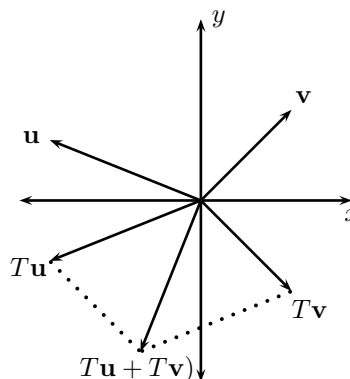
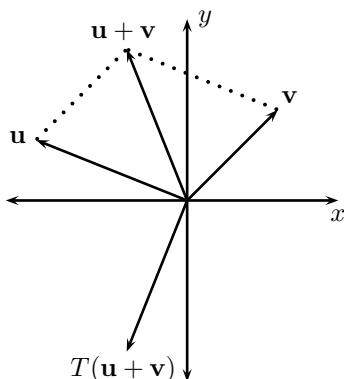


10.2 Linear Transformations

Performance Criteria:

10. (c) Determine whether a given transformation from \mathbb{R}^m to \mathbb{R}^n is linear. If it isn't, give a counterexample; if it is, demonstrate this algebraically and/or give the standard matrix representation of the transformation.
- (d) Draw an arrow diagram illustrating a transformation that is linear, or that is not linear.
- (e) For a transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, show the graphical differences between (a) $T(c\mathbf{u})$ and $cT\mathbf{u}$, and (b) $T(\mathbf{u} + \mathbf{v})$ and $T\mathbf{u} + T\mathbf{v}$.

To begin this section, recall the transformation from Example 10.1(b) that reflects vectors in \mathbb{R}^2 across the x -axis. In the drawing below and to the left we see two vectors \mathbf{u} and \mathbf{v} that are added, and then the vector $\mathbf{u} + \mathbf{v}$ is reflected across the x -axis. In the drawing below and to the right the same vectors \mathbf{u} and \mathbf{v} are reflected across the x -axis *first*, then the resulting vectors $T\mathbf{u}$ and $T\mathbf{v}$ are added.



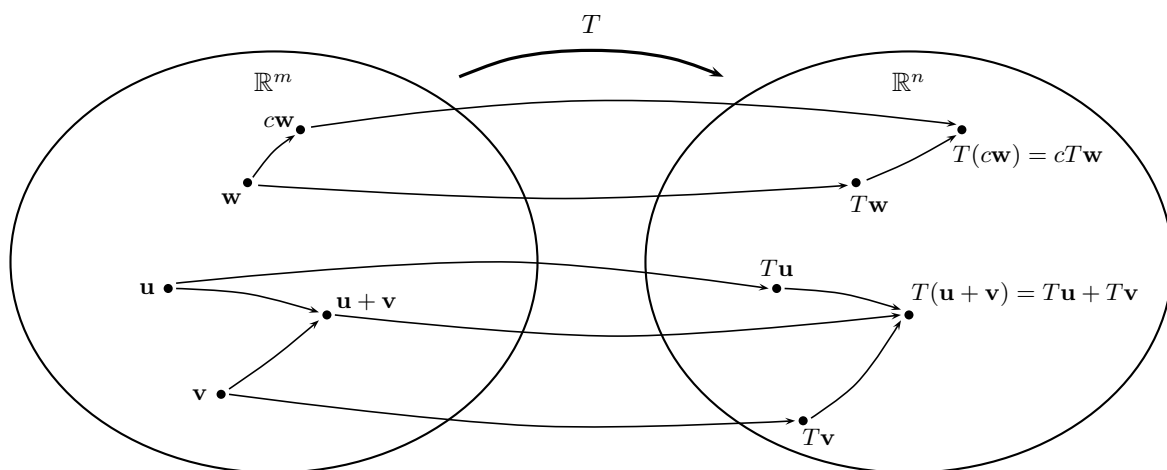
Note that $T(\mathbf{u} + \mathbf{v}) = T\mathbf{u} + T\mathbf{v}$. Not all transformations have this property, but those that do have it, along with an additional property, are very important:

