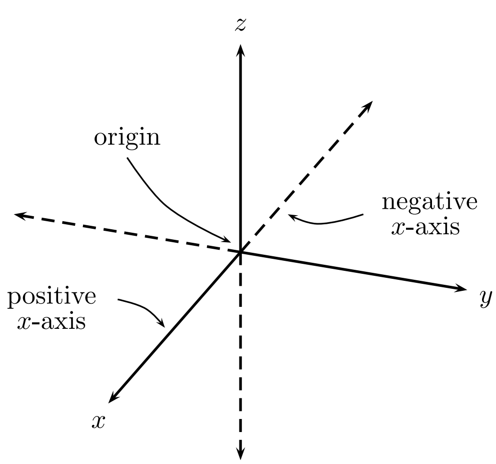
3.1 Euclidean Space

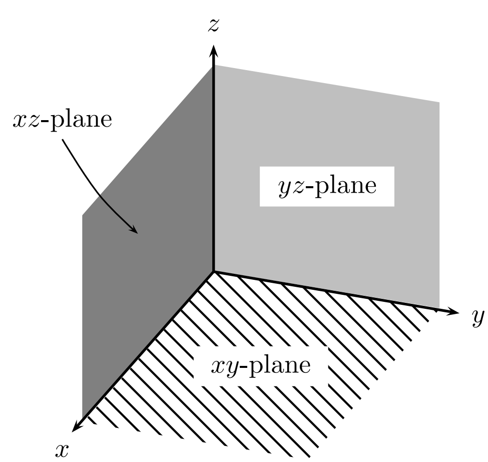
> **Performance Criteria:**
>
> 3. (a) Recognize the equation of a plane in $\mathbb{R}^3$ and determine where the plane intersects each of the three axes. Give the equation of any one of the three coordinate planes or any plane parallel to one of the coordinate planes.
>
>   (b) Find the distance between two points in $\mathbb{R}^n$.

It is often taken for granted that everyone knows what we mean by the **real numbers**. To actually define the real numbers precisely is a bit of a chore and very technical. Suffice it to say that the real numbers include all numbers other than complex numbers (numbers containing $\sqrt{-1} = i$ or, for electrical engineers, j) that a scientist or engineer is likely to run into. The numbers 5, -31.2, π, $\sqrt{2}$, $\frac{2}{7}$, and e are all real numbers. We denote the set of all real numbers with the symbol $\mathbb{R}$, and the geometric representation of the real numbers is the familiar **real number line**, a horizontal line on which every real number has a place. This is possible because the real numbers are ordered: given any two real numbers, either they are equal to each other, one is less than the other, or vice-versa.

As mentioned previously, the set $\mathbb{R}^2$ is the set of all ordered pairs of real numbers. Geometrically, every such pair corresponds to a point in the **Cartesian plane**, which is the familiar xy-plane. $\mathbb{R}^3$ is the set of all ordered triples, each of which represents a point in three-dimensional space. We can continue on - $\mathbb{R}^4$ is the set of all ordered "4-tuples", and can be thought of geometrically as four dimensional space. Continuing further, an "n-tuple" is n real numbers, in a specific order; each n-tuple can be thought of as representing a point in n-dimensional space. These spaces are sometimes called "two-space," "three-space" and "n-space" for short.

Two-space is fairly simple, with the only major features being the two axes and the four quadrants that the axes divide the space into. Three-space is a bit more complicated. Obviously there are three coordinate axes instead of two. In addition to those axes, there are also three coordinate planes as well, the xy-plane, the xz-plane and the yz-plane. Finally the three coordinate planes divide the space into eight **octants**. The pictures below illustrate the coordinate axes and planes. The first octant is the one we are looking into, where all three coordinates are positive. It is not important that we know the numbering of the other octants.





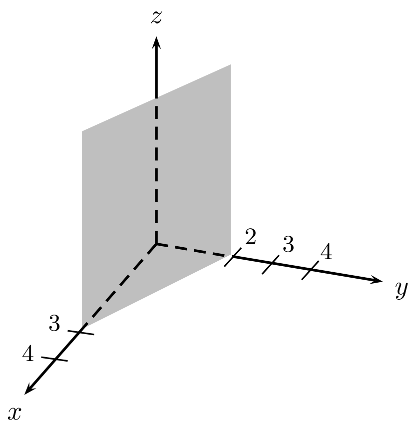
Every plane in $\mathbb{R}^3$ (we will be discussing only $\mathbb{R}^3$ for now) consists of a set of points that behave in an orderly mathematical manner, described here:

> **Equation of a Plane in $\mathbb{R}^3$:** The ordered triples corresponding to all the points in a plane satisfy an equation of the form
>
> $$ax + by + cz = d,$$
>
> where a, b, c and d are constants, and not all of a, b and c are zero.

