Linear Algebra I

Skills, Concepts and Applications

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1 Systems of Linear Equations

Learning Outcome:

1. Solve systems of linear equations using Gaussian elimination, use systems of linear equations to solve problems.

   Performance Criteria:

   (a) Determine whether an equation in n unknowns is linear.

   (b) Set up a system of linear equations to find coefficients of a line or polynomial through a given set of points, or to model flow in a network or equilibrium temperatures in a solid object.

   (c) Determine whether an n-tuple is a solution to a linear equation or a system of linear equations.

   (d) Solve a system of two linear equations by the addition method.

   (e) Give the coefficient matrix and augmented matrix for a system of equations.

   (f) Determine whether a matrix is in row-echelon form. Perform, by hand, elementary row operations to reduce a matrix to row-echelon form.

   (g) Determine whether a matrix is in reduced row-echelon form. Use technology to reduce a matrix to reduced row-echelon form.

   (h) For a system of equations having a unique solution, determine the solution from either the row-echelon form or reduced row-echelon form of the augmented matrix for the system.

   (i) Use a calculator to solve a system of linear equations having a unique solution.

   (j) Given the row-echelon or reduced row-echelon form of an augmented matrix for a system of equations, determine the leading variables and free variables of the system.

   (k) Given the row-echelon or reduced row-echelon form for a system of equations:

   - Determine whether the system has a unique solution, and give the solution if it does.
   - If the system does not have a unique solution, determine whether it is inconsistent (no solution) or dependent (infinitely many solutions).
   - If the system is dependent, give the general form of a solution and give some particular solutions.

   (l) Use systems of equations to solve network problems.

1.1 Linear Equations and Systems of Linear Equations

Performance Criteria:

1. (a) Determine whether an equation in n unknowns is linear.

   (b) Set up a system of linear equations to find coefficients of a line or polynomial through a given set of points, or to model flow in a network or equilibrium temperatures in a solid object.

   (c) Determine whether an n-tuple is a solution to a linear equation or a system of linear equations.

Linear Equations and Their Solutions

It is natural to begin our study of linear algebra with the process of solving systems of linear equations, and applications of such systems.

DEFINITION 1.1.1: A **linear equation** in n unknowns is an equation that can be put in the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b, \tag{1}$$

where a_1, a_2, ..., a_n and b are *known* constants and x_1, x_2, ... , x_n are unknown values. A **solution to a linear equation** is a collection of values for the unknowns that makes the equation true.

◇ **Example 1.1(a):** Which of the equations

$$y = -\tfrac{2}{3}x + 4 \qquad y = -16t^2 + 61t + 7 \qquad 5.3x + 7.2y + 1.4z = 16.9$$

$$a_{41}x_1 + a_{42}x_2 + \cdots + a_{4n}x_n = b_4,$$

where a_{11}, a_{12}, ..., a_{1n}, b_1 are all known numbers, are linear equations?

Solution: The first equation can be rewritten as $\frac{2}{3}x + y = 4$, so it is a linear equation. The second equation can be written $-16t^2 + 61t - y = -4$, but the t^2 prevents this from being a linear equation. The third and fourth equations are in exactly the form (1), so they are linear.

A few comments are in order:

- For the third equation above we see that

$$5.3(1) + 7.2(2) + 1.4(-2) = 16.9,$$

  so $x = 1, y = 2$ and $z = -2$ is a solution to $5.3x + 7.2y + 1.4z = 16.9$. To save writing we usually write such a solution as $(1, 2, -2)$, a form you are likely familiar with.

- The equation $y = -\frac{2}{3}x + 4$ can also be rewritten as $2x + 3y = 12$ instead of $\frac{2}{3}x + y = 4$. An (x, y) pair that is a solution to any one of the forms is also a solution to the other two (and any pair that is *NOT* a solution to any one of them will not be a solution to the other two either). We can multiply or divide both sides of a linear equation by a value in order to make it easier to work with, if we wish.

- Although you may have used x and y, or x, y and z as the unknown quantities in the past, like in the third equation above, we will often use x_1, x_2, ..., x_n instead. Thus the third equation could be written
$$5.3x_1 + 7.2x_2 + 1.4x_3 = 15.9,$$
which is equivalent to the fourth equation with $a_{41} = 5.3$, $a_{42} = 7.2$, $a_{43} = 1.4$ and $b_4 = 15.9$. One obvious advantage to using the letter a for all of the numbers is that we don't have to fret about what letters to use, and there is no danger of running out of letters! You will eventually see that there is also a very good mathematical reason for using just x (or some other single letter), with subscripts denoting different values.

It is important that you easily recognize the form (1) from the definition of a linear equation. Soon we will be interested in similar equations, but of the form

$$number \cdot vector + number \cdot vector + \cdots + number \cdot vector = vector.$$

We now move on to the concept that forms the beginning of our study of linear algebra:

DEFINITION 1.1.2: A **system of linear equations** is a set of linear equations containing the same unknowns. (Not every equation needs to contain every unknown.) A **solution to a system of linear equations** is a collection of values for the unknowns that makes every equation of the system true.

◇ **Example 1.1(b):** Which of the following are systems of linear equations?

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned} \qquad \begin{aligned} x + y^2 &= 3 \\ x^2 + y^2 &= 5 \end{aligned} \qquad \begin{aligned} 4t_1 - t_2 - t_3 &= 108 \\ -t_1 + 4t_2 - t_4 &= 106 \\ -t_1 + 4t_3 - t_4 &= 94 \\ -t_2 - t_3 + 4t_4 &= 96 \end{aligned}$$

Solution: The first and third systems are systems of linear equations, the second is not. The second is a system of *nonlinear* equations. One can verify that $(3, -1, 2)$ is a solution to the first system of equations.

Here we note the following:

- The first system in the previous example is a system of three equations in three unknowns. We will spend a lot of time with such systems because they exhibit just about everything that we would like to see but are small enough to be manageable to work with. As noted before, we will often use x_1, x_2 and x_3 instead of x, y and z for the unknowns.

- The numbers that the unknowns are multiplied by are **coefficients** of the system. It is customary to get the coefficient/unknown terms on the left, and the numbers not multiplying an unknown on the right, as shown in the first and third (and second, for that matter) examples above. The numbers without unknowns are often referred to as the "right hand sides."