DEFINITION 10.2.1: Linear Transformation

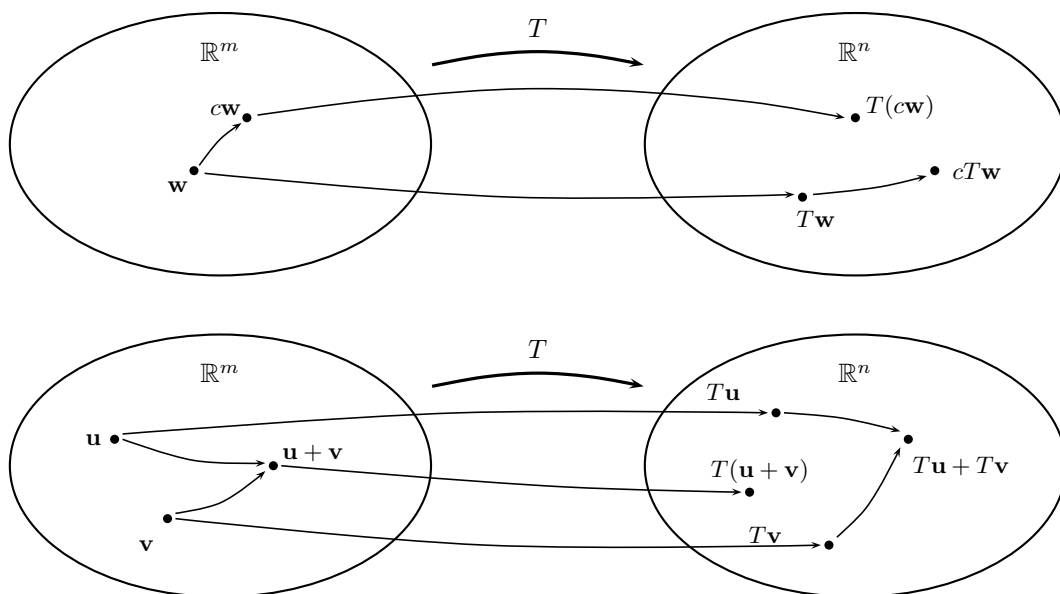
A transformation $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is called a **linear transformation** if, for every scalar c and every pair of vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^m

- 1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and
- 2) $T(c\mathbf{u}) = cT(\mathbf{u})$.

Note that the above statement describes how a transformation T interacts with the two operations of vectors, addition and scalar multiplication. It tells us that if we take two vectors in the domain and add them in the domain, then transform the result, we will get the same thing as if we transform the vectors individually first, then add the results in the codomain. We will also get the same thing if we multiply a vector by a scalar and then transform as we will if we transform first, *then* multiply by the scalar. This is illustrated in the *mapping diagram* at the top of the next page.



The following two mapping diagrams are for transformations R and S that *ARE NOT* linear:



◇ **Example 10.2(a):** Let A be an $m \times n$ matrix. Is $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $T_A \mathbf{x} = A\mathbf{x}$ a linear transformation?

We know from properties of multiplying a vector by a matrix that

$$T_A(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T_A\mathbf{u} + T_A\mathbf{v}, \quad T_A(c\mathbf{u}) = A(c\mathbf{u}) = cA\mathbf{u} = cT_A\mathbf{u}.$$

Therefore T_A is a linear transformation. ♠

◇ **Example 10.2(b):** Is $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \\ x_1^2 \end{bmatrix}$ a linear transformation? If so, show that it is; if not, give a counterexample demonstrating that.

A good way to begin such an exercise is to try the two properties of a linear transformation for some specific vectors and scalars. If either condition is not met, then we have our counterexample, and if both hold we need to show

they hold in general. Usually it is a bit simpler to check the condition $T(c\mathbf{u}) = cT\mathbf{u}$. In our case, if $c = 2$ and $\text{vecu} = [2, 3, 4]$,

$$T\left(2\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = T\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ 36 \end{bmatrix} \quad \text{and} \quad 2T\begin{bmatrix} 3 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 7 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \\ 18 \end{bmatrix}$$

Because $T(c\mathbf{u}) \neq cT\mathbf{u}$ for our choices of c and \mathbf{u} , T is not a linear transformation. ♠

The next example shows the process required to show in general that a transformation is linear.

◇ **Example 10.2(c):** Determine whether $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \end{bmatrix}$ is linear. If it is, prove it in general; if it isn't, give a specific counterexample.

