9.1 Linear Independence

Performance Criterion:

9. (a) Determine whether a set $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$ of vectors is a linearly independent or linearly dependent. If the vectors are linearly dependent, (1) give a non-trivial linear combination of them that equals the zero vector, (2) give any one as a linear combination of the others, when possible.

Suppose that we are trying to create a set $\mathcal{S}$ of vectors that spans $\mathbb{R}^3$. We might begin with one vector, say $\mathbf{u}_1 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$, in $\mathcal{S}$. We know by now that the span of this single vector is all scalar multiples of it, which is a line in $\mathbb{R}^3$. If we wish to increase the span, we would add another vector to $\mathcal{S}$. If we were to add a vector like $\begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}$ to $\mathcal{S}$, we would not increase the span, because this new vector is a scalar multiple of $\mathbf{u}_1$, so it is on the line we already have and would contribute nothing new to the span of $\mathcal{S}$. To increase the span, we need to add to $\mathcal{S}$ a second vector $\mathbf{u}_2$ that is not a scalar multiple of the vector $\mathbf{u}_1$ that we already have. It should be clear that the vector $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not a scalar multiple of $\mathbf{u}_1$, so adding it to $\mathcal{S}$ would increase it's span.

The span of $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2\}$ is a plane. When $\mathcal{S}$ included only a single vector, it was relatively easy to determine a second vector that, when added to $\mathcal{S}$, would increase it's span. Now we wish to add a third vector to $\mathcal{S}$ to further increase its span. Geometrically it is clear that we need a third vector that is *not in the plane spanned by* $\{\mathbf{u}_1, \mathbf{u}_2\}$. Probabilistically, just about any vector in $\mathbb{R}^3$ would do, but what we would like to do here is create an algebraic condition that needs to be met by a third vector so that adding it to $\mathcal{S}$ will increase the span of $\mathcal{S}$.

Let's begin with what we *DON'T* want: we don't want the new vector to be in the plane spanned by $\{\mathbf{u}_1, \mathbf{u}_2\}$. Now every vector $\mathbf{v}$ in that plane is of the form $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ for some scalars c_1 and c_2. We say the vector $\mathbf{v}$ created this way is "dependent" on $\mathbf{u}_1$ and $\mathbf{u}_2$, and that is what causes it to not be helpful in increasing the span of a set that already contains those two vectors. Assuming that neither of c_1 and c_2 is zero, we could also write

$$\mathbf{u}_1 = \frac{c_2}{c_1}\mathbf{u}_2 - \frac{1}{c_1}\mathbf{v} \qquad \text{and} \qquad \mathbf{u}_2 = \frac{c_1}{c_2}\mathbf{u}_1 - \frac{1}{c_2}\mathbf{v},$$

showing that $\mathbf{u}_1$ is "dependent" on $\mathbf{u}_2$ and $\mathbf{v}$, and $\mathbf{u}_2$ is "dependent" on $\mathbf{u}_1$ and $\mathbf{v}$. So whatever "dependent" means (we'll define it more formally soon) all three vectors are dependent on each other. We can create another equation that is equivalent to all three of the ones given so far, and that does not "favor" any particular one of the three vectors:

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{v} = \mathbf{0},$$

where $c_3 = -1$.

Of course, if we want a third vector $\mathbf{u}_3$ to add to $\{\mathbf{u}_1, \mathbf{u}_2\}$ to increase its span, we would not want to choose $\mathbf{u}_3 = \mathbf{v}$; instead we would want a third vector that is "independent" of the two we already have. Based on what we have been doing, we would suspect that we would want

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 \neq \mathbf{0}. \tag{1}$$

Of course even if $\mathbf{u}_3$ was not in the plane spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$, (1) would be true if $c_1 = c_2 = c_3 = 0$, but we want that to be the only choice of scalars that makes (1) true.

We now make the following definition, based on our discussion:

Definition 9.1.1: Linear Dependence and Independence

A set $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ of vectors is **linearly dependent** if there exist scalars $c_1, c_2, ..., c_k$, *not all equal to zero* such that

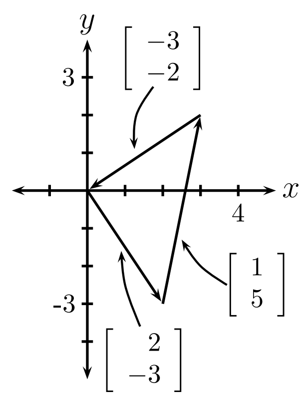
$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}. \tag{2}$$

If (2) only holds for $c_1 = c_2 = \cdots = c_k = 0$ the set $\mathcal{S}$ is **linearly independent**.

We can state linear dependence (independence) in either of two ways. We can say that the set is linearly dependent, or the vectors are linearly dependent. Either way is acceptable. Often we will get lazy and leave off the "linear" of linear dependence or linear independence. This does no harm, as there is no other kind of dependence/independence that we will be interested in.