- One should note carefully the coefficients of the third system and how they are arranged, as shown to the right. Later we will put some brackets around such an array and call it a **matrix**. The fours are on what we will call the **diagonal** of the matrix. (It seems that there is another diagonal with zeros on it, but that diagonal, from lower left to upper right, has no real significance. We therefore make no special note of what is going on there.) In addition to noting the fours on the diagonal, we also need to make special note of the way that the zeros and negative ones are arranged symmetrically across the diagonal. That sort of pattern is commonly encountered in physical applications of systems of linear equations.

$$\begin{array}{rrrr} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{array}$$

- When discussing a system of linear equations in general, we often use the following notation, given for a system of m equations in n unknowns:

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array} \tag{2}$$

  Here the a_{ij}s are the coefficients, with the first subscript of each giving the equation and the second subscript giving which unknown x_1, x_2, ... it is with.

Systems of equations arise naturally in many engineering applications (as well as applications in other areas like business). Some of the uses of systems of equations that we'll work with are

- analysis of electric circuits
- equilibrium distribution of heat in solid materials
- stress and strain in solid materials
- linear regression (least-squares approximation)

You will begin exploring some of these applications in the exercises for this section and the next. There are applications of other linear algebra concepts that we'll see later, such as

- robotics and computer graphics
- sports and internet search rankings
- air travel routing
- signal processing

Section 1.1 Exercises To Solutions

1. Which of the following equations are linear equations?

   (a) $4x^2 + 3y^2 = 5$ (b) $\dfrac{t_1 + t_2 + 83}{4} = t_3$ (c) $3x_1 - x_2 + 4x_3 = x_2$

   (d) $4.3 = 1.7m + b$ (e) $y = \sqrt{10 - x}$ (f) $\dfrac{5}{x} + \dfrac{2}{y} = 7$

2. Which of the systems of equations below are linear?

$$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= 7 \end{aligned} \qquad \begin{aligned} 3x + y - 2z &= -4 \\ 5x + 4z &= 3 \\ x - y + 2z &= 0 \end{aligned} \qquad \begin{aligned} x^2 - y &= 3 \\ x - y &= 1 \end{aligned}$$

3. (a) Determine which of the following are solutions to the first system of equations in the previous exercise: $(5, -2, 4)$, $(-2, -3, 2)$, $(7, 3, 1)$

   (b) Determine which of the following are solutions to the second system of equations in the previous exercise: $(3, -19, -3)$, $(-1, 3, 2)$, $(5, -2, 4)$

   (c) Determine which of the following are solutions to the third system of equations in the previous exercise: $(2, 1)$, $(3, 5)$, $(-1, -2)$

4. Consider the equation $y = ax^3 + bx^2 + cx + d$, representing a third degree polynomial.

   (a) Substitute the value -2 for x and 5 for y into $y = ax^3 + bx^2 + cx + d$, and simplify the result. Is the resulting equation linear?

   (b) Substitute the values $a = 7$, $b = -2$, $c = -5$ and $d = 1$ into $y = ax^3 + bx^2 + cx + d$. Is the resulting equation linear?

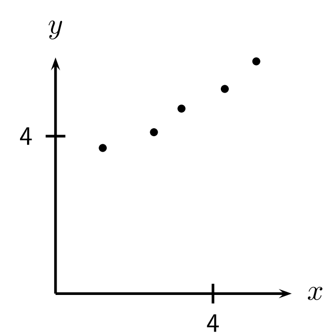
5. It turns out that there is exactly one third degree polynomial with equation $y = ax^3 + bx^2 + cx + d$ whose graph goes through the four points

$$(-2, 5) \qquad (-1, 2) \qquad (1, 3) \qquad (3, 0)$$

   Substitute each of those pairs into the equation (one pair at a time) to obtain four equations in the four unknowns a, b, c and d. Give your final system in the form (2). Once we know how to solve such a system we can determine the values of a, b, c and d.

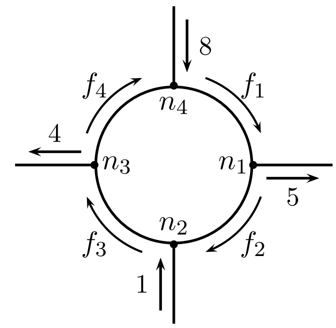
6. The graph to the right shows a plot of the five points with coordinates

$$(1.2, 3.7) \quad (2.5, 4.1) \quad (3.2, 4.7) \quad (4.3, 5.2) \quad (5.1, 5.9).$$

   You can see that there is no line containing them, but the points are arranged somewhat linearly. In many applications it is desirable to find the line that comes closest (in a sense to be described later) to passing through all of the points. Substitute each of the points individually into the equation $y = mx + b$ to obtain a system of five equations. (What are the unknowns?) Give the system in the form (2).

7. In many engineering applications we are interested in flow through a **network**. The flow could consist of water or air through pipes or ductwork (mechanical engineering), electrons through a circuit (or current, electrical engineering) or automobiles on a raodway (civil engineering). Such a network can be modeled by a set of **nodes** or **vertices** connected by arcs or line segments, typically called **edges**. The guiding principle for most such networks is simple: *the flow into any node must equal the flow out*. The network to the right represents a traffic circle. The numbers next to each of the paths leading into or out of the circle are the net flows in the directions of the arrows, in vehicles per minute, during the first part of lunch hour. The unknowns f_1, f_2, f_3 and f_4 represent the flows in the corresponding arcs of the traffic circle.



(a) The nodes of the network have been labeled n_1, n_2, n_3 and n_4. At node n_1, "flow in equals flow out" gives us $f_1 = f_2 + 5$. Rearranging this to get the unknowns on one side with f_1 positive, we get $f_1 - f_2 = 5$. Repeat at nodes n_2, n_3 and n_4, with the corresponding flows being positive in each case. That is, f_2 should be positive in the equation obtained at n_2, and so on. Give the resulting system of equations.

