Performance Criteria:

8. (e) Determine whether a vector is in the column space or null space of a matrix, based only on the definitions of those spaces.

In this section we will define two important subspace associated with a matrix $A$, its column space and its null space.

**Definition 8.4.1: Column Space of a Matrix**

The column space of an $m \times n$ matrix $A$ is the span of the columns of $A$. It is a subspace of $\mathbb{R}^m$ and we denote it by $\text{col}(A)$.

\[ \text{Example 8.4(a):} \text{ Determine whether } u = \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix} \text{ and } v = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \text{ are in the column space of } A = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix}. \]

We need to solve the two vector equations of the form

\[
c_1 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix} = b, \tag{1}
\]

with $b$ first being $u$, then $v$. The respective reduced row-echelon forms of the augmented matrices corresponding to the two systems are

\[
\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

Therefore we can find scalars $c_1$, $c_2$, and $c_3$ for which (1) holds when $b = u$, but not when $b = v$. From this we deduce that $u$ is in $\text{col}(A)$, but $v$ is not.

Recall that the system $Ax = b$ of $m$ linear equations in $n$ unknowns can be written in linear combination form:

\[
\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
\]

Note that the left side of this equation is simply a linear combination of the columns of $A$, with the scalars being the components of $x$. The system will have a solution if, and only if, $b$ can be written as a linear combination of the columns of $A$. Stated another way, we have the following:

**Theorem 8.4.2:** A system $Ax = b$ has a solution (meaning at least one solution) if, and only if, $b$ is in the column space of $A$. 

115
Let’s consider now only the case where \( m = n \), so we have \( n \) linear equations in \( n \) unknowns. We have the following facts:

- If \( \text{col}(A) \) is all of \( \mathbb{R}^n \), then \( Ax = b \) will have a solution for any vector \( b \). What’s more, the solution will be unique.

- If \( \text{col}(A) \) is a proper subspace of \( \mathbb{R}^n \) (that is, it is not all of \( \mathbb{R}^n \)), then the equation \( Ax = b \) will have a solution if, and only if, \( b \) is in \( \text{col}(A) \). If \( b \) is in \( \text{col}(A) \) the system will have infinitely many solutions.

Next we define the **null space** of a matrix.

**Definition 8.4.3: Null Space of a Matrix**

The null space of an \( m \times n \) matrix \( A \) is the set of all solutions to \( Ax = 0 \). It is a subspace of \( \mathbb{R}^n \) and is denoted by \( \text{null}(A) \).

\[ \begin{align*}
\circ \text{ Example 8.4(b): } & \text{ Determine whether } u = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \text{ and } v = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \text{ are in the null space of } A = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix}. \\
\text{ A vector } x \text{ is in the null space of a matrix } A \text{ if } Ax = 0. \text{ We see that } \\
Au = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -21 \\ 11 \end{bmatrix} \text{ and } Av = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{align*} \]

so \( v \) is in the \( \text{null}(A) \) and \( u \) is not. ☀️

Still considering only the case where \( m = n \), we have the following fact about the null space:

- If \( \text{null}(A) \) is just the zero vector, \( A \) is invertible and \( Ax = b \) has a unique solution for any vector \( b \).

We conclude by pointing out the important fact that for an \( m \times n \) matrix \( A \), the null space of \( A \) is a subspace of \( \mathbb{R}^n \) and the column space of \( A \) is a subspace of \( \mathbb{R}^n \).

### Section 8.4 Exercises

1. Let

\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 2 \\ 9 \\ -17 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 8 \\ -8 \\ -4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}. \]

(a) The column space of \( A \) is the set of all vectors that are linear combinations of the columns of \( A \). Determine whether the vector \( u_1 \) is in the column space of \( A \) by determining whether \( u_1 \) is a linear combination of the columns of \( A \). Give the vector equation that you are trying to solve, and your row reduced augmented matrix. Be sure to tell whether \( u_1 \) is in the column space of \( A \) or not! Do this with a brief sentence.

(b) If \( u_1 \) IS in the column space of \( A \), give a specific linear combination of the columns of \( A \) that equals \( u_1 \).

(c) Repeat parts (a) and (b) for the vector \( u_2 \).

2. Again let

\[ A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 2 \\ 9 \\ -17 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 8 \\ -8 \\ -4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}. \]

The null space of \( A \) is all the vectors \( x \) for which \( Ax = 0 \), and it is denoted by \( \text{null}(A) \). This means that to check to see if a vector \( x \) is in the null space we need only to compute \( Ax \) and see if it is the zero vector. Use this method to determine whether either of the vectors \( v_1 \) and \( v_2 \) is in \( \text{null}(A) \). Give your answer as a brief sentence.