3.3 Operations With Vectors, Linear Combinations

Performance Criteria:

3. (d) Multiply vectors by scalars and add vectors, algebraically. Find linear combinations of vectors algebraically.
   (e) Illustrate the parallelogram method and tip-to-tail method for finding a linear combination of two vectors.
   (f) Find a linear combination of vectors equalling a given vector.

In the previous section a vector $\mathbf{x} = [x_1, x_2, ..., x_n]$ in n dimensions was starting to look suspiciously like an n-tuple $(x_1, x_2, ..., x_n)$ and we established a correspondence between any point and the position vector with its tip at that point. One might wonder why we bother then with vectors at all! The reason is that we can perform algebraic operations with vectors that make sense physically, and such operations make no sense with n-tuples. The two most basic things we can do with vectors are add two of them or multiply one by a scalar, and both are done component-wise:

<u>Definition 3.3.1</u>: **Addition and Scalar Multiplication of Vectors**

Let $\mathbf{u} = [u_1, u_2, ..., u_n]$ and $\mathbf{v} = [v_1, v_2, ..., n_n]$, and let c be a scalar. Then we define the vectors $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ by

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} \quad \text{and} \quad c\mathbf{u} = c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$$

Note that result of adding two vectors or multiplying a vector by a scalar is also a vector. It clearly follows from these that we can get subtraction of vectors by first multiplying the second vector by the scalar -1, then adding the vectors. With just a little thought you will recognize that this is the same as just subtracting the corresponding components.

◇ **Example 3.3(a):** For $\mathbf{u} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix}$, find $\mathbf{u}+\mathbf{v}$, $3\mathbf{u}$ and $\mathbf{u}-\mathbf{v}$.

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 5+(-4) \\ -1+\ 9 \\ 2\ +\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 8 \end{bmatrix}$$

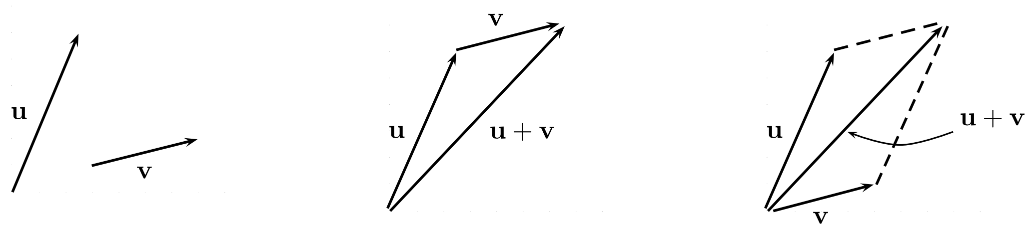
$$3\mathbf{u} = 3\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3(5) \\ 3(-1) \\ 3(2) \end{bmatrix} = \begin{bmatrix} 15 \\ -3 \\ 6 \end{bmatrix}$$

$$\mathbf{u}-\mathbf{v} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 5-(-4) \\ -1-\ 9 \\ 2\ -\ 6 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ -4 \end{bmatrix} \quad ♠$$

Addition of vectors can be thought of geometrically in two ways, both of which are useful. The first way is what we will call the **tip-to-tail method**, and the second method is called the **parallelogram method**. You should become very familiar with both of these methods, as they each have their advantages; they are illustrated below.

◇ **Example 3.3(b):** Add the two vectors $\mathbf{u}$ and $\mathbf{v}$ shown below and to the left, first by the tip-to-tail method, and second by the parallelogram method.

To add using the tip-to-tail method move the second vector so that its tail is at the tip of the first. (Be sure that its length and direction remain the same!) The vector $\mathbf{u}+\mathbf{v}$ goes from the tail of $\mathbf{u}$ to the tip of $\mathbf{v}$. See in the middle below.



tip-to-tail method   parallelogram method

To add using the parallelogram method, put the vectors $\mathbf{u}$ and $\mathbf{v}$ together at their tails (again being sure to preserve their lengths and directions). Draw a dashed line from the tip of $\mathbf{u}$, parallel to $\mathbf{v}$, and draw another dashed line from the tip of $\mathbf{v}$, parallel to $\mathbf{u}$. $\mathbf{u}+\mathbf{v}$ goes from the tails of $\mathbf{u}$ and $\mathbf{v}$ to the point where the two dashed lines cross. See to the right above. The reason for the name of this method is that the two vectors and the dashed lines create a parallelogram. ♠

