10.3 Compositions of Transformations

Performance Criterion:

10. (f) Find the composition of two transformations.

It is likely that at some point in your past you have seen the concept of the composition of two functions; if the functions were denoted by f and g, one composition of them is the new function $f \circ g$. We call this new function "f of g", and we must describe how it works. This is simple - for any x, $(f \circ g)(x) = f[g(x)]$. That is, g acts on x, and f then acts on the result. There is another composition, $g \circ f$, which is defined the same way (but, of course, in the opposite order). For specific functions, you were probably asked to find the new rule for these two compositions. Here's a reminder of how that is done:

◇ **Example 10.3(a):** For the functions $f(x) = 2x - 1$ and $g(x) = 4x - x^2$, find the formulas for the composition functions $f \circ g$ and $g \circ f$.

Basic algebra gives us

$$(f \circ g)(x) = f[g(x)] = f[4x - x^2] = 2(4x - x^2) - 1 = 8x - 2x^2 - 1 = -2x^2 + 8x - 1$$

and

$$(g \circ f)(x) = g[f(x)] = g[2x - 1] = 4(2x - 1) - (2x - 1)^2 =$$
$$(8x - 4) - (4x^2 - 4x + 1) = 8x - 4 - 4x^2 + 4x - 1 = -4x^2 + 12x - 5$$

The formulas are then $(f \circ g)(x) = -2x^2 + 8x - 1$ and $(g \circ f)(x) = -4x^2 + 12x - 5$. ♠

Worthy of note here is that the two compositions $f \circ g$ and $g \circ f$ are not the same!

One thing that was probably glossed over when you first saw this concept was the fact that the range (all possible outputs) of the first function to act must fall within the domain (allowable inputs) of the second function to act. Suppose, for example, that $f(x) = \sqrt{x - 4}$ and $g(x) = x^2$. The function f will be undefined unless x is at least four; we indicate this by writing $f : [4, \infty) \to \mathbb{R}$. This means that we need to restrict g in such a way as to make sure that $g(x) \geq 4$ if we wish to form the composition $f \circ g$. One simple way to do this is to restrict the domain of g to $[2, \infty)$. (We could include the interval $(-\infty, -2]$ also, but for the sake of simplicity we will just use the positive interval.) The range of g is then $[4, \infty)$, which coincides with the domain of f. We now see how these ideas apply to transformations, and we see how to carry out a process like that of Example 10.3(a) for transformations.

◇ **Example 10.3(b):** Let $S : \mathbb{R}^3 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$S\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1^2 \\ x_2x_3 \end{bmatrix}, \qquad T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_2 - x_1 \end{bmatrix}$$

Determine whether each of the compositions $S \circ T$ and $T \circ S$ exists, and find a formula for either of them that do.

Since the domain of S is $\mathbb{R}^3$ and the range of T is a subset of $\mathbb{R}^2$, the composition $S \circ T$ does not exist. The range of S falls within the domain of T, so the composition $T \circ S$ does exist. Its equation is found by

$$(T \circ S)(\mathbf{x}) = T\left(S\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\begin{bmatrix} x_1^2 \\ x_2x_3 \end{bmatrix} = \begin{bmatrix} x_1^2 + 3x_2x_3 \\ 2x_2x_3 - x_1^2 \end{bmatrix} \quad ♠$$

Let's formally define what we mean by a composition of two transformations.

<u>DEFINITION 10.3.1</u> **Composition of Transformations**

Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^m \to \mathbb{R}^p$ be transformations. The **composition** of S and T, denoted by $S \circ T$, is the transformation $S \circ T : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$(S \circ T)\,\mathbf{x} = S(T\mathbf{x})$$

for all vectors $\mathbf{x}$ in $\mathbb{R}^m$.

Although the above definition is valid for compositions of any transformations between vector spaces, we are primarily interested in linear transformations. Recall that any linear transformation between vector spaces can be represented by matrix multiplication for some matrix. Suppose that $S : \mathbb{R}^3 \to \mathbb{R}^3$ and $T : \mathbb{R}^2 \to \mathbb{R}^3$ are linear transformations that can be represented by the matrices

$$[S] = \begin{bmatrix} 3 & -1 & 5 \\ 0 & 2 & 1 \\ 4 & 0 & -3 \end{bmatrix} \qquad \text{and} \qquad [T] = \begin{bmatrix} 2 & 7 \\ -6 & 1 \\ 1 & -4 \end{bmatrix}$$

respectively.

◇ **Example 10.3(c):** For the transformations S and T just defined, find $(S \circ T)\,\mathbf{x} = (S \circ T)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then find the matrix of the transformation $S \circ T$.

We see that

$$\begin{aligned}
(S \circ T)\,\mathbf{x} = S(T\mathbf{x}) &= S\left(\begin{bmatrix} 2 & 7 \\ -6 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \\
&= S\begin{bmatrix} 2x_1 + 7x_2 \\ -6x_1 + x_2 \\ x_1 - 4x_2 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -1 & 5 \\ 0 & 2 & 1 \\ 4 & 0 & -3 \end{bmatrix}\begin{bmatrix} 2x_1 + 7x_2 \\ -6x_1 + x_2 \\ x_1 - 4x_2 \end{bmatrix} \\
&= \begin{bmatrix} 3(2x_1 + 7x_2) - (-6x_1 + x_2) + 5(x_1 - 4x_2) \\ 0(2x_1 + 7x_2) + 2(-6x_1 + x_2) + (x_1 - 4x_2) \\ 4(2x_1 + 7x_2) + 0(-6x_1 + x_2) - 3(x_1 - 4x_2) \end{bmatrix} \\
&= \begin{bmatrix} 17x_1 + 0x_2 \\ -11x_1 - 2x_2 \\ 5x_1 + 40x_2 \end{bmatrix}
\end{aligned}$$

From this we can see that the matrix of $S \circ T$ is $[S \circ T] = \begin{bmatrix} 17 & 0 \\ -11 & -2 \\ 5 & 40 \end{bmatrix}$. ♠

Recall that the linear transformations of this example have matrices $[S]$ and $[T]$, and we find that

$$[S][T] = \begin{bmatrix} 3 & -1 & 5 \\ 0 & 2 & 1 \\ 4 & 0 & -3 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ -6 & 1 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ -11 & -2 \\ 5 & 40 \end{bmatrix}.$$

This illustrates the following:

> <u>THEOREM 10.3.2</u> **Matrix of a Composition**
>
> Let $S : \mathbb{R}^p \to \mathbb{R}^n$ and $T : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations with matrices $[S]$ and $[T]$. Then
> $$[S \circ T] = [S][T]$$

Section 10.3 Exercises

1. Consider the linear transformations $S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ 2x \\ -3y \end{bmatrix}$, $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x-y \\ x+4y \end{bmatrix}$.

   (a) Since both of these are linear transformations, there are matrices A and B representing them. Give those two matrices (A for S, B for T).

   (b) Give equations for either (or both) of the compositions $S \circ T$ and $T \circ S$ that exist.

   (c) Give the matrix for either (or both) of the compositions that exist.

   (d) Find either (or both) of AB and BA that exist.

   (e) What did you notice in parts (c) and (d)? **Answer this with a complete sentence.**