First let's try a specific scalar $c = 2$ and two specific vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix}$. (I threw the negative in there just in case something funny happens when everything is positive.) Then

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix}\right) = T\begin{bmatrix} 5 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \end{bmatrix}$$

and

$$T\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + T\begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -11 \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \end{bmatrix}$$

so the first condition of linearity *appears* to hold. Let's prove it in general. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be arbitrary (that is, randomly selected) vectors in \mathbb{R}^3 . Then

$$\begin{aligned} T(\mathbf{u} + \mathbf{v}) &= T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = T\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 + u_2 + v_2 \\ (u_2 + v_2) - (u_3 + v_3) \end{bmatrix} = \\ &= \begin{bmatrix} u_1 + u_2 + v_1 + v_2 \\ (u_2 - u_3) + (v_2 - v_3) \end{bmatrix} = \begin{bmatrix} u_1 + u_2 \\ u_2 - u_3 \end{bmatrix} + \begin{bmatrix} v_1 + v_2 \\ v_2 - v_3 \end{bmatrix} = T\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + T\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = T(\mathbf{u}) + T(\mathbf{v}) \end{aligned}$$

This shows that the first condition of linearity holds in general. Let \mathbf{u} again be arbitrary, along with the scalar c . Then

$$\begin{aligned} T(c\mathbf{u}) &= T\left(c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = T\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = \begin{bmatrix} cu_1 + cu_2 \\ cu_2 - cu_3 \end{bmatrix} = \\ &= \begin{bmatrix} c(u_1 + u_2) \\ c(u_2 - u_3) \end{bmatrix} = c\begin{bmatrix} u_1 + u_2 \\ u_2 - u_3 \end{bmatrix} = cT\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = cT(\mathbf{u}) \end{aligned}$$

so the second condition holds as well, and T is a linear transformation. ♠

There is a handy fact associated with linear transformations:

THEOREM 10.2.2: If T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$.

Note that this does not say that if $T(\mathbf{0}) = \mathbf{0}$, then T is a linear transformation, as you will see below. However, the contrapositive of the above statement tells us that *if $T(\mathbf{0}) \neq \mathbf{0}$, then T is not a linear transformation*.

When working with coordinate systems, one operation we often need to carry out is a **translation**, which means a shift of all points the same distance and direction. The transformation in the following example is a translation in \mathbb{R}^2 .

◇ **Example 10.2(d):** Let a and b be any real numbers, with not both of them zero. Define $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + a \\ x_2 + b \end{bmatrix}$. Is T a linear transformation?

Since $T \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (since not both a and b are zero), T is not a linear transformation. ♠

◇ **Example 10.2(e):** Determine whether $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 x_2 \end{bmatrix}$ is linear. If it is, prove it in general; if it isn't, give a specific counterexample.

It is easy to see that $T(\mathbf{0}) = \mathbf{0}$, so we can't immediately rule out T being linear, as we did in the last example. Let's do a quick check of the first condition of the definition of a linear transformation with an example. Let $\mathbf{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. Then

$$T(\mathbf{u} + \mathbf{v}) = T \left(\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right) = T \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -12 \end{bmatrix}$$

and

$$T\mathbf{u} + T\mathbf{v} = T \begin{bmatrix} -3 \\ 2 \end{bmatrix} + T \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \end{bmatrix} + \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Clearly $T(\mathbf{u} + \mathbf{v}) \neq T\mathbf{u} + T\mathbf{v}$, so T is not a linear transformation. ♠

Recall from Example 10.2(a) that if A be an $m \times n$ matrix, then $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation. It turns out that the converse of this is true as well:

THEOREM 10.2.3: Matrix of a Linear Transformation

If $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear transformation, then there is a matrix A such that $T(\mathbf{x}) = A(\mathbf{x})$ for every \mathbf{x} in \mathbb{R}^m . We will call A the matrix that represents the transformation.

As it is cumbersome and confusing to represent a linear transformation by the letter T and the matrix representing the transformation by the letter A , we will instead adopt the following convention: We'll denote the transformation itself by T , and the matrix of the transformation by $[T]$.

◇ **Example 10.2(f):** Find the matrix $[T]$ of the linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ of Example 10.2(c), defined by $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \end{bmatrix}$.

