4.2 Vector Equations of Lines and Planes

> **Performance Criterion:**
>
> 4\. (c) Give the vector equation of a line through two points in $\mathbb{R}^2$ or $\mathbb{R}^3$ or the vector equation of a plane through three points in $\mathbb{R}^3$.

The idea of a linear combination does more for us than just give another way to interpret a system of equations. The set of points in $\mathbb{R}^2$ satisfying an equation of the form $y = mx+b$ is a line; any such equation can be rearranged into the form $ax + by = c$. (The values of b in the two equations are clearly not the same.) But if we add one more term to get $ax + by + cz = d$, with the (x, y, z) representing the coordinates of a point in $\mathbb{R}^3$, *we get the equation of a plane, not a line!* In fact, we cannot represent a line in $\mathbb{R}^3$ with a single *scalar* equation. The object of this section is to show how we can represent lines, planes and higher dimensional objects called *hyperplanes* using linear combinations of vectors.

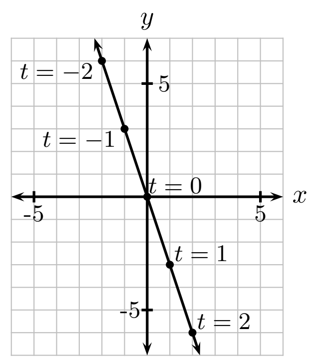
For the bulk of this course, we will think of most vectors as position vectors. (Remember, this means their tails are at the origin.) We will also think of each position vector as corresponding to the point at its tip, so the coordinates of the point will be the same as the components of the vector. Thus, for example, in $\mathbb{R}^2$ the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ corresponds to the ordered pair $(x_1, x_2) = (1, -3)$.

⋄ **Example 4.1(a):** Graph the set of points corresponding to all vectors $\mathbf{x}$ of the form $\mathbf{x} = t\begin{bmatrix} 1 \\ -3 \end{bmatrix}$, where t represents any real number.

We already know that when $t = 1$ the the vector x corresponds to the point $(1, -3)$. We then let $t = -2, -1, 0, 2$ and determine the corresponding vectors $\mathbf{x}$:

$$t = -2 \Rightarrow x = \begin{bmatrix} -2 \\ 6 \end{bmatrix}, \qquad t = -1 \Rightarrow x = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$t = 0 \Rightarrow x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad t = 2 \Rightarrow x = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$



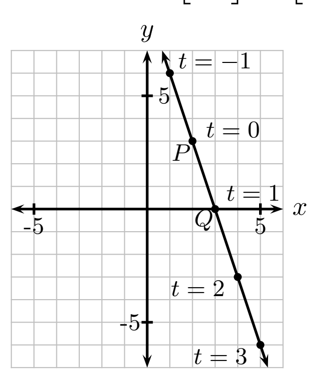
These vectors correspond to the points with ordered pairs $(-2, 6)$, $(-1, 3)$, $(0, 0)$ and $(2, -6)$. When we plot those points and the point $(1, -3)$ that we already had, we get the line shown above and to the right. ♠

It should be clear from the above example that we could create a line through the origin in any direction by simply replacing the vector $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ with a vector in the direction of the desired line. The next question is, "how do we get a line that is *not* through the origin?" The next example illustrates how this is done.

⋄ **Example 4.1(b):** Graph the set of points corresponding to all vectors $\mathbf{x}$ of the form $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} -3 \\ 1 \end{bmatrix}$, where t represents any real number.

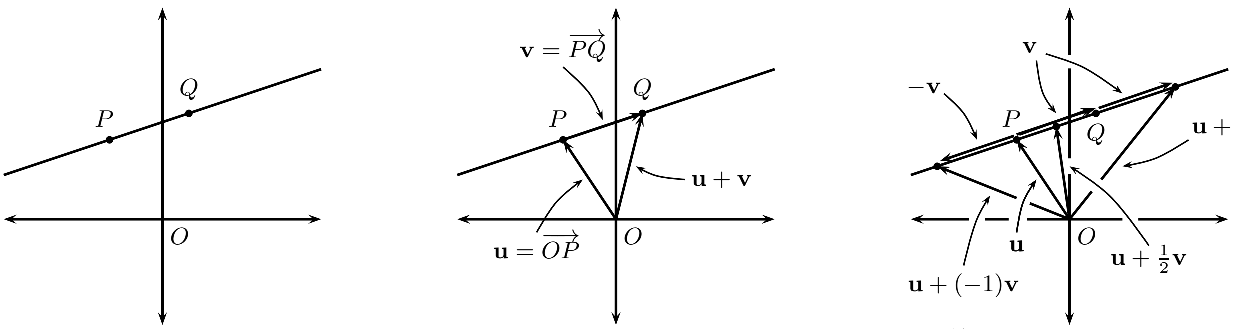
Performing the scalar multiplication by t and adding the two vectors, we get

