8.4 Column Space and Null Space of a Matrix

Performance Criteria:

8. (e) Determine whether a vector is in the column space or null space of a matrix, based only on the definitions of those spaces.

In this section we will define two important subspace associated with a matrix A, its **column space** and its **null space**.

DEFINITION 8.4.1: **Column Space of a Matrix**

The **column space** of an $m \times n$ matrix A is the span of the columns of A. It is a subspace of $\mathbb{R}^m$ and we denote it by $\text{col}(A)$.

◇ **Example 8.4(a):** Determine whether $\mathbf{u} = \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$ are in the column space of $A = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix}$.

We need to solve the two vector equations of the form

$$c_1 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix} = \mathbf{b}, \qquad (1)$$

with $\mathbf{b}$ first being $\mathbf{u}$, then $\mathbf{v}$. The respective reduced row-echelon forms of the augmented matrices corresponding to the two systems are

$$\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore we can find scalars c_1, c_2 and c_3 for which (1) holds when $\mathbf{b} = \mathbf{u}$, but not when $\mathbf{b} = \mathbf{v}$. From this we deduce that $\mathbf{u}$ is in $\text{col}(A)$, but $\mathbf{v}$ is not. ♠

Recall that the system $A\mathbf{x} = \mathbf{b}$ of m linear equations in n unknowns can be written in linear combination form:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Note that the left side of this equation is simply a linear combination of the columns of A, with the scalars being the components of $\mathbf{x}$. The system will have a solution if, and only if, $\mathbf{b}$ can be written as a linear combination of the columns of A. Stated another way, we have the following:

THEOREM 8.4.2: A system $A\mathbf{x} = \mathbf{b}$ has a solution (meaning *at least* one solution) if, and only if, $\mathbf{b}$ is in the column space of A.

Let's consider now only the case where $m = n$, so we have n linear equations in n unknowns. We have the following facts:

- If $\text{col}(A)$ is all of $\mathbb{R}^n$, then $A\mathbf{x} = \mathbf{b}$ will have a solution for any vector $\mathbf{b}$. What's more, *the solution will be unique.*
- If $\text{col}(A)$ is a proper subspace of $\mathbb{R}^n$ (that is, it is not all of $\mathbb{R}^n$), then the equation $A\mathbf{x} = \mathbf{b}$ will have a solution if, and only if, $\mathbf{b}$ is in $\text{col}(A)$. If $\mathbf{b}$ is in $\text{col}(A)$ the system will have infinitely many solutions.

Next we define the **null space** of a matrix.

> Definition 8.4.3: **Null Space of a Matrix**
>
> The **null space** of an $m \times n$ matrix A is the set of all solutions to $A\mathbf{x} = \mathbf{0}$. It is a subspace of $\mathbb{R}^n$ and is denoted by $\text{null}(A)$.

◇ **Example 8.4(b):** Determine whether $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ are in the null space of $A = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix}$.

A vector $\mathbf{x}$ is in the null space of a matrix A if $A\mathbf{x} = \mathbf{0}$. We see that

$$A\mathbf{u} = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -21 \\ 11 \end{bmatrix} \quad \text{and} \quad A\mathbf{v} = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so $\mathbf{v}$ is in the $\text{null}(A)$ and $\mathbf{u}$ is not. ♠

Still considering only the case where $m = n$, we have the following fact about the null space:

- If $\text{null}(A)$ is just the zero vector, A is invertible and $A\mathbf{x} = \mathbf{b}$ has a unique solution for any vector $\mathbf{b}$.

We conclude by pointing out the important fact that for an $m \times n$ matrix A, the null space of A is a subspace of $\mathbb{R}^n$ and the column space of A is a subspace of .

Section 8.4 Exercises

1. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 9 \\ -17 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 8 \\ -8 \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}.$$

   (a) The column space of A is the set of all vectors that are linear combinations of the columns of A. Determine whether the vector $\mathbf{u}_1$ is in the column space of A by determining whether $\mathbf{u}_1$ is a linear combination of the columns of A. Give the vector equation that you are trying to solve, and your row reduced augmented matrix. **Be sure to tell whether $\mathbf{u}_1$ is in the column space of A or not! Do this with a brief sentence.**

   (b) If $\mathbf{u}_1$ *IS* in the column space of A, give a *specific* linear combination of the columns of A that equals $\mathbf{u}_1$.

   (c) Repeat parts (a) and (b) for the vector $\mathbf{u}_2$.

2. Again let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 9 \\ -17 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 8 \\ -8 \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}.$$

   The null space of A is all the vectors $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{0}$, and it is denoted by $\text{null}(A)$. This means that to check to see if a vector $\mathbf{x}$ is in the null space we need only to compute $A\mathbf{x}$ and see if it is the zero vector. Use this method to determine whether either of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is in $\text{null}(A)$. Give your answer as a brief sentence.