

### 1.3 Solving With Matrices

**Performance Criteria:**

1. (f) Give the coefficient matrix and augmented matrix for a system of equations.
- (g) Determine whether a matrix is in row-echelon form. Perform, by hand, elementary row operations to reduce a matrix to row-echelon form.
- (h) Determine whether a matrix is in reduced row-echelon form. Use a calculator or software to reduce a matrix to reduced row-echelon form.
- (f) For a system of equations having a unique solution, determine the solution from either the row-echelon form or reduced row-echelon form of the augmented matrix for the system.
- (g) Use a calculator to solve a system of linear equations having a unique solution.

Note that when using the addition method for solving the system of three equations in three unknowns in the previous section, the symbols  $x$ ,  $y$  and  $z$  and the equal signs are simply “placeholders” that are “along for the ride.” To make the process cleaner we can simply arrange the constants  $a$ ,  $b$ ,  $c$  and  $d$  for each equation  $ax+by+cz = d$  in an array form called a **matrix**, which is simply a table of values like

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix}.$$

Each number in a matrix is called an **entry** of the matrix. Each horizontal line of numbers in a matrix is a **row** of the matrix, and each vertical line of numbers is a **column**. The **size** or **dimensions** of a matrix is (are) given by giving first the number of rows, then the number of columns, with the  $\times$  symbol between them. The size of the above matrix is  $3 \times 4$ , which we say as “three by four.”

Suppose that the above matrix came from the system of equations

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned}$$

When a matrix represents a system of equations, as this one does, it is called the **augmented matrix** of the system. The matrix consisting of just the coefficients of  $x$ ,  $y$  and  $z$  from each equation is called the **coefficient matrix**:

$$\begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix}$$

We are not interested in the coefficient matrix at this time, but we will be later. The reason for the name “augmented matrix” will also be seen later.

Once we have the augmented matrix, we can perform a process called **row-reduction**, which is essentially what we did in the previous section, but we work with just the matrix rather than the system of equations. The following example shows how this is done for the above matrix.

◇ **Example 1.3(a):** Solve the system (1) from the previous section by row-reduction.

We begin by adding negative three times the first row to the second, and put the result in the second row. Then we add two times the first row to the third, and place the result in the third. Using the notation  $R_n$  (not to be confused with  $\mathbb{R}^n$ !) to represent the  $n$ th row of the matrix, we can symbolize these two operations as shown in the middle below. The matrix to the right below is the result of those operations.

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix} \quad \begin{array}{l} -3R_1 + R_2 \rightarrow R_2 \\ \implies \\ 2R_1 + R_3 \rightarrow R_3 \end{array} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 7 & 3 & -1 \end{bmatrix}$$

Next we finish with the following:

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 7 & 3 & -1 \end{bmatrix} \quad \frac{7}{2}R_2 + 2R_3 \rightarrow R_3 \quad \Rightarrow \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 0 & \frac{55}{2} & 55 \end{bmatrix}$$

The process just outlined is called **row reduction**. At this point we return to the equation form

$$\begin{aligned} x + 3y - 2z &= -4 \\ 0x - 2y + 7z &= 16 \\ 0x + 0y + 55z &= 110 \end{aligned}$$

and perform back-substitution as before (see the top of page 6) to obtain  $z = 2$ ,  $y = -1$  and  $x = 3$ . ♠

The final form of the matrix before we went back to equation form is something called **row-echelon form**. (The word “echelon” is pronounced “esh-el-on.”) The first non-zero entry in each row is called a **leading entry**; in this case the leading entries are the numbers 1,  $-2$  and  $\frac{55}{2}$ . To be in row-echelon form means that

- any rows containing all zeros are at the bottom of the matrix and
- the leading entry in any row is to the right of any leading entries above it.

◇ **Example 1.3(b)**: Which of the matrices below are in row-echelon form?