Each of these two methods has a natural physical interpretation. For the tip-to-tail method, imagine an object that gets *displaced* by the direction and amount shown by the vector $\mathbf{u}$. Then suppose that it gets displaced by the direction and amount given by $\mathbf{v}$ after that. Then the vector $\mathbf{u}+\mathbf{v}$ gives the *net* (total) displacement of the object. Now look at that picture for the parallelogram method, and imagine that there is an object at the tails of the two vectors. If we were then to have two forces acting on the object, one in the direction of $\mathbf{u}$ and with an amount (magnitude) indicated by the length of $\mathbf{u}$, and another with amount and direction indicated by $\mathbf{v}$, then $\mathbf{u}+\mathbf{v}$ would represent the net force. (In a statics or physics course you might call this the **resultant** force.)

A very important concept in linear algebra is that of a **linear combination**. Let me say it again:

> **Linear combinations are one of the most important concepts in linear algebra! You need to recognize them when you see them and learn how to create them. They will be central to almost everything that we will do from here on.**

A linear combination of a set of vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ (note that the subscripts now distinguish different *vectors*, not the components of a single vector) is obtained when each of the vectors is multiplied by a scalar, and the resulting vectors are added up. So if $c_1, c_2, ..., c_n$ are the scalars that $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ are multiplied by, the resulting linear combination is the *single vector* $\mathbf{v}$ given by

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n.$$

Emphasizing again the importance of this concept, let's provide a slightly more concise and formal definition:

> <u>DEFINITION 3.3.2</u>: **Linear Combination**
>
> A **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$, all in $\mathbb{R}^n$, is any vector of the form
>
> $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_n\mathbf{v}_n,$$
>
> where $c_1, c_2, ..., c_n$ are scalars.

Note that when we create a linear combination of a set of vectors we are doing virtually everything possible algebraically with those vectors, which is just addition and scalar multiplication!

You have seen this idea before; every polynomial like $5x^3 - 7x^2 + \frac{1}{2}x - 1$ is a linear combination of $1, x, x^2, x^3, ...$ Those of you who have had a differential equations class have seen things like $ds\frac{dy}{dt} + 3\frac{dy}{dt} + 2y$, which is a linear combination of the second, first and "zeroth" derivatives of a function $y = y(t)$. Here is why linear combinations are so important: In many applications we seek to have a basic set of objects (vectors) from which all other objects can be built as linear combinations of objects from our basic set. A large part of our study will be centered around this idea. This may not make any sense to you now, but hopefully it will by the end of the course.

◇ **Example 3.3(c):** For the vectors $\mathbf{v}_1 = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix}$, give the linear combination $2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$ as one vector.

$$2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3 = 2\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 4 \end{bmatrix} - \begin{bmatrix} -12 \\ 27 \\ 18 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ -26 \\ 30 \end{bmatrix}$$ ♠

◇ **Example 3.3(d):** For the same vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ as in the previous exercise and scalars c_1, c_2 and c_3, give the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ as one vector.

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5c_1 \\ -1c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} -4c_2 \\ 9c_2 \\ 6c_2 \end{bmatrix} + \begin{bmatrix} 0c_3 \\ 3c_3 \\ 8c_3 \end{bmatrix}$$

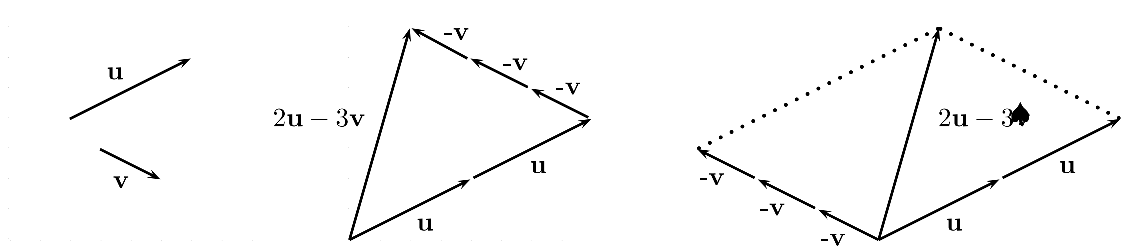
$$= \begin{bmatrix} 5c_1 - 4c_2 + 0c_3 \\ -1c_1 + 9c_2 + 3c_3 \\ 2c_1 + 6c_2 + 8c_3 \end{bmatrix}$$

Note that the final result is a single vector with three components that look suspiciously like the left sides of a system of three equations in three unknowns! ♠

In the previous two examples we found linear combinations algebraically; in the next example we find a linear combination geometrically.

