7.1 Matrix Equation Form of a System

Performance Criteria:

7. (a) Express a system of equations as a coefficient matrix times a vector equalling another vector.

Definition 7.1.1 **Matrix Equation Form of a System**

A system of m linear equations in n unknowns (note that m and n need not be equal) can be written as $A\mathbf{x} = \mathbf{b}$ where A is the $m \times n$ coefficient matrix of the system, $\mathbf{x}$ is the vector consisting of the n unknowns and $\mathbf{b}$ is the vector consisting of the m right-hand sides of the equations, as shown below.

$$\begin{array}{rcl} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & b_m \end{array} \iff \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

We will refer to this as the **matrix form of a system of equations**.

This form of a system of equations can be used, as you will soon see, in another method (besides row-reduction) for solving a system of equations. That method is occasionally useful, though not generally used in practice due to algorithmic inefficiency. The main benefit of this idea is that it allows us to write a system of equations in the very compact form $A\mathbf{x} = \mathbf{b}$ that allows us to discuss both concepts and practical methods in a way that is much less cumbersome than the systems themselves.

◇ **Example 7.1(a):** Give the matrix form of the system
$$\begin{array}{rcr} x_1 + 3x_2 - 2x_3 & = & -4 \\ 3x_1 + 7x_2 + x_3 & = & 4 \\ -2x_1 + x_2 + 7x_3 & = & 7 \end{array}$$

The matrix form of the system is $$\begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$$ ♠

We now have three interpretations of the solution (x_1, x_2, x_3) to a system $A\mathbf{x} = \mathbf{b}$ of three equations in three unknowns, like the one above, assuming that we have a unique solution:

1) (x_1, x_2, x_3) is the point where the planes with the three equations intersect.

2) x_1, x_2 and x_3 are the three scalars for a linear combination of the columns of A that equals the vector $\mathbf{b}$.

3) $\mathbf{x} = [x_1, x_2, x_3]$ is the vector that A transforms into the vector $\mathbf{b}$.

Section 7.1 Exercises

1. Multiply $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

2. Give the matrix form $A\mathbf{x} = \mathbf{b}$ of each system of equations.

(a) $$\begin{aligned} x + y - 3z &= 1 \\ -3x + 2y - z &= 7 \\ 2x + y - 4z &= 0 \end{aligned}$$

(b) $$\begin{aligned} 5x - 3y + z &= -4 \\ x + y - 7z &= 2 \end{aligned}$$