$$\mathbf{x} = \begin{bmatrix} 2 - 3t \\ 3 + \ t \end{bmatrix}.$$

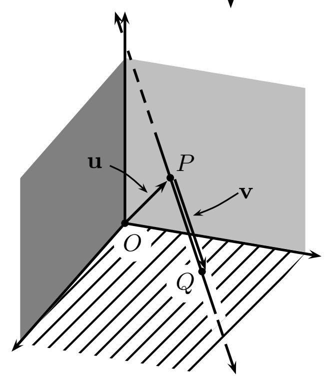


These vectors then correspond to all points of the form $(2-3t, 3+t)$. When $t = 0$ this is the point $(2, 3)$ so our line clearly passes through that point. Plotting the points obtained when we let $t = -1,\ 1,\ 2$ and 3, we see that we will get the line shown to the right. ♠

Now let's make two observations about the set of points represented by the set of all vectors $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, where t again represents any real number. These vectors correspond to the ordered pairs of the form $(4-3t, -2+t)$. Plotting these results in the line through the point $(2, 3)$ and in the direction of the vector $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. This is not a coincidence. Consider the line shown below and to the left, containing the points P and Q. If we let $\mathbf{u} = \overrightarrow{OP}$ and $\mathbf{v} = \overrightarrow{PQ}$, then the points P and Q correspond to the vectors $\mathbf{u}$ and $\mathbf{u}+\mathbf{v}$ (in standard position, which you should assume we mean from here on), as shown in the second picture. From this you should be able to see that if we consider all the vectors $\mathbf{x}$ defined by $\mathbf{x} = \mathbf{u} + t\mathbf{v}$ as t ranges over all real numbers, the resulting set of points is our line! This is shown in the third picture, where t is given the values -1, $\frac{1}{2}$ and 2.



Now this may seem like an overly complicated way to describe a line, but with a little thought you should see that the idea translates directly to three (and more!) dimensions, as shown in the picture to the right. This is all summarized below:



Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$

The equation of a line through two points P and Q in $\mathbb{R}^2$ and $\mathbb{R}^3$ is given by the vector equation

$$\mathbf{x} = \overrightarrow{OP} + t\,\overrightarrow{PQ}.$$

By this we mean that the line consists of all the points corresponding to the position vectors $\mathbf{x}$ as t varies over all real numbers. The vector $\overrightarrow{PQ}$ is called the **direction vector** of the line.

◇ **Example 4.2(c):** Give the vector equation of the line in $\mathbb{R}^2$ through the points $P(-4, 1)$ and $Q(5, 3)$.

We need two vectors, one from the origin out to the line, and one in the direction of the line. For the first we will use $\overrightarrow{OP}$, and for the second we will use $\overrightarrow{PQ} = [9, 2]$. We then have

$$\mathbf{x} = \overrightarrow{OP} + t\overrightarrow{PQ} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 9 \\ 2 \end{bmatrix},$$

where $\mathbf{x} = [x_1, x_2]$ is the position vector corresponding to any point (x_1, x_2) on the line. ♠

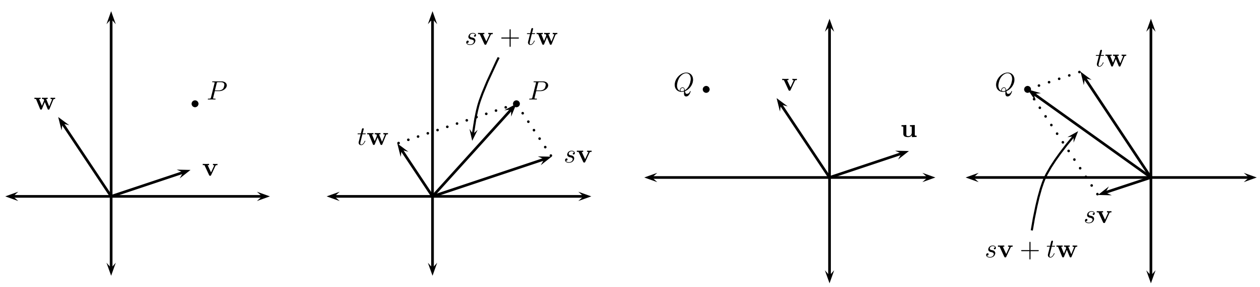
◇ **Example 4.2(d):** Give a vector equation of the line in $\mathbb{R}^3$ through the points $(-5,1,2)$ and $(4,6,-3)$.

