7.2 Solving a System With An LU-Factorization

Performance Criterion:

7. (b) Use LU-factorization to solve a system of equations, given the LU-factorization of its coefficient matrix.

In many cases a square matrix A can be "factored" into a product of a lower triangular matrix and an upper triangular matrix, in that order. That is, $A = LU$ where L is lower triangular and U is upper triangular. In that case, for a system $A\mathbf{x} = \mathbf{b}$ that we are trying to solve for $\mathbf{x}$ we have

$$A\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad (LU)\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad L(U\mathbf{x}) = \mathbf{b}$$

Note that $U\mathbf{x}$ is simply a vector; let's call it $\mathbf{y}$. We then have two systems, $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. To solve the system $A\mathbf{x} = \mathbf{b}$ we first solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$. Once we know $\mathbf{y}$ we can then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, which was our original goal. Here is an example:

◇ **Example 7.2(a):** Solve the system of equations $\begin{array}{rcl} 7x_1 - 2x_2 + x_3 &=& 12 \\ 14x_1 - 7x_2 - 3x_3 &=& 17 \\ -7x_1 + 11x_2 + 18x_3 &=& 5 \end{array}$, given that the coefficient matrix factors as

$$\begin{bmatrix} 7 & -2 & 1 \\ 14 & -7 & -3 \\ -7 & 11 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix}$$

Because of the above factorization we can write the system in matrix form as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 17 \\ 5 \end{bmatrix}$$

We now let $\begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $(*)$ and the above system becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 17 \\ 5 \end{bmatrix} \quad (**)$$

The system $(**)$ is easily solved for the vector $\mathbf{y} = [y_1, y_2, y_3]$ by **forward-substitution**. From the first row we see that $y_1 = 12$; from that it follows that $y_2 = 17 - 2y_1 = 17 - 24 = -7$. Finally, $y_3 = 5 + y_1 + 3y_2 = -4$.

Now that we know $\mathbf{y}$, the system $(*)$ becomes

$$\begin{bmatrix} 7 & -2 & 1 \\ 0 & -3 & -5 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -7 \\ -4 \end{bmatrix}$$

This is now solved by back-substitution. We can see that $x_3 = -1$, so

$$-3x_2 - 5x_3 = -7 \quad \Longrightarrow \quad -3x_2 + 5 = -7 \quad \Longrightarrow \quad x_2 = 4$$

Finally,

$$7x_1 - 2x_2 + x_3 = 12 \quad \Longrightarrow \quad 7x_1 - 9 = 12 \quad \Longrightarrow \quad x_1 = 3$$

The solution to the original system of equations is $(3, 4, -1)$. ♠

This may seem overly complicated, but the factorization of A into LU is done by row reducing, so this method is no more costly than row-reduction in terms of operations used. An added benefit is that if we wish to find $\mathbf{x}$ for various vectors $\mathbf{b}$, we do not have to row-reduce all over again each time. Here are a few additional comments about this method:

- We will see how the LU-factorization is obtained through a series of exercises.
- The LU-factorization of a matrix is not unique; that is, there are different ways to factor a given matrix.
- LU-factorization can be done with non-square matrices, but we are not concerned with that idea.

Section 7.2 Exercises

1. In this exercise you will be working again with the system $\begin{array}{rcr} x_1 + 3x_2 - 2x_3 & = & -4 \\ 3x_1 + 7x_2 + x_3 & = & 4 \\ -2x_1 + x_2 + 7x_3 & = & 7 \end{array}$.

   For the purposes of the exercise we will let

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -\frac{7}{2} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 3 & -2 \\ 0 & -2 & 7 \\ 0 & 0 & \frac{55}{2} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$$

   (a) Write the system $L\mathbf{y} = \mathbf{b}$ as a system of three equations in the three unknowns y_1, y_2, y_3. Then solve the system by hand, showing clearly how it is done. In the end, give the vector $\mathbf{y}$.

   (b) Write the system $U\mathbf{x} = \mathbf{y}$ as a system of three equations in the three unknowns x_1, x_2, x_3. Then solve the system by hand, showing clearly how it is done. In the end, give the vector $\mathbf{x}$.

   (c) Use the linear combination of vectors interpretation of the system to show that the x_1, x_2, x_3 you found in part (b) is a solution to the system of equations. Show the scalar multiplication and vector addition as two separate steps.

   (d) Multiply L times U, in that order. What do you notice about the result? If you don't see something, you may have gone astray somewhere!

2. Let A be the coefficient matrix for the system from the previous exercise.

   (a) Give the matrix E_1 be the matrix for which E_1A is the result of the first row operation used to reduce A to U. Give the matrix E_1A.

   (b) Give the matrix E_2 such that $E_2(E_1A)$ is the result after the second row operation used to reduce A to U. Give the matrix E_2E_1A.

   (c) Give the matrix E_3 such that $E_3(E_2E_1A)$ is U.

   (d) Find the matrix $B = E_3E_2E_1$, then use your calculator to find B^{-1}. What is it? If you don't recognize it, you are asleep or you did something wrong!

3. (a) Fill in the blanks of the second matrix below with the entries from E_1. Then, without using your calculator, fill in the blanks in the first matrix so that the product of the first two matrices is the 3×3 identity, as shown.

$$\begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix} \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   Call the matrix you found F_1. Do the same thing with E_2 and E_3 to find matrices F_2 and F_3.

   (b) Find the product $F_1F_2F_3$, in that order. Again, you should recognize the result.