(b) Determine which of the following "4-tuples" (an n-**tuple** is an ordered "tuple" or collection of values, separated by commas) are solutions to the system that you obtained. *Note that this illustrates that a system can have more than one solution.*

$(12, 7, 8, 4)$ $(7, 2, 3, -1)$ $(10, 5, 6, 2)$ $(9, 4, 3, 1)$

(c) Which of the 4-tuples that you found to be a solution to your system would cause a problem with the traffic circle? Explain.

(d) Suppose that $f_3 = 9$. Just by looking at the traffic circle and using the "flow in equals flow out, determine the other three flows.

1.2 Curve Fitting, Temperature Equilibrium and Electric Circuits

Performance Criterion:

1. (b) Set up a system of linear equations to find coefficients of a line or polynomial through a given set of points, or to model flow in a network or equilibrium temperatures in a solid object.

Curve Fitting

Curve fitting refers to the process of finding a polynomial function of "minimal degree" whose graph contains some given points. We all know that any two distinct points (that is, points that are not the same) in $\mathbb{R}^2$ have exactly one line through them. In a previous course you should have learned how to find the equation of that line in the following manner. Suppose that we wish to find the equation of the line through the points $(2,3)$ and $(6,1)$. We know that the equation of a line looks like $y = mx + b$, where m and b are to be determined. m is the slope, which can be found by $m = \dfrac{3-1}{2-6} = \dfrac{2}{-4} = -\frac{1}{2}$. Therefore the equation of our line looks like $y = -\frac{1}{2}x + b$. To find b we simply substitute either of the given ordered pairs into our equation (the fact that both pairs lie on the line means that either pair is a solution to the equation) and solve for b: $3 = -\frac{1}{2}(2) + b \implies b = 4$. The equation of the line through $(2,3)$ and $(6,1)$ is then $y = -\frac{1}{2}x + 4$.

We will now solve the same problem in a different way. A student should understand that whenever a new approach to a familiar exercise is taken, there is something to be gained by it. Usually the new method is in some way more powerful, and allows the solving of additional problems. This will be the case with the following example, which uses a process you should have seen in a previous course, and that we will review in detail in the next section.

◇ **Example 1.2(a):** Find the equation of the line containing the points $(6,1)$ and $(2,3)$.

Solution: We are again trying to find the two constants m and b of the equation $y = mx + b$. Here we substitute the values of x and y from each of the two points into the equation $y = mx+b$ (separately, of course!) to get two equations in the two unknowns m and b. The resulting system is then solved for m, then b.

$$\begin{array}{rcl} 1 &=& 6m+b \\ 3 &=& 2m+b \end{array} \quad\Longrightarrow\quad \begin{array}{rcl} 1 &=& 6m+b \\ -(3 &=& 2m+b) \\ \hline -2 &=& 4m \\ -\frac{1}{2} &=& m \end{array} \quad\Longrightarrow\quad \begin{array}{rcl} 3 &=& 2(-\frac{1}{2})+b \\ 3 &=& -1+b \\ 4 &=& b \end{array}$$

The equation of the line containing $(6,1)$ and $(2,3)$ is $y = -\frac{1}{2}x + 4$.

The process of solving systems of two linear equations in two unknowns will be covered in more detail in the next section.

The equation of a line is considered to be a first-degree polynomial, since the power of x in $y = mx+b$ is one. Note that when we have two points in the xy-plane we can find a first-degree polynomial whose graph contains the points, and there is only one such line. Similarly, when given three points we can find a second-degree polynomial (quadratic polynomial) whose graph contains the three points. In general, THEOREM 1.2.1: Given n points in the plane such that (a) no two of them have the same x-coordinate and (b) they are not collinear, we can find a *unique* polynomial function of degree $n-1$ whose graph contains the n points.

Often in mathematics we are looking for some object (solution) and we wish to be certain that such an object exists. In addition, it is generally preferable that *only one* such object exists. We refer to the first of these wishes as "existence," and the second is "uniqueness." If we have, for example, four points meeting the two conditions of the above theorem, there would be infinitely many fourth degree polynomials whose graphs would contain them, and the same would be true for fifth degree, sixth degree, and so on. Additionally, a set of four points meeting the above conditions will likely *NOT* not have a polynomial of degree two whose graph passes through all of them. But the theorem guarantees us that there is one, and only one, third degree polynomial whose graph contains the four points. In Exercise 3 of the previous section you saw how to construct a system of linear equations whose solution gives us the coefficients of the third degree polynomial whose graph contains four given points. In Example 1.4(e) we'll see how to find such a polynomial, from start to finish.

Temperature Equilibrium

Consider the following hypothetical situation: We have a plate of metal that is perfectly insulated on both of its faces so that no heat can get in or out of the faces. Each point on the edge (which we will call the **boundary**), however, is held at a constant temperature (constant at that point, but possibly differing from point to point). The temperatures at points on the boundary affect the temperatures at interior points. If the plate is left alone for a long time ("infinitely long"), the temperature at each point in the interior of the plate will reach a constant temperature, called the "equilibrium temperature." This equilibrium temperature at any given interior point is a **weighted average** of the temperatures at all the boundary points, with temperatures at closer boundary points being weighted more heavily in the average than the temperatures at boundary points that are farther away.

The task of trying to determine those interior temperatures based on the edge temperatures is a famous problem of applied mathematics, called the **Dirichlet problem** (pronounced "dir-i-shlay"). Finding the exact solution involves methods beyond the scope of this course, but we will use systems of equations to solve the problem "numerically," which means to approximate the exact solution, usually by some non-calculus method. The key to solving the Dirichlet problem is the following:

THEOREM 1.2.2: **Mean Value Property**

The equilibrium temperature at any interior point P is the average of the temperatures of all interior points on *any* circle centered at P.