Letting P be the point $(-5,1,2)$ and Q be the point $(4,6,-3)$, we get $\overrightarrow{PQ} = \langle 9,5,-5\rangle$. The vector equation of the line is then

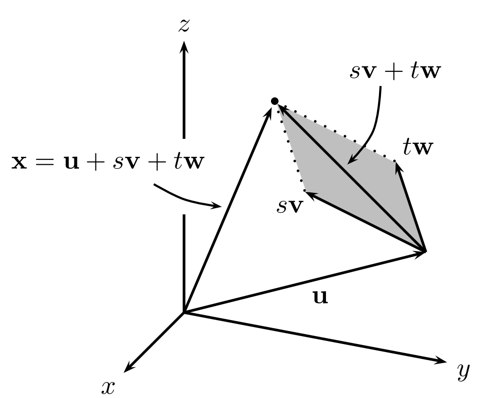
$$\mathbf{x} = \overrightarrow{OP} + t\overrightarrow{PQ} = \begin{bmatrix} -5 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 9 \\ 5 \\ -5 \end{bmatrix},$$

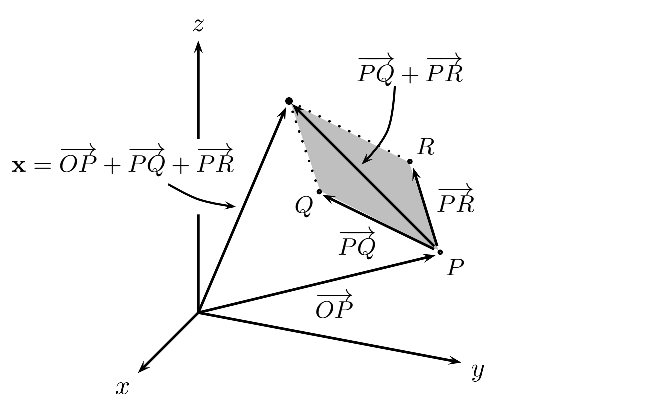
where $\mathbf{x} = [x_1, x_2, x_3]$ is the position vector corresponding to any point (x_1, x_2, x_3) on the line. The first vector can be any point on the line, so it could be the vector $[4,6,-3]$ instead of $[-5,1,2]$, and the second vector is a direction vector, so can be any scalar multiple of $\mathbf{d} = [9,5,-5]$. ♠

The same general idea can be used to describe a plane in $\mathbb{R}^3$. Before seeing how that works, let's define something and look at a situation in $\mathbb{R}^2$. We say that two vectors are **parallel** if one is a scalar multiple of the other. Now suppose that $\mathbf{v}$ and $\mathbf{w}$ are two vectors in $\mathbb{R}^2$ that are *not* parallel (and neither is the zero vector either), as shown in the picture to the left below, and let P be the randomly chosen point in $\mathbb{R}^2$ shown in the same picture. The next picture shows that a linear combination of $\mathbf{v}$ and $\mathbf{w}$ can be formed that gives us a vector $s\mathbf{v} + t\mathbf{w}$ corresponding to the point P. In this case the scalar s is positive and less than one, and t is positive and greater than one. The third and fourth pictures show the same thing for another point Q, with both s being negative and t positive in that case. It should now be clear that *any* point in $\mathbb{R}^2$ can be obtained in this manner.



Now let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be three vectors in $\mathbb{R}^3$, and consider the vector $\mathbf{x} = \mathbf{u} + s\mathbf{v} + t\mathbf{w}$, where s and t are scalars that are allowed to take all real numbers as values. The vectors $s\mathbf{v} + t\mathbf{w}$ all lie in the plane containing $\mathbf{v}$ and $\mathbf{w}$. Adding $\mathbf{u}$ "moves the plane off the origin" to where it passes through the tip of $\mathbf{u}$ (again, in standard position). This is probably best visualized by thinking of adding $s\mathbf{v}$ and $t\mathbf{w}$ with the parallelogram method, then adding the result to $\mathbf{u}$ with the tip-to-tail method. I have attempted to illustrate this below and to the left , with the gray parallelogram being part of the plane created by all the points corresponding to the vectors $\mathbf{x}$.





