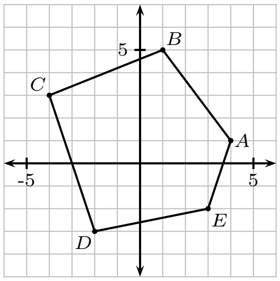
10.4 Transformations in $\mathbb{R}^2$, Homogeneous Coordinates

Performance Criteria:

10. (g) Give the vertex matrix of a polygon in $\mathbb{R}^2$ using either standard coordinates or homogeneous coordinates.
    (h) Give the matrix of a dilation, rotation, reflection or translation in $\mathbb{R}^2$ using homogeneous coordinates.
    (i) Give the matrix that can be used to perform composition of dilations, rotations, reflections and translations to transform a polygon P to another polygon P' that is similar to P.

In Section 5.3 we studied dilations, rotations, reflections and projections in $\mathbb{R}^2$. In this section we continue that study, with the addition of another type of transformation called a **translation**. As we will see, translations are not linear transformations, so they can't be represented by matrices. A clever idea allows us to get around that problem by working in a horizontal plane one unit above the xy-pane in $\mathbb{R}^3$.



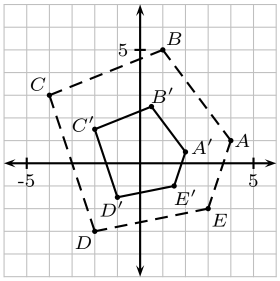
First we introduce a representation for a polygonal object in $\mathbb{R}^2$ like the pentagon $ABCDE$ shown to the right. We simply consider position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$ representing the vertices of the polygon and form a 2×5 matrix P (for polygon, not to be confused with our representation of a point) by augmenting the vector $\mathbf{a}$ with each of the remaining vectors, in order. So we have

$$P = [\,\mathbf{a}\,|\,\mathbf{b}\,|\,\mathbf{c}\,|\,\mathbf{d}\,|\,\mathbf{e}\,] = \begin{bmatrix} 4 & 1 & -4 & -2 & 3 \\ 1 & 5 & 3 & -3 & -2 \end{bmatrix}$$

We will call this the vertex matrix of the polygon.

If we recall the linear combination form of a matrix times a vector and the linear combination form of matrix multiplication, we can see that if we multiply a transformation matrix times a vertex matrix, it applies the transformation to each of the vertices of the polygon, which are represented by columns in the vertex matrix. This results in a new polygon matrix P' which describes the transformed polygon. The following examples demonstrate this.

◇ **Example 10.4(a):** Apply a dilation by $\frac{1}{2}$ to the polygon matrix P above and plot the resulting polygon P'.



The matrix that performs a dilation by $\frac{1}{2}$ is $D = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. Applying this to P we get

$$P' = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 & 1 & -4 & -2 & 3 \\ 1 & 5 & 3 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 2.0 & 0.5 & -2.0 & -1.0 & 1.5 \\ 0.5 & 2.5 & 1.5 & -1.5 & -1 \end{bmatrix}$$

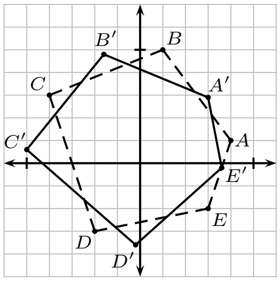
The transformed polygon is shown to the right as $A'B'C'D'E'$, with the original polygon P shown with dashed sides. ♠

◇ **Example 10.4(b):** Apply a rotation of 30° counterclockwise to the polygon matrix P above and plot the resulting polygon P'.

The matrix $A = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ performs a counterclockwise rotation of 30°. Applying this to P we get

$$P' = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 4 & 1 & -4 & -2 & 3 \\ 1 & 5 & 3 & -3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3.0 & -1.6 & -5.0 & -0.2 & 3.6 \\ 2.9 & 4.8 & 0.6 & -3.6 & -0.2 \end{bmatrix}$$

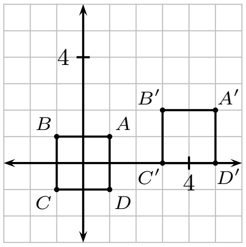


The transformed polygon is shown to the right, with the original shown with dashed sides. ♠

In the next example we see a translation, the one transformation we need that we haven't really looked at yet.

◇ **Example 10.4(c):** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3 \\ x_2 + 1 \end{bmatrix}$. Apply T to the square $ABCD$ with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$ and $(1,-1)$, respectively. Plot the square $ABCD$ and the transformed square $A'B'C'D'$ on the same coordiante grid.