We will solve what are called **discrete** versions of the Dirichlet problem, which means that we only know the temperatures at a finite number of points on the boundary of our metal plate, and we will only find the equilibrium temperatures at a finite number of the interior points. These finite points, both on the boundary and in the interior, are usually evenly spaced on a rectangular grid. Consider the plate shown in Figure 1.2(a) on the next page, with boundary temperatures known at the indicated points. We can then construct a square grid in the interior of the plate, as shown in Figure 1.2(b). The unknown temperatures at the **mesh points** of the grid are denoted by t_1, t_2, t_3 and t_4, as shown in Figure 1.2(b). By the mean value property, the temperature t_1 is the average of the temperatures at all points on the circle shown in Figure 1.2(c). Such an average is obtained by an integral, but in our case we will simply average the temperatures at the four boundary and mesh points that are on the circle. This gives us the equation

$$t_1 = \frac{61 + 68 + t_2 + t_3}{4}.$$

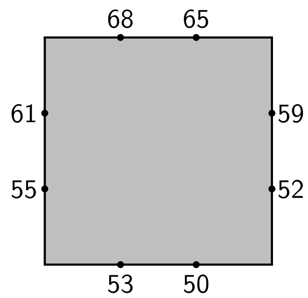


Figure 1.2(a)

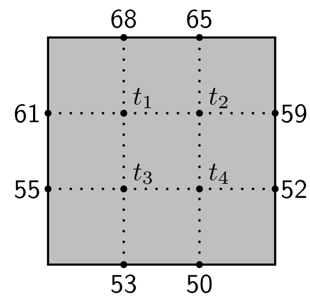


Figure 1.2(b)

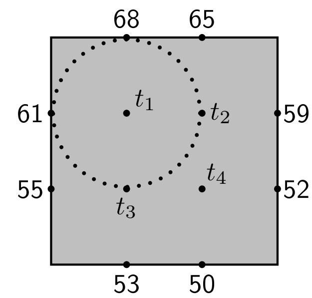
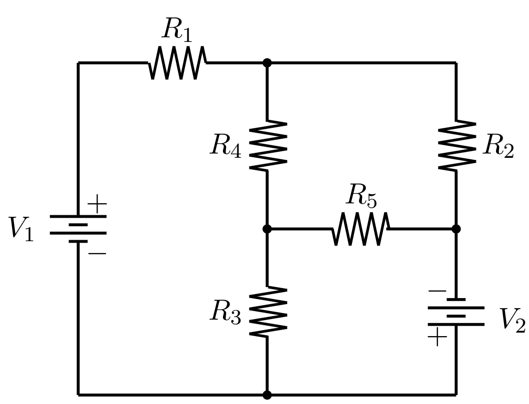


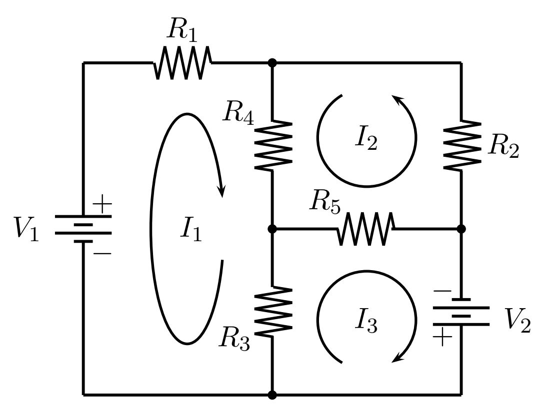
Figure 1.2(c)

We can then find three more such equations for circles centered at the mesh points with temperatures t_2, t_3 and t_4. If we then multiply both sides of each equation by four, combine the two known numerical values and get all of the unknowns on one side of each equation, we can obtain a system of linear equations in the standard form. You will do this in the exercises at the end of the section.

Electric Circuits

We could spend a great deal of time that we don't have on electric circuits, so here we'll just learn one method to get a system of linear equations modeling a circuit with constant **voltage sources** (batteries) and **resistors**. An example of such a circuit is shown below and to the left. The lines indicate wires that are connected at the dots, which are called **principal nodes**, and the portion of the circuit between two principle nodes we'll call a **branch**. A set of branches that can be placed end to end to get from one node back to itself is called a **loop**.





Each "zig-zag" is a resistor and the parallel long and short lines with $+$ and $-$ at either side are batteries. Each resistor has a characteristic called its **resistance**, which is measured in **ohms**. Similarly, each battery has a **voltage**, measured in ... **volts**! We will let V_1 and V_2 represent both the voltage sources and their voltages, and R_1 through R_5 will represent both the indicated resistors and their resistances.

The voltage sources cause something called **current** to flow in the circuit. Intuitively, we can think of the voltage sources as "pumps" pushing current through the wires, like pushing water through pipes. The resistors "resist" the flow of current. *Our objective is to find the current in each branch of the circuit.* To find the current in each branch we will proceed as follows:

1) Establish a clockwise or counterclockwise direction of current in each loop. If there is a voltage source in a loop, establish the current in the direction from the negative side to the positive side. If there is no voltage source, the current can be in either direction that you wish. (If you choose the "wrong" direction you will simply obtain a negative value for the current.) The diagram above and to the right shows currents established for each loop - we will use I for current.

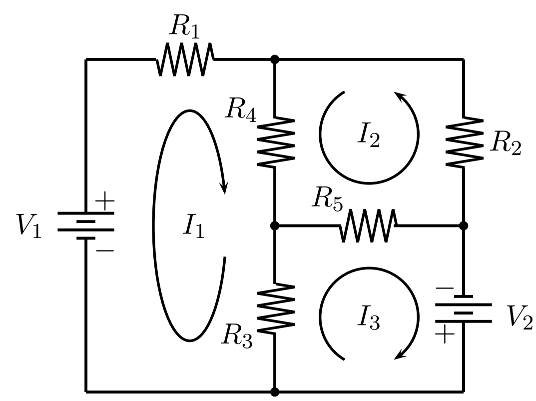
2) The "voltage drop" across each resistor is given by **Ohm's Law**, $V = IR$. **Kirchoff's Voltage Law** then tells us that the voltage supplied in a loop is equal to the sum of the voltage drops across each of the resistors in the loop. *When working in a loop and calculating the voltage drop across a resistor shared with another loop, the current used is the one for the loop under consideration plus or minus the current from the adjacent loop, depending on whether that current is going the same, or the opposite, direction as the current in the loop under consideration.* Write an equation for each loop based on Kirchoff's Voltage Law. If there is no voltage source in a loop, the voltage supplied is zero.

3) Get each equation in the form $aI_1 + bI_2 + cI_3 = V_k$, where k is the loop the equation was obtained from.

4) Solve the system of equations.