We can see that $[T]$ needs to have three columns and two rows in order for the multiplication to be defined, and that we need to have

$$\begin{bmatrix} _ & _ & _ \\ _ & _ & _ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 - x_3 \end{bmatrix}$$

From this we can see that the first row of the matrix needs to be $1 \ 1 \ 0$ and the second row needs to be $0 \ 1 \ -1$. The matrix representing T is then $[T] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$. ♠

The sort of “visual inspection” method used above can at times be inefficient, especially when trying to find the matrix of a linear transformation based on a geometric description of the action of the transformation. To see a more effective method, let’s look at any linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Suppose that the matrix of the transformation is $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then for the two standard basis vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$T\mathbf{e}_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad T\mathbf{e}_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}.$$

This indicates that the columns of $[T]$ are the vectors $T\mathbf{e}_1$ and $T\mathbf{e}_2$. In general we have the following:

THEOREM 10.2.4

Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m$ be the standard basis vectors of \mathbb{R}^m , and suppose that $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear transformation. Then the columns of $[T]$ are the vectors obtained when T acts on each of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m$. We indicate this by

$$[T] = [T\mathbf{e}_1 \ T\mathbf{e}_2 \ \cdots \ T\mathbf{e}_m]$$

◇ **Example 10.2(g):** Let T be the transformation in \mathbb{R}^2 that rotates all vectors counterclockwise by ninety degrees. This is a linear transformation; use the previous theorem to determine its matrix $[T]$.

It should be clear that $T\mathbf{e}_1 = T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T\mathbf{e}_2 = T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Then

$$[T] = [T\mathbf{e}_1 \ T\mathbf{e}_2] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \spadesuit$$

Section 10.2 Exercises

- Two transformations from \mathbb{R}^3 to \mathbb{R}^2 are given below. One is linear and one is not. For the one that is, give the matrix of the transformation. For the one that is not, give a *specific* counterexample showing that the transformation violates the definition of a linear transformation. ($T(c\mathbf{u}) = cT\mathbf{u}$, $T(\mathbf{u} + \mathbf{v}) = T\mathbf{u} + T\mathbf{v}$)

(a) $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_3 \\ x_1 + x_2 + x_3 \end{bmatrix}$

(b) $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1x_2 + x_3 \\ x_1 \end{bmatrix}$

2. For each of the following transformations, give

- the matrix of the transformation if it is linear,
- a **specific** counterexample showing that it is not linear, if it is not.

(a) The transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_3 \\ x_1 + x_2 \end{bmatrix}$.

(b) The transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 1 \\ x_2 - 1 \end{bmatrix}$

3. Two transformations from \mathbb{R}^3 to \mathbb{R}^2 are given below. One is linear and one is not. For the one that is, give the matrix of the transformation. For the one that is not, give a *specific* counterexample showing that the transformation violates the definition of a linear transformation. ($T(c\mathbf{u}) = cT\mathbf{u}$, $T(\mathbf{u} + \mathbf{v}) = T\mathbf{u} + T\mathbf{v}$)

(a) $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_3 \\ x_1 + x_2 + x_3 \end{bmatrix}$

(b) $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1x_2 + x_3 \\ x_1 \end{bmatrix}$

4. For each of the following transformations, give

- the matrix of the transformation if it is linear,
- a **specific** counterexample showing that it is not linear, if it is not.

(a) The transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_3 \\ x_1 + x_2 \end{bmatrix}$.

(b) The transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 1 \\ x_2 - 1 \end{bmatrix}$

5. (a) The transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_2 - 5x_1 \\ x_1 \end{bmatrix}$ is linear. Show that, for

vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $T(\mathbf{u} + \mathbf{v}) = T\mathbf{u} + T\mathbf{v}$. Do this via a string of equal expressions, beginning with $T(\mathbf{u} + \mathbf{v})$ and ending with $T\mathbf{u} + T\mathbf{v}$.

(b) Give the matrix for the transformation from part (a).

6. For each of the following, a transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by describing its action on a vector $\mathbf{x} = [x_1, x_2]$. For each transformation, determine whether it is linear by

- finding $T(c\mathbf{u})$ and $c(T\mathbf{u})$ and seeing if they are equal,
- finding $T(\mathbf{u} + \mathbf{v})$ and $T(\mathbf{u}) + T(\mathbf{v})$ and seeing if they are equal.

For any that you find to be linear, say so. For any that are not, say so and produce a **specific** counterexample to one of the two conditions for linearity.

(a) $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 + x_2 \end{bmatrix}$

(b) $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1x_2 \end{bmatrix}$

(c) $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}$

(d) $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ x_1 - x_2 \end{bmatrix}$

7. For each of the transformations from the previous exercise that are linear, give the standard matrix of the transformation.