◇ **Example 3.3(e):** In the space below and to the right, sketch the vector $2\mathbf{u} - 3\mathbf{v}$ for the vectors $\mathbf{u}$ and $\mathbf{v}$ shown below and to the left.

In the center below the linear combination is obtained by the tip-to-tail method, and to the right below it is obtained by the parallelogram method.



The last example is probably the most important in this section.

◇ **Example 3.3(f):** Find a linear combination of the vectors $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$ that equals the vector $\mathbf{w} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$.

We are looking for two scalars c_1 and c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{w}$. By the method of Example 3.3(d) we have

$$c_1 \begin{bmatrix} 3 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} 3c_1 \\ -4c_1 \end{bmatrix} + \begin{bmatrix} 7c_2 \\ -3c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} 3c_1 + 7c_2 \\ -4c_1 - 3c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

In the last line above we have two vectors that are equal. It should be intuitively obvious that this can only happen if the individual components of the two vectors are equal. This results in the system $\begin{array}{rcr} 3c_1 + 7c_2 & = & 1 \\ -4c_1 - 3c_2 & = & -14 \end{array}$ of two equations in the two unknowns c_1 and c_2. Solving, we arrive at $c_1 = 5,\ c_2 = -2$. It is easily verified that these are correct:

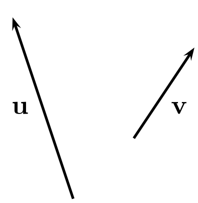
$$5\begin{bmatrix} 3 \\ -4 \end{bmatrix} - 2\begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 15 \\ -20 \end{bmatrix} - \begin{bmatrix} 14 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$ ♠

We now conclude with an important observation. Suppose that we consider all possible linear combinations of a single vector $\mathbf{v}$. That is then the set of all vectors of the form $c\mathbf{v}$ for some scalar c, which is just all scalar multiples of $\mathbf{v}$. At the risk of being redundant, the set of all linear combinations of a single vector is all scalar multiples of that vector.

Section 3.3 Exercises

1. For the two vectors **u** and **v** shown below and to the left, illustrate the tip-to-tail and parallelogram methods for finding $-\mathbf{u} + 2\mathbf{v}$ in the spaces indicated.

Tip-to-tail: Parallelogram:



2. For the vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ and $\mathbf{v}_4 = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$, give the linear combination $5\mathbf{v}_1 + 2\mathbf{v}_2 - 7\mathbf{v}_3 + \mathbf{v}_4$ as one vector.

3. For the vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \\ -6 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -8 \\ 1 \\ 4 \end{bmatrix}$, give the linear combination $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ as one vector.

4. Give a linear combination of $\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$ that equals $\begin{bmatrix} 17 \\ -4 \\ -9 \end{bmatrix}$. Demonstrate that your answer is correct by filling in the blanks:

$$\_\_\_\_ \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} + \_\_\_\_ \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} + \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} + \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} = \begin{bmatrix} 17 \\ -4 \\ -9 \end{bmatrix}$$

5. For each of the following, find a linear combination of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ that equals $\mathbf{v}$. Conclude by giving the actual linear combination, not just some scalars.

   (a) $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

   (b) $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$

6. (a) Consider the vectors $\mathbf{u}_1 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -2 \\ 6 \\ 5 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$.
   If possible, find scalars a_1, a_2 and a_3 such that $a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 = \mathbf{w}$.

   (b) Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -7 \\ 2 \\ 5 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$.
   If possible, find scalars b_1, b_2 and b_3 such that $b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 = \mathbf{w}$.

   (c) To do each of parts (a) and (b) you should have solved a system of equations. Let A be the coefficient matrix for the system in (a) and let B be the coefficient matrix for the system in part (b). Use your calculator to find $\det(A)$ and $\det(B)$, the determinants of matrices A and B. You will probably find the command for the determinant in the same menu as *rref*.