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & 0 & 3 & -5 \\ 0 & 7 & -10 & -1 \end{bmatrix} \quad \begin{bmatrix} 2 & 6 & -1 & 9 & 5 \\ 0 & 0 & -8 & 1 & -3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 7 & -12 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 1 & 8 \end{bmatrix}$$

The leading entries of the rows of the first matrix are 1, 3 and 7. Because the leading entry of the third row (7) is not to the right of the leading entry of the second row (3), the first matrix *is not* in row-echelon form. In the third matrix, there is a row of zeros that is not at the bottom of the matrix, so it *is not* in row-reduced form. The second matrix *is* in row-reduced form. ♠

It is possible to continue with the matrix operations to obtain something called **reduced row-echelon form**, from which it is easier to find the values of the unknowns. The requirements for being in reduced row-echelon form are the same as for row-echelon form, with the addition of the following:

- All leading entries are ones.
- The entries above any leading entry are all zero *except perhaps in the last column*.

Obtaining reduced row-echelon form requires more matrix manipulations, and nothing is really gained by obtaining that form if you are doing this by hand. However, when using software or a calculator it is most convenient to obtain reduced row-echelon form. Here are two examples of matrices in reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 6 & 0 & 9 & 5 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For the first matrix above, one can easily see that if it came from the augmented matrix of a system of three equations in three unknowns, then  $(3, -7, 4)$  would be the solution to the system. We will have to wait a bit before we are ready to interpret what the second matrix would be telling us if it came from a system of equations.

Occasionally we need to exchange two rows when performing row-reduction. The following example shows a situation in which this applies.

◇ **Example 1.3(c):** Row-reduce the matrix  $\begin{bmatrix} 1 & 3 & -2 & -4 \\ 2 & 6 & -1 & -13 \\ -1 & 4 & -8 & 3 \end{bmatrix}$ .

We begin by adding negative three times the first row to the second, and put the result in the second row. Then we add two times the first row to the third, and place the result in the third. Using the notation  $R_n$  (not to be confused with  $\mathbb{R}^n!$ ) to represent the  $n$ th row of the matrix, we can symbolize these two operations as shown in the middle below. The matrix to the right below is the result of those operations.

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 2 & 6 & -1 & -13 \\ -1 & 4 & -8 & 3 \end{bmatrix} \quad \begin{array}{l} -2R_1 + R_2 \rightarrow R_2 \\ \implies \\ R_1 + R_3 \rightarrow R_3 \end{array} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & 0 & 3 & -5 \\ 0 & 7 & -10 & -1 \end{bmatrix}$$

We can see that the matrix would be in row-echelon form if we simply switched the second and third rows (which is equivalent to simply rearranging the order of our original equations), so that's what we do:

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & 0 & 3 & -5 \\ 0 & 7 & -10 & -1 \end{bmatrix} \quad \begin{array}{l} R_2 \leftrightarrow R_3 \\ \implies \end{array} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & 7 & -10 & -1 \\ 0 & 0 & 3 & -5 \end{bmatrix} \quad \spadesuit$$

The act of rearranging rows in a matrix is called **permuting** them. In general, a **permutation** of a set of objects is simply a rearrangement of them.

### Row Reduction Using Technology

The two technologies that we will use in this course are your graphing calculator and the computer software called MATLAB. The row reduction process can be done on a TI-83 calculator as follows; if you have a different calculator you will need to refer to your user's manual to find out how to do this. Practice using the matrix from the Example 1.3(a).

- Find and press the *MATRIX* (or maybe *MATRX*) key.
- Select *EDIT*. At that point you will see something like  $3 \times 3$  somewhere. This is the number of rows and columns your matrix is going to have. We want  $3 \times 4$ .
- After you have told the calculator that you want a  $3 \times 4$  matrix, it will begin prompting you for the entries in the matrix, starting in the upper left corner. Here you will begin entering the values from the augmented matrix, row by row. You should see the entries appear in a matrix as you enter them.
- After you enter the matrix, you need to get to *MATH* under the *MATRIX* menu. Select *rref* (for reduced row-echelon form) and you should see *rref* ( on your calculator screen.
- Select *NAMES* under the *MATRIX* menu. Highlight *A* and hit enter, then enter again.
- Pick off the values of  $x$ ,  $y$  and  $z$ , or  $x_1$ ,  $x_2$  and  $x_3$ , depending on notation. *Use the same letters as are given in the exercise you are doing.*

We will put off learning how to do this with MATLAB until we have some more things we can do with it as well, but the command is the same. Only the method for entering the matrix differs.