Let's explore the idea of linearly dependent vectors a bit more by first looking at a specific example; consider the following sum of vectors in $\mathbb{R}^2$:

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$



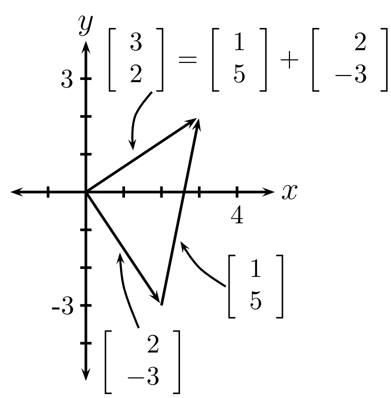
The picture to the right gives us some idea of what is going on here. Recall that when adding two vectors by the tip-to-tail method, the sum is the vector from the tail of the first vector to the tip of the second. We can add three vectors in the same way, putting the tail of the second at the tip of the first, and the tail of the third at the tip of the second. The sum is then the vector from the tail of the first vector to the tip of the third; in this case it is the zero vector since both the tail of the first vector and the tip of the third are at the origin. Letting $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, equation (3) above becomes

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = 0,$$

where $c_1 = c_2 = c_3 = 1$. Therefore the three vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ are linearly dependent.

Now if we add the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ to both sides of equation (3) we obtain the equation

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



The geometry of this equation can be seen in the picture to the right. We have basically "reversed" the vector $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$, and we can now see that the "reversed" vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is a linear combination of the two vectors $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$. This indicates that if three vectors are linearly dependent, then one of them can be written as a linear combination of the others.

Let's consider the more general case of a set $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ of linearly dependent vectors in $\mathbb{R}^n$. By definitions, there are scalars $c_1, c_2, ..., c_k$, not all equal to zero, such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}$$

Let c_j, for some j between 1 and k, be one of the non-zero scalars. (By definition there has to be at least one such scalar.) Then we can do the following:

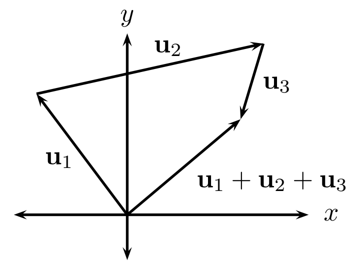
$$
\begin{aligned}
c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_j\mathbf{u}_j + \cdots + c_k\mathbf{u}_k &= \mathbf{0} \\
c_j\mathbf{u}_j &= -c_1\mathbf{u}_1 - c_2\mathbf{u}_2 - \cdots - c_k\mathbf{u}_k \\
\mathbf{u}_j &= -\tfrac{c_1}{c_j}\mathbf{u}_1 - \tfrac{c_2}{c_j}\mathbf{u}_2 - \cdots - \tfrac{c_k}{c_j}\mathbf{u}_k \\
\mathbf{u}_j &= d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + \cdots + d_k\mathbf{u}_k
\end{aligned}
$$

This, along with the previous specific example in $\mathbb{R}^2$, gives us the following:

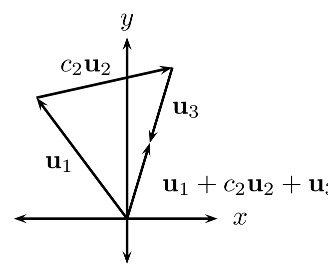
> <u>THEOREM 9.1.2</u>: If a set $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is linearly dependent, then at least one of these vectors can be written as a linear combination of the remaining vectors.

The importance of this, which we'll reiterate again later, is that *if we have a set of linearly dependent vectors with a certain span, we can eliminate at least one vector from our original set without reducing the span of the set.* If, on the other hand, we have a set of linearly *independent* vectors, eliminating any vector from the set will reduce the span of the set.

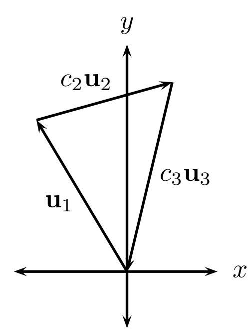
We now consider three vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ in $\mathbb{R}^2$ whose sum is not the zero vector, and for which no two of the vectors are parallel. I have arranged these to show the tip-to-tail sum in the top diagram to the right; clearly their sum is not the zero vector.



At this point if we were to multiply $\mathbf{u}_2$ by some scalar c_2 less than one we could shorten it to the point that after adding it to $\mathbf{u}_1$ the tip of $c_2\mathbf{u}_2$ would be in such a position as to line up $\mathbf{u}_3$ with the origin. This is shown in the bottom diagram to the right.