In Section 1.4 we'll see how to solve such systems; for now we will only complete steps 1, 2 and 3 above.

◇ **Example 1.2(b):** Use the steps above to obtain a system of three equations that models the circuit shown below and to the right.



Solution: For the loop with current I_1 the voltage supplied is V_1. Going around the loop from the battery the voltage drops are

$$I_1R_1, \qquad (I_1 + I_2)R_4, \qquad (I_1 - I_3)R_3.$$

Kirchoff's Voltage Law then gives us

$$I_1R_1 + (I_1 + I_2)R_4 + (I_1 - I_3)R_3 = V_1.$$

The equations for the other two loops are

$$I_2R_2 + (I_2 + I_1)R_4 + (I_2 + I_3)R_5 = 0$$

and

$$(I_3 - I_1)R_3 + (I_3 + I_2)R_5 = V_2.$$

Putting each of our three equations in the form $aI_1 + bI_2 + cI_3 = V_k$ gives us

$$\begin{array}{rcrcrcl} (R_1 + R_3 + R_4)I_1 & + & R_4I_2 & - & R_3I_3 & = & V_1 \\ R_4I_1 & + & (R_2 + R_4 + R_5)I_2 & + & R_5I_3 & = & 0 \\ -R_3I_1 & + & R_5I_2 & + & (R_3 + R_5)I_3 & = & V_2 \end{array}$$

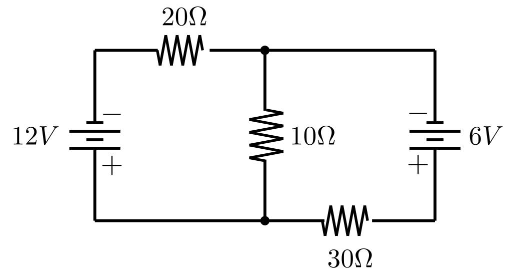
It is worth noting the array of coefficients of the three unknowns I_1, I_2 and I_3:

$$\begin{array}{ccc} R_1+R_3+R_4 & R_4 & -R_3 \\ R_4 & R_2+R_4+R_5 & R_5 \\ -R_3 & R_5 & R_3+R_5 \end{array}$$

Once again we see symmetry across the diagonal!

Let's do another example with numerical values for the voltage and resistances:

◇ **Example 1.2(c):** Find a system of equations that models the circuit below and to the right.

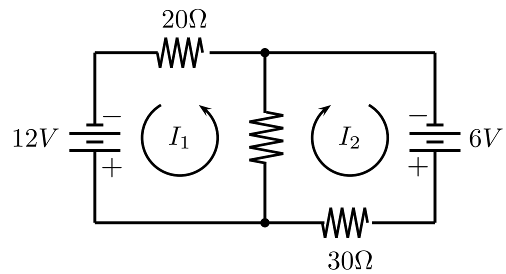


Solution: Here we establish the current I_1 in a counterclockwise direction in the left loop, and I_2 in a clockwise direction in the right loop, as shown in the lower picture to the right. For the left loop we get the equation

$$(I_1+I_2)10+20I_1=12$$

and, from the right loop,

$$30I_2+(I_2+I_1)10=6.$$



Distributing the resistances and regrouping gives us the system

$$\begin{aligned} 30I_1+10I_2 &= 12 \\ 10I_1+40I_2 &= 6 \end{aligned}$$

Section 1.2 Exercises To Solutions

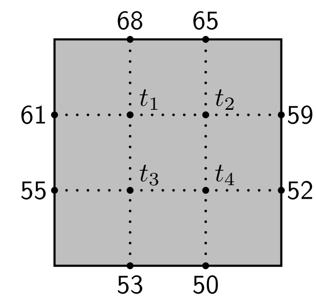
1. Consider the four points $(-1,3)$, $(1,5)$, $(2,4)$ and $(4,-1)$. By Theorem 1.2.1, there is a unique third degree polynomial of the form
$$y=a+bx+cx^2+dx^3 \tag{1}$$
whose graph contains those four points.

   (a) Substitute the x and y values from the first ordered pair into (1) and rearrange the resulting equation so that it has all of the unknowns on the left and a number on the right, like all of the linear equations we have worked with so far.

   (b) Repeat (a) for the other 3 ordered pairs, and give the system of equations whose solution is the four coefficients a, b, c and d.

2. Give a system of equations that can be solved to find the values of a, b and c for the quadratic polynomial $y=ax^2+bx+c$ whose graph is the parabola passing through the points $(-1,-4)$, $(1,1)$ and $(3,0)$.

3. To the right is a diagram of the metal plate described in the discussion of the mean value property for temperature equilibrium. In this exercise you will set up a system of equations whose solution gives the unknown temperatures t_1, t_2, t_3 and t_4 at the four interior points.



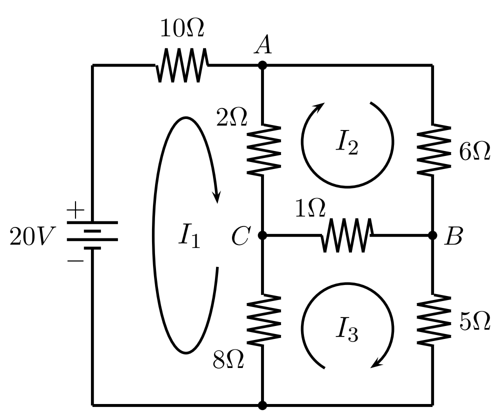
   (a) Applying the mean value property at the mesh point with temperature t_1 gave us the equation $t_1 = \dfrac{61+68+t_2+t_3}{4}$. Multiply both sides by four to eliminate the fraction, subtract t_2 and t_3 from both sides and add 61 and 68. You should end up with an equation of the form $at_1 + bt_2 + ct_3 + dt_4 = e$, where $d = 0$.

   (b) Follow a similar process for each of the other three interior mesh points in order to obtain a system of four equations in the four unknowns t_1, t_2, t_3 and t_4. In each equation one of a, b, c or d will be zero, so each equation will actually only contain three of t_1, t_2, t_3 and t_4.

