## 9.3 Bases for the Column Space and Null Space of a Matrix

**Performance Criteria:**

9. (c) Find the dimension and bases for the column space and null space of a given matrix.

   (d) Given the dimension of the column space and/or null space of the coefficient matrix for a system of equations, say as much as you can about how many solutions the system has.

In a previous section you learned about two special subspaces related to a matrix $A$, the column space of $A$ and the null space of $A$. Remember the importance of those two spaces:

A system $A\mathbf{x} = \mathbf{b}$ has a solution if, and only if, $\mathbf{b}$ is in the column space of $A$.

If the null space of a square matrix $A$ is just the zero vector, $A$ is invertible and $A\mathbf{x} = \mathbf{b}$ has a unique solution for any vector $\mathbf{b}$.

We would now like to be able to find bases for the column space and null space of a given vector $A$. The following describes how to do this:

Theorem 9.3.1: **Bases for Null Space and Column Space**

- A basis for the column space of a matrix $A$ is the columns of $A$ corresponding to columns of $rref(A)$ that contain leading ones.
- The solution to $A\mathbf{x} = \mathbf{0}$ (which can be easily obtained from $rref(A)$ by augmenting it with a column of zeros) will be an arbitrary linear combination of vectors. Those vectors form a basis for null($A$).

◇ **Example 9.3(a):** Find bases for the null space and column space of $A = \begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix}$.

The reduced row-echelon form of $A$ is shown below and to the left. We can see that the first through third columns contain leading ones, so a basis for the column space of $A$ is the set shown below and to the right.

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \right\}$$

If we were to augment $A$ with a column of zeros to represent the system $A\mathbf{x} = \mathbf{0}$ and row reduce we'd get the matrix shown above and to the left but with an additional column of zeros on the right. We'd then have $x_4$ as a free variable $t$, with $x_1 = -3t$, $x_2 = 7t$ and $x_3 = -4t$. The solution to $A\mathbf{x} = \mathbf{0}$ is any scalar multiple of $\begin{bmatrix} -3 \\ 7 \\ -4 \\ 1 \end{bmatrix}$, so that vector is a basis for the null space of $A$. ♠

$\diamond$ **Example 9.3(b):** Find a basis for the null space and column space of $A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix}$.

The reduced row-echelon form of this matrix is the identity, so a basis for the column space consists of all the columns of $A$. If we augment $A$ with the zero vector and row reduce we get a solution of the zero vector, so the null space is just the zero vector (which is of course a basis for itself). ♠

We should note in the last example that the column space is all of $\mathbb{R}^3$, so for any vector $\mathbf{b}$ in $\mathbb{R}^3$ there is a vector $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{b}$. Thus $A\mathbf{x} = \mathbf{b}$ has a solution for every choice of $\mathbf{b}$.

There is an important distinction to be made between a subspace and a basis for a subspace:

- Other than the trivial subspace consisting of the zero vector, *a subspace is an infinite set of vectors.*
- A basis for a subspace *is a finite set of vectors.* In fact a basis consists of relatively few vectors; the basis for any subspace of $\mathbb{R}^n$ contains at most $n$ vectors (and it only contains $n$ vectors if the subspace is all of $\mathbb{R}^n$).

To illustrate, consider the matrix $A = \begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix}$ from Example 9.3(a). The set $\left\{ \begin{bmatrix} -3 \\ 7 \\ -4 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space of $A$, whereas the set $\left\{ t \begin{bmatrix} -3 \\ 7 \\ -4 \\ 1 \end{bmatrix} \right\}$ *IS* the null space of $A$.

We finish this section with a couple definitions and a major theorem of linear algebra. The importance of these will be seen in the next section.

<u>Definition 9.3.2</u>: **Rank and Nullity of a Matrix**

- The **rank** of a matrix $A$, denoted $\text{rank}(A)$, is the dimension of its column space.
- The **nullity** of a matrix $A$, denoted $\text{nullity}(A)$, is the dimension of its null space.

<u>Theorem 9.3.3</u>: **The Rank Theorem**

For an $m \times n$ matrix $A$, $\text{rank}(A) + \text{nullity}(A) = n$.

### Section 9.3 Exercises

1. Consider the matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & 6 \\ 2 & 2 & -4 \end{bmatrix}$, which has row-reduced form $\begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. In this exercise you will see how to find a basis for the null space of $A$. All this means is that *you are looking for a "minimal" set of vectors whose span (all possible linear combinations of them) give all the vectors* $\mathbf{x}$ *for which* $A\mathbf{x} = \mathbf{0}$.

   (a) Give the augmented matrix for the system of equations $A\mathbf{x} = \mathbf{0}$, then give its row reduced form.

   (b) There are two free variables, $x_3$ and $x_2$. Let $x_3 = t$ and $x_2 = s$, then find $x_1$ (in terms of $s$ and $t$). Give the vector $\mathbf{x}$, in terms of $s$ and $t$.

(c) Write $\mathbf{x}$ as the sum of two vectors, one containing only the parameter $s$ and the other containing only the parameter $t$. Then factor $s$ out of the first vector and $t$ out of the second vector. You now have $\mathbf{x}$ as all linear combinations of two vectors.

(d) Those two vectors are linearly independent, since neither of them is a scalar multiple of the other, so both are essential in the linear combination you found in (c). They then form a basis for the null space of $A$. Write this out as a full sentence, "A basis for ...". *A basis is technically a set of vectors, so use the set brackets* $\{\ \ \}$ *appropriately.*

2. Consider the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & 11 \\ 2 & 5 & 3 \end{bmatrix}$

(a) Solve the system $A\mathbf{x} = \mathbf{0}$. You should get infinitely many solutions containing one or more parameters. Give the general solution, in terms of the parameters. **Give all values in exact form**.

(b) If you didn't already, you should be able to give the general solution as a linear combination of vectors, with the scalars multiplying them being the parameter(s). Do this.

(c) The vector or vectors you see in (c) is (are) a basis for the null space of $A$. Give the basis.

3. When doing part (a) of the previous exercise you should have obtained the row reduced form of the matrix $A$ (of course you augmented it). A basis for the column space of $A$ is the columns of $A$ (*NOT* the columns of the row reduced form of $A$!) corresponding to the leading variables in the row reduced form of $A$. Give the basis for the column space of $A$.

4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 9 \\ -17 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 8 \\ -8 \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}.$$

(a) Determine whether each of $\mathbf{u}_1$ and $\mathbf{u}_2$ is in the column space of $A$.

(b) Find a basis for col($A$). **Give your answer with a brief sentence, and indicate that the basis is a set of vectors.**

(c) One of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ *IS* in the column space of $A$. Give a linear combination of the *basis vectors* that equals that vector.

5. Again let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 9 \\ -17 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 8 \\ -8 \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}.$$

(a) Determine whether each of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is in null($A$). Give your answer as a brief sentence.

(b) Determine a basis for null($A$), giving your answer in a brief sentence.

(c) Give the linear combinations of the basis vectors of the null space for either of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ that are in the null space.

6. Give a sentence telling the dimensions of the column space and null space of the matrix $A$ from the previous two exercises.