8.2 Closure of a Set Under an Operation

> **Performance Criteria:**
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> 8. (c) Determine whether a set is closed under an operation. If it is, prove that it is; if it is not, give a counterexample.

Consider the set $\{0, 1, 2, 3, ...\}$, which are called the whole numbers. Notice that if we add or multiply any two whole numbers the result is also a whole number, but if we try subtracting two such numbers it is possible to get a number that is not in the set. We say that the whole numbers are **closed under addition and multiplication**, but the set of whole numbers is not closed under subtraction. If we enlarge our set to be the integers $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ we get a set that is closed under addition, subtraction and multiplication. These operations we are considering are called **binary operations** because they take two elements of the set and create a single new element. An operation that takes just one element of the set and gives another (possibly the same) element of the set is called a **unary operation**. An example would be absolute value; note that the set of integers is closed under absolute value.

> DEFINITION 8.2.1: **Closed Under an Operation**
>
> A set $\mathcal{S}$ is said to be closed under a binary operation $*$ if for every s and t in $\mathcal{S}$, $s * t$ is in $\mathcal{S}$. $\mathcal{S}$ is closed under a unary operation $\langle\ \rangle$ if for every s in $\mathcal{S}$, $\langle s \rangle$ is in $\mathcal{S}$.

Notice that the term "closed," as defined here, only makes sense in the context of a set with an operation. Notice also that *it is the set that is closed*, not the operation. The operation is important as well; as we have seen, a given set can be closed under one operation but not another.

When considering closure of a set $\mathcal{S}$ under a binary operation $*$, our considerations are as follows:

- We first wish to determine whether we think $\mathcal{S}$ *IS* closed under $*$.

- If we do think that $\mathcal{S}$ is closed under $*$, we then need to prove that it is. To do this, we need to take two general, or *arbitrary* elements x and y of $\mathcal{S}$ and show that $x * y$ is in $\mathcal{S}$.

- If we think that $\mathcal{S}$ is not closed under $*$, we need to take two *specific* elements x and y of $\mathcal{S}$ and show that $x * y$ is not in $\mathcal{S}$.

$\diamond$ **Example 8.2(a):** The odd integers are the numbers $..., -5, -3, -1, 1, 3, 5, ...$ Are the odd integers closed under addition? Multiplication?

We see that $3 + 5 = 8$. Because 3 and 5 are both odd but their sum isn't, the odd integers are not closed under addition. Let's try multiplying some odds:

$$3 \times 5 = 15 \qquad -7 \times 9 = -63 \qquad -1 \times -7 = 7$$

Based on these three examples, it appears that the odd integers are perhaps closed under multiplication. Let's attempt to prove it. First we observe that any number of the form $2n + 1$, where n is any integer, is odd. (This is in fact the definition of an odd integer.) So if we have two *possibly different* odd integers, we can write them as $2m + 1$ and $2n + 1$, where m and n are not necessarily the same integers. Their product is

$$(2m + 1)(2n + 1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1.$$

Because the integers are closed under multiplication and addition, $2mn + m + n$ is an integer and the product of $2m + 1$ and $2n + 1$ is of the form two times an integer, plus one, so it is odd as well. Therefore the odd integers are closed under multiplication. ♠

Closure of a set under an operation is a fairly general concept; let's narrow our focus to what is important to us in linear algebra.

◇ **Example 8.2(b):** Prove that then span of a set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, ..., \mathbf{v}_k\}$ in $\mathbb{R}^n$ is closed under addition and scalar multiplication.

Suppose that $\mathbf{u}$ and $\mathbf{w}$ are in $\mathrm{span}(S)$. Then there are scalars $c_1, c_2, c_3, ..., c_k$ and $d_1, d_2, d_3, ..., d_k$ such that

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k \qquad \text{and} \qquad \mathbf{w} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 + \cdots + d_k\mathbf{v}_k.$$

Therefore

$$
\begin{aligned}
\mathbf{u} + \mathbf{w} &= (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k) + (d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3 + \cdots + d_k\mathbf{v}_k) \\
&= (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + (c_3 + d_3)\mathbf{v}_3 + \cdots + (c_k + d_k)\mathbf{v}_k
\end{aligned}
$$

This last expression is a linear combination of the vectors in S, so it is in $\mathrm{span}(S)$. Therefore $\mathrm{span}(S)$ is closed under addition. Now suppose that $\mathbf{u}$ is as above and a is any scalar. Then

$$
\begin{aligned}
a\mathbf{u} &= a(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_k\mathbf{v}_k) \\
&= ac_1\mathbf{v}_1 + ac_2\mathbf{v}_2 + ac_3\mathbf{v}_3 + \cdots + ac_k\mathbf{v}_k
\end{aligned}
$$

which is also a linear combination of the vectors in S, so it is also in $\mathrm{span}(S)$. Thus $\mathrm{span}(S)$ is closed under multiplication by scalars. ♠

The result of the above Example is that

> THEOREM 8.2.2: The span of a set S of vectors is closed under vector addition and scalar multiplication.

This seemingly simple observation is the beginning of one of the most important stories in the subject of linear algebra. The remainder of this chapter and all of the next will fill out the rest of that story.