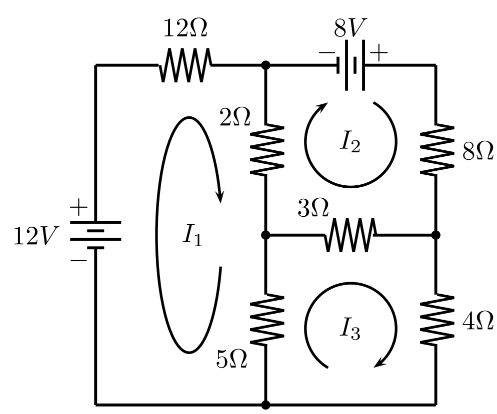
   (c) Although we can put the four equations in any order we want, arrange them so that the coefficients of four are along the diagonal of the left side, as we saw in third system of Example 1.1(b). Be sure that the remaining coefficients are symmetric about the diagonal. If they are not, find and correct your error.

4. (a) Give the system of equations modeling the circuit below and to the left.

   (b) Give the system of equations modeling the circuit below and to the right.



Exercise 4(a)



Exercise 4(b)

   (c) The solution to the circuit for Exercise 4(a) is $I_1 = 1.36$ amperes, $I_2 = 0.39$ amperes and $I_3 = 0.81$ amperes. Given this information, what is the current in the branch from point A to point C? (Note that it is I_1 and I_2 combined, with the direction of each taken into account.) Does the current flow from A to C, or from C to A?

   (d) Again considering the circuit for Exercise 4(a) with the current values given above, what is the current in the branch from point B to point C? Does the current flow from B to C, or from C to B?

1.3 Solving Systems of Linear Equations

Performance Criteria:

1. (d) Solve a system of two linear equations by the addition method.

Now that we know some applications of systems of equations, and how to set up systems for an application, it is time we learn how to solve a system. In this section we remember how to solve a system of two equations in two unknowns by the addition method, and extend the method to a system of three equations in three unknowns. Then in Section 1.4, we will introduce the method that we will use throughout the rest of the course.

Consider the system $\begin{aligned} x - 3y &= 6 \\ -2x + 5y &= -5 \end{aligned}$ of linear equations. In this case a solution to the system is an **ordered pair** (x, y) that makes *both* equations true. In the past you should have learned two methods for solving such systems, the **addition method** and the **substitution method**. The method we want to focus on is the addition method. In this case we could multiply the first equation by two and add the resulting equation to the second. The result is $\begin{aligned} x - 3y &= 6 \\ -y &= 7 \end{aligned}$; from this we can see that $y = -7$. This value is then substituted into the first equation to get $x = -15$.

Sometimes we have to do something a little more complicated:

◇ **Example 1.3(a):** Solve the system $\begin{aligned} 2x - 4y &= 18 \\ 3x + 5y &= 5 \end{aligned}$ using the addition method.

Solution: Here we can eliminate x by multiplying the first equation by 3 and the second by -2, then adding:

$$\begin{aligned} 2x - 4y &= 18 \\ 3x + 5y &= 5 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} 6x - 12y &= 54 \\ -6x - 10y &= -10 \\ \hline -22y &= 44 \\ y &= -2 \end{aligned}$$

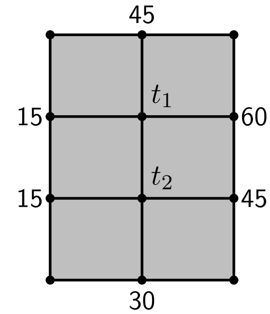
Now we can substitute this value of y back into either equation to find x:

$$\begin{aligned} 2x - 4(-2) &= 18 \\ 2x + 8 &= 18 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

The solution to the system is then $x = 5$, $y = -2$, which we usually write as the ordered pair $(5, -2)$. It can be easily verified that this pair is a solution to both equations.

Let's now solve an applied problem that uses a system of two equations.

◇ **Example 1.3(b):** The temperatures (in degrees Fahrenheit) at six points on the edge of a rectangular plate are shown to the right. Assuming that the temperatures in the plate have reached equilibrium, find the interior temperatures t_1 and t_2 at their indicated "mesh points."



Solution: The discrete version of the mean value property tells us that the equilibrium temperature at any interior point of the mesh is the average of the four adjacent points. This gives us the two equations

$$t_1 = \frac{15+45+60+t_2}{4} \qquad \text{and} \qquad t_2 = \frac{15+t_1+45+30}{4}$$

If we multiply both sides of each equation by four, combine the constants and get the t_1 and t_2 terms on the left side we get the system of equations $\begin{array}{rcl} 4t_1 - \ t_2 & = & 120 \\ -t_1 + 4t_2 & = & 90 \end{array}$. Multiplying the first equation by four and adding the result to the second gives us $15t_1 = 570$, from which we find that $t_1 = 38$. Substituting that into either equation and solving for t_2 gives $t_2 = 32$. These values can easily be shown to verify our discrete mean value property:

$$\frac{15+45+60+t_2}{4} = \frac{15+45+60+32}{4} = 38 = t_1\,,$$

$$\frac{15+t_1+45+30}{4} = \frac{15+38+45+30}{4} = 32 = t_2$$

Geometric Interpretation

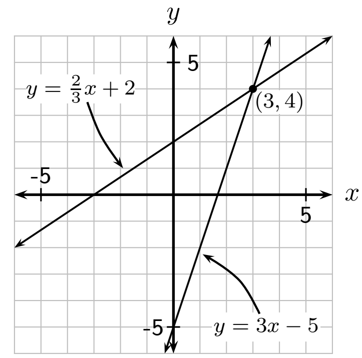
At the start of this section we saw that the system $\begin{array}{rcl} x-3y & = & 6 \\ -2x+5y & = & -5 \end{array}$ has the solution $(-15,-7)$. You should be aware that if we graph the equation $x-3y=6$ we get a line. Technically speaking, what we have graphed is the **solution set**, the set of all pairs (x,y) that make the equation true. *Any pair* (x,y) *of numbers that makes the equation true is on the line, and the* (x,y) *representing any point on the line will make the equation true.* If we plot the solution sets of both equations in the system $\begin{array}{rcl} x-3y & = & 6 \\ -2x+5y & = & -5 \end{array}$ together in the coordinate plane we will get two lines. Since $(-15,-7)$ is a solution to both equations, *the two lines cross at the point with those coordinates!* We could use this idea to (somewhat inefficiently and possibly inaccurately) solve a system of two equations in two unknowns:

◇ **Example 1.3(c):** Solve the system $\begin{array}{rcl} 2x-3y & = & -6 \\ 3x-y & = & 5 \end{array}$ graphically.