The same diagram above and to the right shows how all of the previous discussion relates to the plane through three points P, Q and R in $\mathbb{R}^3$. This leads us to the description of a plane in $\mathbb{R}^3$ given at the top of the next page.

Planes in $\mathbb{R}^3$

The equation of a plane through three points P, Q and R in $\mathbb{R}^3$ is given by the vector equation

$$\mathbf{x} = \overrightarrow{OP} + s\,\overrightarrow{PQ} + t\,\overrightarrow{PR}.$$

By this we mean that the plane consists of all the points corresponding to the position vectors $\mathbf{x}$ as s and t vary over all real numbers.

◇ **Example 4.2(e):** Give a vector equation of the plane in $\mathbb{R}^3$ through the points $(2,-1,3)$, $(-5,1,2)$ and $(4,6,-3)$. What values of s and t give the point R?

Letting P be the point $(2,-1,3)$, Q be $(-5,1,2)$ and R be $(4,6,-3)$, we get $\overrightarrow{PQ} = [-7,2,-1]$ and $\overrightarrow{PR} = [2,7,-6]$. The vector equation of the plane is then

$$\mathbf{x} = \overrightarrow{OP} + s\overrightarrow{PQ} + t\overrightarrow{PR} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + s\begin{bmatrix} -7 \\ 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 7 \\ -6 \end{bmatrix},$$

where $\mathbf{x} = [x_1, x_2, x_3]$ is the position vector corresponding to any point (x_1, x_2, x_3) on the plane. It should be clear that there are other possibilities for this. The first vector in the equation could be any of the three position vectors for P, Q or R. The other two vectors could be any two vectors from one of the points to another.

The vector corresponding to point R is $\overrightarrow{OR}$, which is equal to $\mathbf{x} = \overrightarrow{OP} + \overrightarrow{PR}$ (think about that), so $s = 0$ and $t = 1$. ♠

We now summarize all of the ideas from this section.

Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$, Planes in $\mathbb{R}^3$

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ with $\mathbf{v} \neq \mathbf{0}$. Then the set of points corresponding to the vector $\mathbf{x} = \mathbf{u} + t\mathbf{v}$ as t ranges over all real numbers is a line through the point corresponding to $\mathbf{u}$ and in the direction of $\mathbf{v}$. (So if $\mathbf{u} = 0$ the line passes through the origin.)

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors $\mathbb{R}^3$, with $\mathbf{v}$ and $\mathbf{w}$ being nonzero and not parallel. (That is, not scalar multiples of each other.) Then the set of points corresponding to the vector $\mathbf{x} = \mathbf{u} + s\mathbf{v} + t\mathbf{w}$ as s and t range over all real numbers is a plane through the point corresponding to $\mathbf{u}$ and containing the vectors $\mathbf{v}$ and $\mathbf{w}$. (If $\mathbf{u} = 0$ the plane passes through the origin.)

Section 4.2 Exercises

1. For each of the following, give the vector equation of the line or plane described.

   (a) The line through the two points $P(3,-1,4)$ and $Q(2,6,0)$ in $\mathbb{R}^3$.

   (b) The plane through the points $P(3,-1,4)$, $Q(2,6,0)$ and $R(-1,0,3)$ in $\mathbb{R}^3$.

   (c) The line through the points $P(3,-1)$ and $Q(6,0)$ in $\mathbb{R}^2$.

2. Find another point in the plane containing $P_1(-2,1,5)$, $P_2(3,2,1)$ and $P_3(4,-2,-3)$. **Show clearly how you do it**. (**Hint:** Find and use the vector equation of the plane.)

3. "Usually" a vector equation of the form $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$ gives the equation of a plane in $\mathbf{R}^3$.

   (a) Under what conditions on $\mathbf{p}$ and/or $\mathbf{u}$ and/or $\mathbf{v}$ would this be the equation of a line?

   (b) Under what conditions on $\mathbf{p}$ and/or $\mathbf{u}$ and/or $\mathbf{v}$ would this be the equation of a plane through the origin?