1$ and $\vec{x}_2$ are linearly independent.

Test for idependence of $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$

① Form $\bar{\bar{X}} = \begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{bmatrix}$

② Find $\det(\bar{\bar{X}})$, called the Wronskian

If it is not zero for any t, $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are linearly independent

$\bar{\bar{X}}$ in that case is called a fundamental matrix.

Suppose $\vec{x}\,' = \underbrace{A}_{n\times n}\vec{x}$

If $\mathbf{X}$ is an $n\times n$ fundamental matrix of solutions, then

$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$ is <u>the general solution</u> to $\vec{x}\,' = A\vec{x}$.

Suppose $\lambda = -1, 2$ with eigenvectors

$\vec{k} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Solutions: $\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-t} = \begin{bmatrix} -e^{-t} \\ 3e^{-t} \end{bmatrix}$

$\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$

$W = \det \begin{bmatrix} -e^{-t} & e^{2t} \\ 3e^{-t} & e^{2t} \end{bmatrix}$

$= -e^{t} - 3e^{t} = -4e^{t} \neq 0$ for any t

$\vec{x}_1, \vec{x}_2$ are linearly independent

$$\vec{x}' = A\vec{x} + \vec{f} \quad (1)$$

If $\vec{x}_p' = A\vec{x}_p + \vec{f}$, then $\vec{x}_p$ is a particular solution to (1).

Suppose $\vec{x}_h' = A\vec{x}_h$, consider $\vec{x} = \vec{x}_h + \vec{x}_p$

$$\begin{aligned}(\underline{\underline{\vec{x}_h + \vec{x}_p}})' &= \vec{x}_h' + \vec{x}_p' \\ &= A\vec{x}_h + A\vec{x}_p + \vec{f} \\ &= A(\underline{\underline{\vec{x}_h + \vec{x}_p}}) + \vec{f}\end{aligned}$$

Series Solutions

Power series

Ordinary point

Singular point

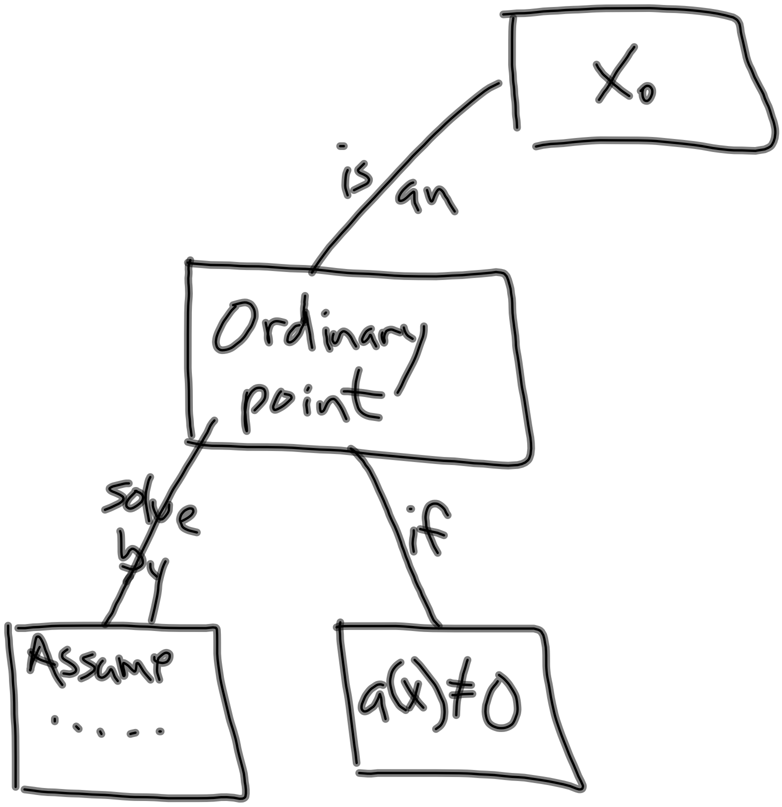
Regular singular point

Method of Frobenius

Recurrence relation

Indicial equation

$$a(x)y'' + b(x)y' + c(x) = 0$$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$a(x) \neq 0$