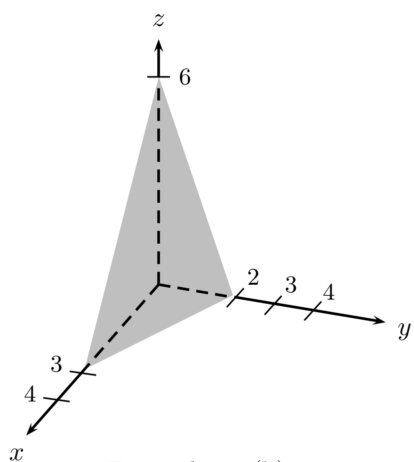
The xy-plane is the plane containing the x and y-axes. The only condition on points in that plane is that $z = 0$, so that is the equation of that plane. (Here the constants a, b and d are all zero, and $c = 1$.) The plane $z = 5$ is a horizontal plane that is five units above the xy-plane.

◇ **Example 3.1(a):** Graph the equation $2x + 3y = 6$ in the first octant. Indicate clearly where it intersects each of the coordinate axes, if it does.

Some points that satisfy the equation are $(3, 0, 0)$, $(6, -2, 5)$, and so on. Since z is not included in the equation, there are no restrictions on z; it can take any value. If we were to fix z at zero and plot all points that satisfy the equation, we would get a line in the xy-plane through the two points $(3, 0, 0)$ and $(0, 2, 0)$. These points are obtained by first letting y and z be zero, then by letting x and z be zero. Since z can be anything, the set of points satisfying $2x + 3y = 6$ is a vertical plane intersecting the xy-plane in that line. The plane is shown below and to the left. ♠



Example 3.1(a)



Example 3.1(b)

◇ **Example 3.1(b):** Graph the equation $2x + 3y + z = 6$ in the first octant. Indicate clearly where it intersects each of the coordinate axes, if it does.

The set of points satisfying this equation is also a plane, but z is no longer free to take any value. The simplest way to "get a handle on" such a plane is to find where it intercepts the three axes. Note that every point on the x-axis has y- and z-coordinates of zero. So to find where the plane intersects the x-axis we put zero into the equation for y and z, then solve for x, getting $x = 3$. The plane then intersects the x-axis at $(3, 0, 0)$. A similar process gives us that the plane intersects the y and z axes at $(0, 2, 0)$ and $(0, 0, 6)$. The graph of the plane is shown in the drawing above and to the right. ♠

Consider now a system of equations like

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned} \quad ,$$

which has solution $(3, -1, 2)$. Each of the three equations represents a plane in $\mathbb{R}^3$, and the point $(3, -1, 2)$ is where the three planes intersect. This is completely analogous to the interpretation of the solution of a system of two linear equations in two unknowns as the point where the two lines representing the equations cross. This is the first of three interpretations we'll have for the solution to a system of equations.

The only other basic geometric fact we need about three-space is this:

Distance Between Points: The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbb{R}^n$ is given by

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

This is simply a three-dimensional version of the Pythagorean Theorem. This is in fact used in even higher dimensional spaces; even though we cannot visualize the distance geometrically, this idea is both mathematically valid and useful.

⋄ **Example 3.1(c):** Find the distance in $\mathbb{R}^3$ between the points $(-4,7,1)$ and $(13,0,-6)$.

Using the above formula we get

$$d = \sqrt{(-4-13)^2+(7-0)^2+(1-(-6))^2} = \sqrt{(-17)^2+7^2+7^2} = \sqrt{387} \approx 19.7 \quad ♠$$

We will now move on to our main tool for working in Euclidean space, vectors.

Section 3.1 Exercises

1. Determine whether each of the equations given describes a plane in $\mathbb{R}^3$. If not, say so. If it does describe a plane, give the points where it intersects each axis. If it doesn't intersect an axis, say so.

   (a) $-2x-y+3z=-6$ (b) $x+3z=6$ (c) $y=-6$

   (d) $x+3z^2=12$ (e) $x-2y+3z=-6$

2. Give the equation of a plane in $\mathbb{R}^3$ that does not intersect the y-axis but does intersect the other two axes. Give the points at which it intersects the x- and z-axes.

3. Give the equation of the plane that intersects the y-axis at 4 and does not intersect either of the other two axes.