4.4 The Dot Product of Vectors, Projections

Performance Criteria:

4. (d) Find the dot product of two vectors, determine the length of a single vector.
   (e) Determine whether two vectors are orthogonal (perpendicular).
   (f) Find the projection of one vector onto another, graphically or algebraically.

The Dot Product and Orthogonality

There are two ways to "multiply" vectors, both of which you have likely seen before. One is called the **cross product**, and only applies to vectors in $\mathbb{R}^3$. It is quite useful and meaningful in certain physical situations, but it will be of no use to us here. More useful is the other method, called the **dot product**, which is valid in all dimensions.

DEFINITION 4.4.1: **Dot Product**

Let $\mathbf{u} = [u_1, u_2, ..., u_n]$ and $\mathbf{v} = [v_1, v_2, ..., v_n]$. The **dot product** of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \cdot \mathbf{v}$, is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 + \cdots + u_n v_n$$

The dot product is useful for a variety of things. Recall that the length of a vector $\mathbf{v} = [v_1, v_2, ..., v_n]$ is given by $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. Note also that $v_1^2 + v_2^2 + \cdots + v_n^2 = \mathbf{v} \cdot \mathbf{v}$, which implies that $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. Perhaps the most important thing about the dot product is that the dot product of two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ is zero if, and only if, the two vectors are perpendicular. In general, we make the following definition.

DEFINITION 4.4.2: **Orthogonal Vectors**

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are said to be **orthogonal** if, and only if, $\mathbf{u} \cdot \mathbf{v} = 0$.

◇ **Example 4.4(a):** For the three vectors $\mathbf{u} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$,

find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \cdot \mathbf{w}$ and $\mathbf{v} \cdot \mathbf{w}$. Are any of the vectors orthogonal to each other?

We find that

$$\mathbf{u} \cdot \mathbf{v} = (5)(-1) + (-1)(3) + (2)(4) = -5 + (-3) + 8 = 0,$$

$$\mathbf{u} \cdot \mathbf{w} = (5)(2) + (-1)(-1) + (2)(-3) = 10 + 1 + (-6) = 5,$$

$$\mathbf{v} \cdot \mathbf{w} = (-1)(2) + (3)(-1) + (4)(-3) = -2 + (-3) + (-12) = -17$$

From the first computation we can see that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. ♠

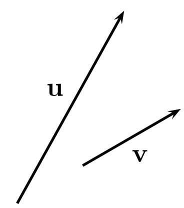
Projections

Given two vectors $\mathbf{u}$ and $\mathbf{v}$, we can create a new vector $\mathbf{w}$ called the **projection of $\mathbf{u}$ onto $\mathbf{v}$**, denoted by $\text{proj}_{\mathbf{v}}\mathbf{u}$. This is a very useful idea, in many ways. Geometrically, we can find $\text{proj}_{\mathbf{v}}\mathbf{u}$ as follows:

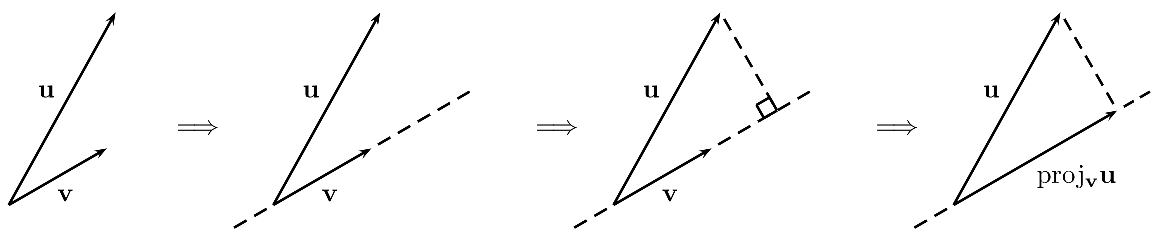
- Bring $\mathbf{u}$ and $\mathbf{v}$ together tail-to-tail.
- Sketch in the line containing $\mathbf{v}$, as a dashed line.
- Sketch in a dashed line segment from the tip of $\mathbf{u}$ to the dashed line containing $\mathbf{v}$, *perpendicular to that line.*
- Draw the vector $\text{proj}_{\mathbf{v}}\mathbf{u}$ from the point at the tails of $\mathbf{u}$ and $\mathbf{v}$ to the point where the dashed line segment meets $\mathbf{v}$ or the dashed line containing $\mathbf{v}$.

Note that $\text{proj}_{\mathbf{v}}\mathbf{u}$ is parallel to $\mathbf{v}$; if we were to find $\text{proj}_{\mathbf{u}}\mathbf{v}$ instead, the result would be parallel to $\mathbf{u}$ in that case. The above steps are illustrated in the following example.

◇ **Example 4.4(b):** For the vectors $\mathbf{u}$ and $\mathbf{v}$ shown to the right, find the projection $\text{proj}_{\mathbf{v}}\mathbf{u}$.