Solution: We begin by solving each of the equations for y; this will give us the equations in $y=mx+b$ form, for easy graphing. The results are

$$y = \tfrac{2}{3}x + 2 \qquad \text{and} \qquad y = 3x - 5$$

If we graph these two equations on the same graph, we get the picture to the right. Note that the two lines cross at the point $(3,4)$, so the solution to the system of equations is $(3,4)$, or $x=3,\ y=4$.



It is possible that two lines in the standard two-dimensional plane might be parallel; in that case a system consisting of the two equations representing those lines will have no solution. It is also possible that two equations might actually represent the same line, in which case the system consisting of those two equations will have infinitely many solutions. Investigation of those two cases will lead us to more complex considerations that we will avoid for now.

A System of Equations in Three Unknowns

The previous examples were two linear equations with two unknowns. Now we consider the following system of *three linear equations in three unknowns*.

$$\begin{aligned} x+3y-2z &= -4 \\ 3x+7y+z &= 4 \\ -2x+y+7z &= 7 \end{aligned} \tag{1}$$

We can use the addition method here as well; first we multiply the first equation by negative three and add it to the second. We then multiply the first equation by two and add it to the third. This eliminates the unknown x from the second and third equations, giving the second system of equations shown below. We can then add $\frac{7}{2}$ times the second equation to the third to obtain a new third equation in which the unknown y has been eliminated. This “final” system of equations is shown to the right below.

$$\begin{aligned} x+3y-2z &= -4 \\ 3x+7y+z &= 4 \\ -2x+y+7z &= 7 \end{aligned} \quad\Longrightarrow\quad \begin{aligned} x+3y-2z &= -4 \\ -2y+7z &= 16 \\ 7y+3z &= -1 \end{aligned} \quad\Longrightarrow\quad \begin{aligned} x+3y-2z &= -4 \\ -2y+7z &= 16 \\ \tfrac{55}{2}z &= 55 \end{aligned} \tag{2}$$

Above we have three different systems, each with three equations. The three systems are *equivalent*, meaning that a solution to any one of them is also a solution to the other two. The point of the above process is to obtain a system that is equivalent to the original but easier to solve. We can see that the solution to the last equation of the third system is $z=2$. That result is then substituted into the second equation in the last system to get $y=-1$. Finally, we substitute the values of y and z into the first equation to get $x=3$. The solution to the system is then the **ordered triple** $(3,-1,2)$. The process of finding the last unknown first, substituting it to find the next to last, and so on, is called **back substitution**. The word “back” here means that we find the last unknown (in the order they appear in the equations) first, then the next to last, and so on.

You might note that we could eliminate any of the three unknowns from any two equations, then use the addition method with those two to eliminate another variable. However, we will always follow a process that first uses the first equation to eliminate the first unknown from all equations but the first one itself. After that we use the second equation to eliminate the second unknown from all equations from the third on, and so on. One reason for this is that if we were to create a computer algorithm to solve systems, it would need a consistent method to proceed, and what we have done is as good as any.

What is the geometric interpretation of this? Since there are three unknowns, the appropriate geometric setting is three-dimensional space. *The solution set to any equation* $ax + by + cz = d$ *is a plane* in three-dimensional space, as long as not all of a, b and c are zero. Therefore, a solution to the system is a point that lies on each of the planes representing the solution sets of the three equations. For our example, then, the planes representing the three equations intersect at the point $(3, -1, 2)$.

In the study of linear algebra we will be defining new concepts and developing corresponding notation. We begin the development of notation with the following. The set of all real numbers is denoted by $\mathbb{R}$, and the set of all ordered pairs of real numbers is $\mathbb{R}^2$, spoken as "R-two." Geometrically, $\mathbb{R}^2$ is the familiar Cartesian coordinate plane. Similarly, the set of all ordered triples of real numbers is the three-dimensional space referred to as $\mathbb{R}^3$, "R-three."

All of the algebra that we will be doing using equations with two or three unknowns can easily be done with more unknowns. In general, when we are working with n unknowns, we will get solutions that are n-**tuples** of numbers. Any such n-tuple represents a location in n-dimensional space, denoted $\mathbb{R}^n$. Note that a linear equation in two unknowns represents a line in $\mathbb{R}^2$, in the sense that the set of solutions to the equation forms a line. We consider a line to be a one-dimensional object, so the linear equation represents a one-dimensional object in two-dimensional space. The solution set to a linear equation in three unknowns is a plane in three-dimensional space. The plane itself is two-dimensional, so we have a two-dimensional "flat" object in three dimensional space.

Similarly, when we consider the solution set of a linear equation in n unknowns, its solution set represents an $n-1$-dimensional "flat" object in n-dimensional space. When such an object has more than two dimensions, we usually call it a **hyperplane**. Although such objects can't be visualized, they certainly exist in a mathematical sense.

Section 1.3 Exercises To Solutions

1. Solve each of the following systems by the addition method.

(a) $\begin{aligned} 2x - 3y &= -7 \\ -2x + 5y &= 9 \end{aligned}$ (b) $\begin{aligned} 2x - 3y &= -6 \\ 3x - y &= 5 \end{aligned}$ (c) $\begin{aligned} 4x + y &= 14 \\ 2x + 3y &= 12 \end{aligned}$

(d) $\begin{aligned} 7x - 6y &= 13 \\ 6x - 5y &= 11 \end{aligned}$ (e) $\begin{aligned} 5x + 3y &= 7 \\ 3x - 5y &= -23 \end{aligned}$ (f) $\begin{aligned} 5x - 3y &= -11 \\ 7x + 6y &= -12 \end{aligned}$

2. Solve each of the following systems by graphing, as done in Example 1.3(c).

(a) $\begin{aligned} 3x - 4y &= 8 \\ x + 2y &= 6 \end{aligned}$ (b) $\begin{aligned} 4x - 3y &= 9 \\ x + 2y &= -6 \end{aligned}$ (c) $\begin{aligned} 5x + y &= 12 \\ 7x - 2y &= 10 \end{aligned}$

1.4 Solving With Matrices

Performance Criteria:

1. (e) Give the coefficient matrix and augmented matrix for a system of equations.
   (f) Determine whether a matrix is in row-echelon form. Perform, by hand, elementary row operations to reduce a matrix to row-echelon form.
   (g) Determine whether a matrix is in reduced row-echelon form. Use technology to reduce a matrix to reduced row-echelon form.
   (h) For a system of equations having a unique solution, determine the solution from either the row-echelon form or reduced row-echelon form of the augmented matrix for the system.
   (i) Use a calculator to solve a system of linear equations having a unique solution.

