7.4 Determinants and Matrix Form

Performance Criterion:

7. (d) Find the determinant of a $2 \times 2$ or $3 \times 3$ matrix by hand. Use a calculator to find the determinant of an $n \times n$ matrix.
(e) Use the determinant to determine whether a system of equations has a unique solution.

Associated with every square matrix is a scalar that is called the determinant of the matrix, and determinants have numerous conceptual and practical uses. For a square matrix $A$, the determinant is denoted by $\det(A)$. This notation implies that the determinant is a function that takes a matrix and returns a scalar. Another notation is that the determinant of a specific matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ or $\left| \begin{array}{cc} a & b \\ c & d \end{array} \right|$.

There is a simple formula for finding the determinant of a $2 \times 2$ matrix:

**Definition 7.4.1: Determinant of a $2 \times 2$ Matrix**

The determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$.

*Example 7.4(a):* Find the determinant of $A = \begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}$

\[ \det(A) = (5)(-2) - (-2)(4) = -10 + 8 = -2 \]

There is a fairly involved method of breaking the determinant of a larger matrix down to where it is a linear combination of determinants of $2 \times 2$ matrices, but we will not go into that here. It is called the cofactor expansion of the determinant, and can be found in most any other linear algebra book, or online. Of course your calculator will find determinants of matrices whose entries are numbers, as will online matrix calculators and various software like MATLAB.

Later we will need to be able to find determinants of matrices containing an unknown parameter, and it will be necessary to find determinants of $3 \times 3$ matrices. For that reason, we now show a relatively simple method for finding the determinant of a $3 \times 3$ matrix. (This will not look simple here, but it is once you are familiar with it.) *This method only works for $3 \times 3$ matrices.*

\[
\begin{vmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33}
\end{vmatrix} \quad \Rightarrow
\begin{vmatrix}
 a_{11}a_{22}a_{33} & a_{12}a_{23}a_{31} & a_{13}a_{21}a_{32} \\
 a_{11}a_{22}a_{32} & a_{12}a_{23}a_{33} & a_{13}a_{21}a_{31} \\
 a_{11}a_{23}a_{32} & a_{12}a_{21}a_{33} & a_{13}a_{22}a_{31}
\end{vmatrix}
\]

We get the determinant by adding up each of the results of the downward multiplications and then subtracting each of the results of the upward multiplications. This is shown below.

\[
\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}
\]

*Example 7.4(b):* Find the determinant of $A = \begin{bmatrix} -1 & 5 & 2 \\ 3 & 1 & 6 \\ -5 & 2 & 4 \end{bmatrix}$.
\[
\begin{vmatrix}
-1 & 5 & 2 \\
3 & 1 & 6 \\
-5 & 2 & 4
\end{vmatrix} = \begin{vmatrix}
-10 & -12 \\
3 & 1 \\
-5 & 2
\end{vmatrix}
\]
\[
\begin{vmatrix}
-4 & 5 & 2 \\
3 & 4 & 6 \\
-5 & 2 & 4
\end{vmatrix}
\]

\[
\det(A) = (-4) + (-150) + 12 - (-10) - (-12) - 60 = -4 - 150 + 12 + 10 + 12 - 60 = -180
\]

In the future we will need to compute determinants like the following.

\[\text{Example 7.4(c):} \text{ Find the determinant of } B = \begin{vmatrix}
1 - \lambda & 0 & 3 \\
1 & -1 - \lambda & 2 \\
-1 & 1 & -2 - \lambda
\end{vmatrix}.\]

\[
\begin{align*}
\det(B) &= (1 - \lambda)(-1 - \lambda)(-2 - \lambda) + (0)(2)(-1) + (3)(1)(1) \\
&\quad - (-1)(-1 - \lambda)(3) - (1)(2)(1 - \lambda) - (-2 - \lambda)(1)(0) \\
&= (-1 + \lambda^2)(-2 - \lambda) + 3 - 3\lambda - 2 + 2\lambda \\
&= 2 + \lambda - 2\lambda^2 - \lambda^3 - \lambda - 2 \\
&= -\lambda^3 - 2\lambda^2
\end{align*}
\]

Here is why we care about determinants right now:

**Theorem 7.4.2: Determinants and Invertibility, Systems**

Let \( A \) be a square matrix.

(a) \( A \) is invertible if, and only if, \( \det(A) \neq 0 \).

(b) The system \( Ax = b \) has a unique solution if, and only if, \( A \) is invertible.

(c) If \( A \) is not invertible, the system \( Ax = b \) will have either no solution or infinitely many solutions.

Recall that when things are “nice” the system \( Ax = b \) can be solved as follows:

\[
\begin{align*}
Ax &= b \\
A^{-1}(Ax) &= A^{-1}b \\
(A^{-1}A)x &= A^{-1}b \\
Ix &= A^{-1}b \\
x &= A^{-1}b
\end{align*}
\]

In this case the system will have the unique solution \( x = A^{-1}b \). (When we say unique, we mean only one.)

If \( A \) is not invertible, the above process cannot be carried out, and the system will not have a single unique solution. In that case there will either be no solution or infinitely many solutions.
We previously discussed the fact that the above computation is analogous to the following ones involving simple numbers and an unknown number $x$:

\[
3x = 5
\]

\[
\frac{1}{3}(3x) = \frac{1}{3} \cdot 5
\]

\[
\left(\frac{1}{3} \cdot 3\right)x = \frac{5}{3}
\]

\[
1x = \frac{5}{3}
\]

\[
x = \frac{5}{3}
\]

Now let’s consider the following two equations, of the same form $ax = b$ but for which $a = 0$:

\[
0x = 5
\]

\[
0x = 0
\]

We first recognize that we can’t do as before and multiply both sides of each by $\frac{1}{a}$, since that is undefined. The first equation has no solution, since there is no number $x$ that can be multiplied by zero and result in five! In the second case, every number is a solution, so the system has infinitely many solutions. These equations are analogous to $Ax = b$ when $\det(A) = 0$. The one difference is that $Ax = b$ can have infinitely many solutions even when $b$ is NOT the zero vector.

Section 7.4 Exercises

1. Explain/show how to use the determinant to determine whether

\[
\begin{align*}
x + 3y - 3z &= -5 \\
2x - y + z &= -3 \\
-6x + 3y - 3z &= 4
\end{align*}
\]

has a unique solution. You may use your calculator for finding determinants - be sure to conclude by saying whether or not this particular system has a solution!

2. Suppose that you hope to solve a system $Ax = b$ of $n$ equations in $n$ unknowns.

(a) If the determinant of $A$ is zero, what does it tell you about the nature of the solution? (By “the nature of the solution” I mean no solution, a unique solution or infinitely many solutions.)

(b) If the determinant of $A$ is NOT zero, what does it tell you about the nature of the solution?

3. Suppose that you hope to solve a system $Ax = 0$ of $n$ equations in $n$ unknowns.

(a) If the determinant of $A$ is zero, what does it tell you about the nature of the solution? (By “the nature of the solution” I mean no solution, a unique solution or infinitely many solutions.)

(b) If the determinant of $A$ is NOT zero, what does it tell you about the nature of the solution?