The transformed vertices are given by the position vectors



$$\mathbf{a}' = T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \qquad \mathbf{b}' = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

$$\mathbf{c}' = T\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad \mathbf{d}' = T\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

The original and transformed squares are shown to the right. ♠

The general form of a translation in $\mathbb{R}^2$ is $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + a \\ x_2 + b \end{bmatrix}$, where a and b be any real numbers, with not both of them zero. (Why not?) As pointed out in Example 10.2(d), $T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so T is not a linear transformation. Therefore T cannot be performed by multiplication by a matrix.

We would like to remedy this unfortunate situation, and some clever mathematicians of the past came up with a way to do it. To each point P in $\mathbb{R}^2$ with position vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ we assign a vector $\mathbf{u}_h = \begin{bmatrix} u_1 \\ u_2 \\ 1 \end{bmatrix}$ whose components are called the **homogeneous coordinates** of P. (We will relax our language a bit to refer to the vector $\mathbf{u}_h$ as the homogeneous coordinates.) We can then see that the matrix $[T]_h = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$ acts on $\mathbf{u}_h$ to give the homogenous coordinates of $T\mathbf{u}$:

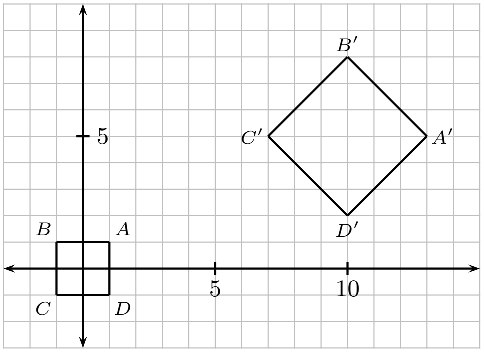
$$[T]_h\mathbf{u}_h = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ 1 \end{bmatrix} = \begin{bmatrix} u_1 + a \\ u_2 + b \\ 1 \end{bmatrix} = (T\mathbf{u})_h$$

◇ **Example 10.4(d):** Give the homogeneous coordinates $\mathbf{v}_h$ of the point $Q(-3,2)$ in $\mathbb{R}^2$ and the matrix $[T]_h$ for the translation from Example 10.4(c). Then give the point Q' resulting from translating Q with that translation.

$$\mathbf{v}_h = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}, \qquad [T]_h = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [T]_h\mathbf{v}_h = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$ ♠

When performing a rotation or dilation in homogeneous coordinates we add a row and a column to the 2×2 rotation or dilation matrix. Each (the row and the column) consists of two zeros followed by a one.

Suppose now that we wish to transform the square $ABCD$ to the square $A'B'C'D'$, both shown on the coordinate grid to the right. We can see that this will require a dilation, a rotation and a translation. The translation must be done last, for the reason that will be shown in Example 10.4(f). The order of the rotation and dilation does not matter, but *both must be done while the square is still centered at the origin.* So we can find matrices $[D]$, $[R]$ and $[T]$ that perform the dilation, rotation and translation and compose them to obtain a single matrix that will transform $ABCD$ to $A'B'C'D'$. We do this in the next example.



◇ **Example 10.4(e):** Create a single matrix $[S]$ that will transform the square $ABCD$ to the square $A'B'C'D'$.

It is clear from the picture that the rotation is $45°$ clockwise and the translation is ten units in the x_1-direction and five units in the x_2-direction. What is more difficult is determining the amount of the dilation. From the Pythagorean theorem or some trigonometry, each side of the square $A'B'C'D'$ has length $3\sqrt{2}$. Letting d be the factor of the dilation, we then have $2d = 3\sqrt{2}$ and $d = \frac{3\sqrt{2}}{2}$. So the rotation, dilation and translation matrices are then

$$[R] = \begin{bmatrix} \cos(-45°) & -\sin(-45°) & 0 \\ \sin(-45°) & \cos(-45°) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$[D] = \begin{bmatrix} \frac{3\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{3\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [T] = \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}.$$

As stated before, the translation must be performed last. Let's do the dilation first, followed by the rotation and the translation, to find the matrix $[S]$ that performs the entire transformation:

$$\begin{aligned} [S] = [T \circ R \circ D] &= \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{3\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & 0 \\ -\frac{3}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & 10 \\ -\frac{3}{2} & \frac{3}{2} & 5 \\ 0 & 0 & 1 \end{bmatrix} \quad \spadesuit \end{aligned}$$

Now lets see if our matrix "works!" The original polygon has vertex matrix $P = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$. Clearly the dimensions of this matrix will not allow multiplication by the matrix $[S]$. What we need is to have a *homogeneous* vertex we'll call P_h, which is just P with a row of ones added to the bottom. Then we see that

$$[S]P_h = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & 10 \\ -\frac{3}{2} & \frac{3}{2} & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 10 & 7 & 10 \\ 5 & 8 & 5 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} = P'_h,$$

where P'_h is the homogeneous vertex matrix for the transformed polygon $A'B'C'D'$. Thus our matrix $[S]$ performs the desired transformation.