6.2 More Multiplying Matrices

Performance Criteria:

6. (c) Multiply two matrices "by hand" using all three of the linear combination of columns, outer product, and linear combination of rows methods.

Recall the following notation from the previous section: Given a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

we refer to, for example the third row as A_{3*}. Here the first subscript 3 indicates that we are considering the third row, and the $*$ indicates that we are taking the elements from the third row *in all columns*. Therefore A_{3*} refers to a $1 \times n$ matrix. Similarly, $\mathbf{a}_{*2}$ is the vector that is the second column of A. So we have

$$A_{3*} = \begin{bmatrix} a_{31} & a_{32} & \cdots & a_{3n} \end{bmatrix} \qquad\qquad \mathbf{a}_{*2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{2m} \end{bmatrix}$$

Using this notation, if we are multiplying the $m \times n$ matrix A times the $n \times p$ matrix B and $AB = C$, the c_{ij} entry of C is obtained by the product $A_{i*}b_{*j}$.

In some sense the product $A_{i*}\mathbf{b}_{*j}$ is performed as a dot product. Another name for the dot product is **inner product** and this method of multiplying two matrices we will call the inner product method. We will take it to be the the definition of the product of two matrices.

Definition 6.2.1: **Matrix Multiplication, Inner Product Method**

Let A and B be $m \times n$ and $n \times p$ matrices respectively. We define the product AB to be the $m \times p$ matrix C whose (i, j) entry is given by

$$c_{ij} = A_{i*}b_{*j},$$

where A_{i*} and b_{*j} are as defined above. That is, each element c_{ij} of C is the product of the ith row of A times the jth column of B.

◇ **Example 6.2(a):** For the matrices $A = \begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 4 \\ 7 & -2 \end{bmatrix}$, find $C = AB$ by the inner product method.

Here the matrix C will also be 2×2, with

$$c_{11} = \begin{bmatrix} 3 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = 18 + (-7) = 11, \qquad c_{12} = \begin{bmatrix} 3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = 12 + 2 = 14,$$

$$c_{21} = \begin{bmatrix} -2 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = -12 + 35 = 23, \qquad c_{22} = \begin{bmatrix} -2 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = -8 + (-10) = -18,$$

so

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ 23 & -18 \end{bmatrix} \quad ♠$$

We will now see three other methods for multiplying matrices. All three are perhaps more complicated than the above, but their value is not in computation of matrix products but rather in giving us conceptual tools that are useful when examining certain ideas in the subject of linear algebra. The first of these other methods uses the ever so important idea of linear combinations.

<u>THEOREM 6.2.2</u>: **Matrix Multiplication, Linear Combination of Columns Method**

Let A and B be $m \times n$ and $n \times p$ matrices respectively. The product $C = AB$ is the matrix for which

$$\mathbf{c}_{*j} = b_{1j}\mathbf{a}_{*1} + b_{2j}\mathbf{a}_{*2} + b_{3j}\mathbf{a}_{*3} + \cdots + b_{nj}\mathbf{a}_{*n}$$

That is, the jth column $\mathbf{c}_{*j}$ of C is the linear combination of all the columns of A, using the entries of the jth column of B as the scalars.

◇ **Example 6.2(b):** For the matrices $A = \begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 4 \\ 7 & -2 \end{bmatrix}$, find $C = AB$ by the linear combination of columns method.

Again C will also be 2×2, with

$$\mathbf{c}_{*1} = 6\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 7\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 23 \end{bmatrix} \qquad \mathbf{c}_{*2} = 4\begin{bmatrix} 3 \\ -2 \end{bmatrix} + (-2)\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 14 \\ -18 \end{bmatrix}$$

so

$$C = \begin{bmatrix} \mathbf{c}_{*1} & \mathbf{c}_{*2} \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ 23 & -18 \end{bmatrix} \quad ♠$$

Suppose that we have two vectors $\mathbf{u} = \begin{bmatrix} -6 \\ 1 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$. Now we see that $\mathbf{u}^T = \begin{bmatrix} -6 & 1 & 4 \end{bmatrix}$, which is a 1×3 matrix. Thinking of the vector $\mathbf{v}$ as a 3×1 matrix, we can use the inner product definition of matrix multiplication to get

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} -6 & 1 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} = (-6)(3) + (1)(-5) + (4)(2) = -15 = \mathbf{u} \cdot \mathbf{v}.$$

As mentioned previously this is sometimes also called the **inner product** of $\mathbf{u}$ and $\mathbf{v}$.