Finally, we could then multiply $\mathbf{u}_3$ by a scalar c_3 greater than one to lengthen it to the point of putting its tip at the origin. We would then have $\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$. You should play around with a few pictures to convince yourself that this can always be done with three vectors in $\mathbb{R}^2$, as long as none of them are parallel (scalar multiples of each other). This shows us that *any three vectors in* $\mathbb{R}^2$ *are always linearly dependent.* In fact, we can say even more:



$\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$

> <u>THEOREM 9.1.3</u>: Any set of more than n vectors in $\mathbb{R}^n$ must be linearly dependent.

Let's start looking at some specific examples now.

◇ **Example 9.1(a):** Determine whether the vectors $\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix}$ are linearly dependent, or linearly independent. If they are dependent, give a non-trivial linear combination of them that equals the zero vector. (Non-trivial means that not all of the scalars are zero!)

To make such a determination we always begin with the vector equation from the definition:

$$c_1 \begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

We recognize this as the linear combination form of a system of equations that has the augmented matrix shown below and to the left, which reduces to the matrix shown below and to the right.

$$\begin{bmatrix} -1 & 1 & 7 & 0 \\ -7 & -3 & -1 & 0 \\ 3 & 2 & 4 & 0 \\ 11 & 5 & 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we see that there are infinitely many solutions, so there are certainly values of c_1, c_2 and c_3, not all zero, that make (4) true, so the set of vectors is linearly dependent. To find a non-trivial linear combination of the vectors that equals the zero vector we let the free variable c_3 *be any value other than zero*. (You should try letting it be zero to see what happens.) If we take c_3 to be one, then $c_2 = -5$ and $c_1 = 2$. Then

$$2\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} - 5\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -14 \\ 6 \\ 22 \end{bmatrix} + \begin{bmatrix} -5 \\ 15 \\ -10 \\ -25 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \spadesuit$$

◇ **Example 9.1(b):** Determine whether the vectors $\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 5 \\ -1 \end{bmatrix}$ are linearly dependent, or linearly independent. If they are dependent, give a non-trivial linear combination of them that equals the zero vector. (Non-trivial means that not all of the scalars are zero!)

To make such a determination we always begin with the vector equation from the definition:

$$c_1 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We recognize this as the linear combination form of a system of equations that has the augmented matrix shown below and to the left, which reduces to the matrix shown below and to the right.

$$\begin{bmatrix} 3 & 4 & -2 & 0 \\ -1 & 7 & 5 & 0 \\ 2 & 0 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We see that the only solution to the system is $c_1 = c_2 = c_3 = 0$, so the vectors are linearly independent. ♠

A comment is in order at this point. The system $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}$ is homogeneous, so it will always have at least the zero vector as a solution. It is precisely *when the only solution is the zero vector that the vectors are linearly independent.*

Here's an example demonstrating the fact that if a set of vectors is linearly dependent, at least one of them can be written as a linear combination of the others:

◇ **Example 9.1(c):** In Example 9.1(a) we determined that the vectors $\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix}$ are linearly dependent. Give one of them as a linear combination of the others.

When testing for linear dependence we found that we could write

$$2\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} - 5\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

The easiest vector to solve for is the third, by simply subtracting the other two and their scalars from both sides of this equation:

$$\begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = -2\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} + 5\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}$$

However, going back to (5) we could have instead subtracted the second and third vectors and their scalars from both sides, then multiplied both sides by $\frac{1}{2}$ to get

$$\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix}$$

Of course we could also solve for the second vector in a similar manner! ♠

Section 9.1 Exercises

1. Consider the vectors $\mathbf{u}_1 = \begin{bmatrix} -5 \\ 9 \\ 4 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix}$.

   (a) Give the *VECTOR* equation that we must consider in order to determine whether the three vectors are linearly independent.

   (b) Your equation has one solution for sure. What is it? What does it mean (in terms of linear dependence or independence) if that is the *ONLY* solution?

   (c) Write your equation from (a) as a system of linear equations. Then give the augmented matrix for the system.

   (d) Does the system have more solutions than the one you gave in (b)? If so, find one of them. (By "one" I mean one ordered triple of three numbers.)

   (e) Find each of the three vectors as a linear combination of the other two.

2. Show that the vectors $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -4 \\ 4 \\ 9 \end{bmatrix}$ are linearly dependent. Then give one of the vectors as a linear combination of the others.

3. For the following, use the vectors $\mathbf{v_1} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$.

   (a) Determine whether $\mathbf{u} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 17 \\ -17 \end{bmatrix}$ are in span$(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$.

   (b) Show that the vectors $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$ are linearly dependent by the definition of linearly dependent. In other words, produce scalars c_1, c_2 and c_3 and demonstrate that they and the vectors satisfy the equation given in the definition.

   (c) Since the vectors are linearly dependent, at least one the vectors can be expressed as a linear combination of the other two. Express $\mathbf{v_1}$ as a linear combination of $\mathbf{v_2}$ and $\mathbf{v_3}$.