5.1 Introduction

A **matrix** is simply an array of numbers arranged in rows and columns. Here are some examples:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix}$$

We will always denote matrices with italicized capital letters. There should be no need to define the **rows** and **columns** of a matrix. The number of rows and number of columns of a matrix are called its **dimensions**. The second matrix above, B, has dimensions 3×2, which we read as "three by two." The numbers in a matrix are called its **entries**. Each entry of a matrix is identified by its row, then column. For example, the $(3,2)$ entry of L is the entry in the 3rd row and second column, -2. In general, we will define the (i,j)th entry of a matrix to be the entry in the ith row and jth column.

There are a few special kinds of matrices that we will run into regularly:

- A matrix with the same number of rows and columns is called a **square matrix**. Matrices A, D and L above are square matrices. The entries that are in the same number row and column of a square matrix are called the **diagonal entries** of the matrix. For example, the diagonal entries of A are 1 and 3.

- A square matrix with zeros "above" the diagonal is called a **lower triangular matrix**; L is an example of a lower triangular matrix. Similarly, an **upper triangular matrix** is one whose entries below the diagonal are all zeros. (Note that the words "lower" and "upper" refer to the triangular parts of the matrices where the entries are *NOT* zero.)

- A square matrix all of whose entries above *AND* below the diagonal are zero is called a **diagonal matrix**. D is an example of a diagonal matrix.

- A diagonal matrix with only ones on the diagonal is called "the" **identity matrix**. We use the word "the" because in a given size there is only one identity matrix. We will soon see why it is called the "identity."

- Given a matrix, we can "flip the matrix over its diagonal," so that the rows of the original matrix become the columns of the new matrix, and vice versa. The new matrix is called the **transpose** of the original. The transposes of the matrices B and L above are denoted by B^T and L^T. They are the matrices

$$B^T = \begin{bmatrix} -5 & 0 & 2 \\ 1 & 4 & -3 \end{bmatrix} \qquad L^T = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

- Notice that $A^T = A$. Such a matrix is called a **symmetric matrix**. One way of thinking of such a matrix is that the entries across the diagonal from each other are equal. Matrix D is also symmetric, as is the matrix

$$\begin{bmatrix} 1 & 5 & 0 & -2 \\ 5 & -4 & 7 & 3 \\ 0 & 7 & 0 & -6 \\ -2 & 3 & -6 & -3 \end{bmatrix}$$

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When discussing an arbitrary matrix A with dimensions $m \times n$ we refer to each entry as a, but with a double subscript with each to indicate its position in the matrix. The first number in the subscript indicates the row of the entry and the second indicates the column of that entry:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} & \cdots & a_{2n} \\ \vdots & \vdots & & & \vdots & & \vdots \\ a_{j1} & a_{j2} & a_{j3} & \cdots & a_{jk} & \cdots & a_{jn} \\ \vdots & \vdots & & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mk} & \cdots & a_{mn} \end{bmatrix}$$

Under the right conditions it is possible to add, subtract and multiply two matrices. We'll save multiplication for a little, but we have the following:

DEFINITION 5.1.1: **Adding and Subtracting Matrices**

When two matrices have the same dimensions, they are added or subtracted by adding or subtracting the corresponding entries.

◇ **Example 5.1(a):** Determine which of the matrices below can be added, and add those that can be.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}, \qquad C = \begin{bmatrix} -7 & 4 \\ 1 & 5 \end{bmatrix}$$

B cannot be added to either A or C, but

$$A + C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -7 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 3 & 8 \end{bmatrix}$$

It should be clear that A and C could be subtracted, and that $A + C = C + A$ but $A - C \neq C - A$.

Section 5.1 Exercises

1. (a) Give the dimensions of matrices A, B and C in Exercise 3 below.

   (b) Give the entries b_{13} and c_{32} of the matrices B and C in Exercise 3 below.

2. Give examples of each of the following types of matrices.

   (a) lower triangular   (b) diagonal

   (c) symmetric   (d) identity

   (e) upper triangular but not diagonal   (f) symmetric but without any zero entries

   (g) symmetric but not diagonal   (h) diagonal but not a multiple of an identity

3. Give the transpose of each matrix. Use the correct notation to denote the transpose.

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -3 & 1 & -2 \\ 4 & 7 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 4 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & -1 & 3 \\ -3 & 1 & 2 & 0 \\ 4 & 7 & 0 & -2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -3 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$$

4. Give all possible sums and differences of matrices from Exercise 3.