We can consider instead $\mathbf{u}$ as a 3×1 matrix and $\mathbf{v}^T$ as a 1×3 matrix and look at the product $\mathbf{u}\mathbf{v}^T$. This is then a 3×3 matrix given by

$$\mathbf{u}\mathbf{v}^T = \begin{bmatrix} -6 \\ 1 \\ 4 \end{bmatrix}\begin{bmatrix} 3 & -5 & 2 \end{bmatrix} = \begin{bmatrix} (-6)(3) & (-6)(-5) & (-6)(2) \\ (1)(3) & (1)(-5) & (1)(2) \\ (4)(3) & (4)(-5) & (4)(2) \end{bmatrix} = \begin{bmatrix} -18 & 30 & -12 \\ 3 & -5 & 2 \\ 12 & -20 & 8 \end{bmatrix}$$

This last result is called the **outer product** of $\mathbf{u}$ and $\mathbf{v}$, and is used in our next method for multiplying two matrices.

Theorem 6.2.3: Matrix Multiplication, Outer Product Method

Let A and B be $m \times n$ and $n \times p$ matrices respectively. The product $C = AB$ is the matrix

$$C = \mathbf{a}_{*1}B_{1*} + \mathbf{a}_{*2}B_{2*} + \mathbf{a}_{*3}B_{3*} + \cdots + \mathbf{a}_{*n}B_{n*}$$

That is, C is the $m \times p$ matrix given by the sum of all the $m \times p$ outer product matrices obtained from multiplying each column of A times the corresponding row of B.

◇ **Example 6.2(c):** For the matrices $A = \begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 4 \\ 7 & -2 \end{bmatrix}$, find $C = AB$ by the outer product method.

$$C = \mathbf{a}_{*1}B_{1*} + \mathbf{a}_{*2}B_{2*} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}\begin{bmatrix} 6 & 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix}\begin{bmatrix} 7 & -2 \end{bmatrix} =$$

$$\begin{bmatrix} 18 & 12 \\ -12 & -8 \end{bmatrix} + \begin{bmatrix} -7 & 2 \\ 35 & -10 \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ 23 & -18 \end{bmatrix} \quad ♠$$

Theorem 6.2.4: Matrix Multiplication, Linear Combination of Rows Method

Let A and B be $m \times n$ and $n \times p$ matrices respectively. The product $C = AB$ is the matrix for which

$$C_{i*} = a_{i1}B_{1j} + a_{i2}B_{2j} + a_{i3}B_{3j} + \cdots + a_{in}B_{nj}.$$

That is, the ith row C_{i*} of C is the linear combination of all the rows of B, using the entries of the ith row of A as the scalars.

◇ **Example 6.2(d):** For the matrices $A = \begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 4 \\ 7 & -2 \end{bmatrix}$, find $C = AB$ by the linear combination of rows method.

By the above theorem we have

$$C_{1*} = 3\begin{bmatrix} 6 & 4 \end{bmatrix} + (-1)\begin{bmatrix} 7 & -2 \end{bmatrix} = \begin{bmatrix} 18 & 12 \end{bmatrix} + \begin{bmatrix} -7 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 14 \end{bmatrix}$$

and

$$C_{2*} = -2\begin{bmatrix} 6 & 4 \end{bmatrix} + 5\begin{bmatrix} 7 & -2 \end{bmatrix} = \begin{bmatrix} -12 & -8 \end{bmatrix} + \begin{bmatrix} 35 & -10 \end{bmatrix} = \begin{bmatrix} 23 & -18 \end{bmatrix}$$

so

$$C = \begin{bmatrix} C_{1*} \\ C_{2*} \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ 23 & -18 \end{bmatrix} \quad ♠$$

Section 6.2 Exercises

1. Let $A = \begin{bmatrix} -5 & 1 & -2 \\ 7 & 0 & 4 \\ 2 & -3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -7 & 0 \\ 5 & 2 & 3 \end{bmatrix}$.

   (a) Find the second column of $C = AB$, using the linear combination of columns method, showing clearly the linear combination used and the resulting column. Label the result using the correct notation for the column of a matrix, as described at the beginning of the section.

   (b) Find the product $C = AB$, showing the sum of outer products, the sum of resulting matrices, and the final result.

   (c) Find the third row of $C = AB$, using the linear combination of rows method, showing clearly the linear combination used and the resulting column. Label the result using the correct notation for the row of a matrix, as described at the beginning of the section.

2. Let $C = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

   (a) Find the product $A = CD$, using the linear combination of rows method. Show clearly how each row of A is obtained, labeling each using the correct notation. Then give the final result A.

   (b) Find the product $A = CD$, using the linear combination of columns method. Show clearly how each column of A is obtained, labeling each using the correct notation. Then give the final result A.

   (c) Find the product $A = CD$ using the outer product method.