### Section 1.3 Exercises

1. Give the coefficient matrix and augmented matrix for the system of equations

$$\begin{array}{rcl} x + y - 3z & = & 1 \\ -3x + 2y - z & = & 7 \\ 2x + y - 4z & = & 0 \end{array} .$$

2. Determine which of the following matrices are in row-echelon form.

$$A = \begin{bmatrix} 3 & -7 & 5 & 0 & 2 & -4 \\ 0 & 0 & 0 & -2 & 5 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 3 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 4 & 4 \\ 0 & 1 & -3 & 2 \\ 6 & 1 & 3 & 5 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 3 & -5 & 10 & -7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Determine which of the matrices in Exercise 2 are in reduced row-echelon form.

4. Perform the first two row operations for the augmented matrix from Exercise 1, to get zeros in the bottom two entries of the first column.

5. Fill in the blanks in the second matrix with the appropriate values after the first step of row-reduction. Fill in the long blanks with the row operations used.

(a)

$$\begin{bmatrix} 1 & 5 & -7 & 3 \\ -5 & 3 & -1 & 0 \\ 4 & 0 & 8 & -1 \end{bmatrix} \xRightarrow{\text{_____}} \begin{bmatrix} \text{---} & \text{---} & \text{---} & \text{---} \\ 0 & \text{---} & \text{---} & \text{---} \\ 0 & \text{---} & \text{---} & \text{---} \end{bmatrix}$$

(b)

$$\begin{bmatrix} 2 & -8 & -1 & 5 \\ 0 & -2 & 0 & 0 \\ 0 & 6 & -5 & 2 \end{bmatrix} \xRightarrow{\text{_____}} \begin{bmatrix} \text{---} & \text{---} & \text{---} & \text{---} \\ 0 & \text{---} & \text{---} & \text{---} \\ 0 & 0 & \text{---} & \text{---} \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 3 & 5 & -2 \\ 0 & 2 & -8 & 1 \end{bmatrix} \xRightarrow{\text{_____}} \begin{bmatrix} \text{---} & \text{---} & \text{---} & \text{---} \\ 0 & \text{---} & \text{---} & \text{---} \\ 0 & 0 & \text{---} & \text{---} \end{bmatrix}$$

6. Find  $x$ ,  $y$  and  $z$  for the system of equations that reduces to the each of the matrices shown.

(a)  $\begin{bmatrix} 1 & 6 & -2 & 7 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & -2 & 8 \end{bmatrix}$       (b)  $\begin{bmatrix} 1 & 6 & -2 & 7 \\ 0 & 2 & -5 & -13 \\ 0 & 0 & 3 & 3 \end{bmatrix}$       (c)  $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -4 & 8 \end{bmatrix}$

7. Use row operations (by hand) on an augmented matrix to solve each system of equations.

(a)  $\begin{aligned} x - 2y - 3z &= -1 \\ 2x + y + z &= 6 \\ x + 3y - 2z &= 13 \end{aligned}$       (b)  $\begin{aligned} -x - y + 2z &= 5 \\ 2x + 3y - z &= -3 \\ 5x - 2y + z &= -10 \end{aligned}$       (c)  $\begin{aligned} x + 2y + 4z &= 7 \\ -x + y + 2z &= 5 \\ 2x + 3y + 3z &= 7 \end{aligned}$

8. Use the *rref* capability of your calculator to solve each of the systems from the previous exercise.