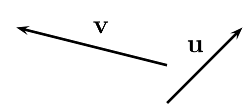
We follow the above steps:



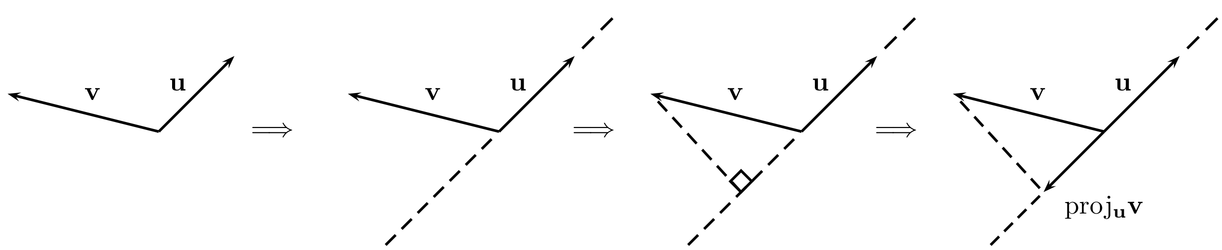
♠

Projections are a bit less intuitive when the angle between the two vectors is obtuse, as seen in the next example.

◇ **Example 4.4(c):** For the vectors $\mathbf{u}$ and $\mathbf{v}$ shown to the right, find the projection $\text{proj}_{\mathbf{u}}\mathbf{v}$.



We follow the steps again, noting that this time we are projecting $\mathbf{v}$ onto $\mathbf{u}$:



Here we see that $\text{proj}_{\mathbf{u}}\mathbf{v}$ is in the direction opposite $\mathbf{u}$. ♠

We will also want to know how to find projections algebraically:

> <u>Definition 4.4.3</u>: **The Projection of One Vector on Another**
>
> For two vectors $\mathbf{u}$ and $\mathbf{v}$, the vector $\text{proj}_{\mathbf{v}}\mathbf{u}$ is given by
>
> $$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}$$

Note that since both $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{v}\cdot\mathbf{v}$ are scalars, so is $\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}$. That scalar is then multiplied times $\mathbf{v}$, resulting in a vector parallel $\mathbf{v}$. if the scalar is positive the projection is in the direction of $\mathbf{v}$, as shown in Example 4.2(b); when the scalar is negative the projection is in the direction opposite the vector being projected onto, as shown in Example 4.3(c).

◇ **Example 4.4(d):** For the vectors $\mathbf{u} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$, find $\text{proj}_{\mathbf{u}}\mathbf{v}$.

Note that here we are projecting $\mathbf{v}$ onto $\mathbf{u}$. first we find

$$\mathbf{v}\cdot\mathbf{u} = (2)(5) + (-1)(-1) + (-3)(2) = 5 \quad \text{and} \quad \mathbf{u}\cdot\mathbf{u} = 5^2 + (-1)^2 + 2^2 = 30$$

Then

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{5}{30}\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ -\frac{1}{6} \\ \frac{1}{3} \end{bmatrix}$$ ♠

As stated before, the idea of projection is extremely important in mathematics, and arises in situations that do not appear to have anything to do with geometry and vectors as we are thinking of them now. You will see a clever geometric use of vectors in one of the exercises.

Section 4.4 Exercises

1. Consider the vectors $\mathbf{v} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

   (a) Draw a *neat and accurate* graph of $\mathbf{v}$ and $\mathbf{b}$, with their tails at the origin, labeling each.

   (b) Use the formula to find $\text{proj}_{\mathbf{b}}\mathbf{v}$, with its components rounded to the nearest tenth.

   (c) Add $\text{proj}_{\mathbf{b}}\mathbf{v}$ to your graph. Does it look correct?

2. For each pair of vectors **v** and **b** below, do each of the following

   i) Sketch **v** and **b** with the same initial point.

   ii) Find $\text{proj}_{\mathbf{b}}\mathbf{v}$ algebraically, using the formula for projections.

   iii) On the same diagram, sketch the $\text{proj}_{\mathbf{b}}\mathbf{v}$ you obtained in part (ii). If it does not look the way it should, find your error.

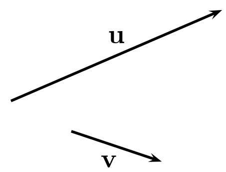
   iv) Find $\text{proj}_{\mathbf{b}}\mathbf{v}$, and sketch it as a new sketch. Compare with your previous sketch.

   (a) $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ (b) $\mathbf{v} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

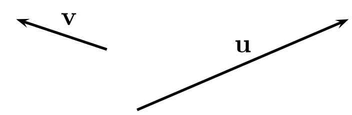
   (c) $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

3. For each pair of vectors $\mathbf{u}$ and $\mathbf{v}$, sketch $\text{proj}_{\mathbf{v}}\mathbf{u}$. Indicate any right angles with the standard symbol.

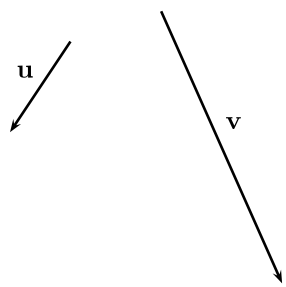
(a)



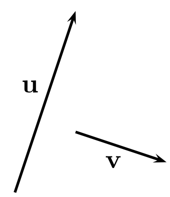
(b)



(c)



(d)



($\mathbf{u}$ and $\mathbf{v}$ are orthogonal)