5.2 Actions of Matrices on Vectors

Performance Criteria:

5. (d) Multiply a matrix times a vector.

   (e) Give the identity matrix (for a given dimensional space) and its effect when a vector is multiplied by it.

In Section 6.1 we will find out how to multiply a matrix times another matrix but, for now we'll multiply only matrices times vectors. This is not to say that doing so is a minor step on the way to learning to multiply matrices; multiplying a matrix times a vector is in some sense *THE* foundational operation of linear algebra.

Before getting into how to do this, we need to devise a useful notation. Consider the matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Each column of A, taken by itself, is a vector. We'll refer to the first column as the vector $\mathbf{a}_{*1}$, with the asterisk $*$ indicating that the row index will range through all values, and the 1 indicating that the values all come out of column one. Of course $\mathbf{a}_{*2}$ denotes the second column, and so on. Similarly, $\mathbf{a}_{1*}$ will denote the first row, $\mathbf{a}_{1*}$ the second row, etc. Technically speaking, the rows are not vectors, but we'll call them **row vectors** and we'll call the columns **column vectors**. If we use just the word *vector*, we will mean a column vector.

◇ **Example 5.2(a):** Give $\mathbf{a}_{2*}$ and $\mathbf{a}_{*3}$ for the matrix $A = \begin{bmatrix} -5 & 3 & 4 & -1 \\ 7 & 5 & 2 & 4 \\ 2 & -1 & -6 & 0 \end{bmatrix}$

$$\mathbf{a}_{2*} = \begin{bmatrix} 7 & 5 & 2 & 4 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_{*3} = \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix} \quad ♠$$

<u>Definition 5.2.1</u>: **Matrix Times a Vector**

An $m \times n$ matrix A can be multiplied times a vector $\mathbf{x}$ with n components. The result is a vector with m components, the ith component being the dot product of the ith row of A with $\mathbf{x}$, as shown below.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_{1*} \cdot \mathbf{x} \\ \mathbf{a}_{2*} \cdot \mathbf{x} \\ \vdots \\ \mathbf{a}_{m*} \cdot \mathbf{x} \end{bmatrix}$$

◇ **Example 5.2(b):** Multiply $\begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \\ 1 & -6 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \\ 1 & -6 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix} = \begin{bmatrix} (3)(2) + (0)(1) + (-1)(-7) \\ (-5)(2) + (2)(1) + (4)(-7) \\ (1)(2) + (-6)(1) + (0)(-7) \end{bmatrix} = \begin{bmatrix} 13 \\ -36 \\ -4 \end{bmatrix} \quad ♠$$

There is no need for the matrix multiplying a vector to be square, but when it is not, the resulting vector is not the same length as the original vector:

◇ **Example 5.2(c):** Multiply $\begin{bmatrix} 7 & -4 & 2 \\ -1 & 0 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 7 & -4 & 2 \\ -1 & 0 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} (7)(3)+(-4)(-5)+(2)(1) \\ (-1)(3)+(0)(-5)+(6)(1) \end{bmatrix} = \begin{bmatrix} 43 \\ 3 \end{bmatrix} \quad \spadesuit$$

Although the above defines a matrix times a vector in a purely computational sense, it is best to think of a matrix as *acting on a vector to create a new vector.* One might also think of this as a matrix *transforming* a vector into another vector. In general, when a matrix acts on a vector the resulting vector will have a different direction and length than the original vector. There are a few notable exceptions to this:

- The matrix that acts on a vector without actually changing it at all is called the **identity matrix**. Clearly, then, when the identity matrix acts on a vector, neither the direction or magnitude is changed.
- A matrix that rotates every vector in $\mathbb{R}^2$ through a fixed angle θ is called a **rotation matrix**. In this case the direction changes, but not the magnitude. (Of course the direction doesn't change if $\theta = 0°$ and, in some sense, if $\theta = 180°$. In the second case, even though the direction is opposite, the resulting vector is still just a scalar multiple of the original.)
- For most matrices there are certain vectors, called **eigenvectors** whose directions don't change (other than perhaps reversing) when acted on by by the matrix under consideration. In those cases, the effect of multiplying such a vector by the matrix is the same as multiplying the vector by a scalar. This has very useful applications.

Multiplication of vectors by matrices has the following important properties, which are easily verified.

<u>Definition 5.2.2</u>

Let A and B be matrices, $\mathbf{x}$ and $\mathbf{y}$ be vectors, and c be any scalar. Assuming that all the indicated operations below are defined (possible), then

(a) $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$ (b) $A(c\mathbf{x}) = c(A\mathbf{x})$

(b) $(A+B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}$

We now come to a very important idea that depends on the first two properties above. When we act on a mathematical object with another object, the object doing the "acting on" is often called an **operator**. Some operators you are familiar with are the derivative operator and the antiderivative operator (indefinite integral), which act on functions to create other functions. Note that the derivative operator has the following two properties, for any functions f and g and real number c:

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}, \qquad \frac{d}{dx}(cf) = c\frac{df}{dx}$$

These are the same as the first two properties for multiplication of a vector by a matrix. A matrix can be thought of as an operator that operates on vectors by multiplying them. The first two properties of multiplication of a vector by a matrix, as well as the corresponding properties of the derivative, are called the **linearity properties**. Both the derivative operator and matrix multiplication operator are then called **linear operators**. This is why this subject is called *linear* algebra!

There is another way to compute a matrix times a vector. It is not as efficient to do by hand as what we have been doing so far, but it will be very important conceptually quite soon. Using our earlier definition of a matrix A times a vector $\mathbf{x}$, we see that

$$
\begin{aligned}
A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} &= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{21}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix} \\
&= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \\
&= x_1\mathbf{a}_{*1} + x_2\mathbf{a}_{*2} + \cdots + x_n\mathbf{a}_{*n}
\end{aligned}
$$

Let's think about what the above shows. It gives us the result below, which is illustrated in Examples 5.2(d) and (e).

Linear Combination Form of a Matrix Times a Vector

The product of a matrix A and a vector $\mathbf{x}$ is a linear combination of the columns of A, with the scalars being the corresponding components of $\mathbf{x}$.

$\diamond$ **Example 5.2(d):** Give the linear combination form of $\begin{bmatrix} 7 & -4 & 2 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 7 & -4 & 2 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 7 \\ -1 \end{bmatrix} - 5\begin{bmatrix} -4 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 6 \end{bmatrix} \qquad \spadesuit$$

$\diamond$ **Example 5.2(e):** Give the linear combination form of $\begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \qquad \spadesuit$$

Section 5.2 Exercises

1. Multiply $\begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 2 \\ 4 & 7 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 & -4 & 0 \\ -1 & 5 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$

2. Find a matrix A such that $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - 5x_2 \\ x_1 + x_2 \end{bmatrix}$.

3. Give the 3×3 identity matrix I. For any vector $\mathbf{x}$, $I\mathbf{x} =$ ______ .