Note that when using the addition method for solving the system of three equations in three unknowns in the previous section, the symbols x, y and z and the equal signs are simply "placeholders" that are "along for the ride." To make the process cleaner we can simply arrange the constants a, b, c and d for each equation $ax+by+cz=d$ in an array form called a **matrix**, which is simply a table of values like

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix}. \tag{1}$$

Each number in a matrix is called an **entry** of the matrix. Each horizontal line of numbers in a matrix is a **row** of the matrix, and each vertical line of numbers is a **column**. The **size** or **dimensions** of a matrix is (are) given by first telling the number of rows, then the number of columns, with the $\times$ symbol between them. The size of the above matrix is 3×4, which we say as "three by four."

Suppose that the above matrix came from the system of equations

$$\begin{aligned} x+3y-2z &= -4 \\ 3x+7y+z &= 4 \\ -2x+y+7z &= 7 \end{aligned}$$

When a matrix represents a system of equations, as (1) does, it is called the **augmented matrix** of the system. The matrix consisting of just the coefficients of x, y and z from each equation is called the **coefficient matrix**:

$$\begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix}$$

We are not interested in the coefficient matrix at this time, but we will be later. The reason for the name "augmented matrix" will also be seen later.

Once we have the augmented matrix, we can perform a process called **row-reduction**, which is essentially what we did in the previous section, but we work with just the matrix rather than the system of equations. The following example shows how this is done for the above matrix.

◇ **Example 1.4(a):** Solve the system

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned}$$

from the previous section by row-reduction.

Solution: We begin with the augmented matrix for the system, shown below and to the left. We then add negative three times the first row to the second, and put the result in the second row. Then we add two times the first row to the third, and place the result in the third. Using the notation R_n (not to be confused with $\mathbb{R}^n$!) to represent the nth row of the matrix, we can symbolize these two operations as shown in the middle below. The matrix to the right below is the result of those operations.

$$\left[\begin{array}{rrrr} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{array}\right] \quad \begin{array}{c} -3R_1 + R_2 \to R_2 \\ \Longrightarrow \\ 2R_1 + R_3 \to R_3 \end{array} \quad \left[\begin{array}{rrrr} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 7 & 3 & -1 \end{array}\right]$$

Next we finish with the following:

$$\left[\begin{array}{rrrr} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 7 & 3 & -1 \end{array}\right] \quad \begin{array}{c} \frac{7}{2}R_2 + R_3 \to R_3 \\ \Longrightarrow \end{array} \quad \left[\begin{array}{rrrr} 1 & 3 & -2 & -4 \\ 0 & -2 & 7 & 16 \\ 0 & 0 & \frac{55}{2} & 55 \end{array}\right]$$

The process just outlined is called **row reduction**. At this point we return to the equation form

$$\begin{aligned} x + 3y - \ 2z &= -4 \\ 0x - 2y + \ 7z &= 16 \\ 0x + 0y + \tfrac{55}{2}z &= 55 \end{aligned}$$

and perform back-substitution. The last equation gives us that $z = 2$. We can than substitute this value into the second equation to get $-2y + 14 = 16$, resulting in $y = -1$. These values of y and z are substituted into the first equation which is then solved to get $x = 3$. The solution to the system is then $(3, -1, 2)$.

The final form of the matrix before we went back to equation form is something called **row-echelon** form. (The word "echelon" is pronounced "esh-el-on.") The first non-zero entry in each row is called a **leading entry**; in this case the leading entries are the numbers 1, -2 and $\frac{55}{2}$. To be in row-echelon form means that

- any rows containing all zeros are at the bottom of the matrix and
- the leading entry in any row is to the right of any leading entries above it.

◇ **Example 1.4(b):** Which of the matrices below are in row-echelon form?

$$\left[\begin{array}{rrrr} 1 & 3 & -2 & -4 \\ 0 & 0 & 3 & -5 \\ 0 & 7 & -10 & -1 \end{array}\right] \quad \left[\begin{array}{rrrrr} 2 & 6 & -1 & 9 & 5 \\ 0 & 0 & -8 & 1 & -3 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right] \quad \left[\begin{array}{rrrr} 7 & -12 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 1 & 8 \end{array}\right]$$

Solution: The leading entries of the rows of the first matrix are 1, 3 and 7. Because the leading entry of the third row (7) is not to the right of the leading entry of the second row (3), the first matrix *is not* in row-echelon form. In the third matrix, there is a row of zeros that is not at the bottom of the matrix, so it *is not* in row-echelon form. The second matrix *is* in row-echelon form.

Note that if we switch the second and third rows of the first and third matrices in the above example, which we are usually allowed to do, then both will then be in row-echelon form.

It is possible to continue with the matrix operations to obtain something called **reduced row-echelon form**, from which it is easier to find the values of the unknowns. The requirements for being in reduced row-echelon form are the same as for row-echelon form, with the addition of the following:

- All leading entries are ones.
- The entries above any leading entry are all zero *except perhaps in the last column*.

Obtaining reduced row-echelon form requires more matrix manipulations, and nothing is really gained by obtaining that form if you are doing this by hand. However, when using software or a calculator it is most convenient to obtain reduced row-echelon form. Here are two examples of matrices in reduced row-echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 6 & 0 & 9 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In the next section we will see how to interpret what the second matrix would be telling us if it came from a system of equations. The next example shows what the first matrix tells us.

◇ **Example 1.4(c):** Suppose that the matrix to the right is the result of row-reduction of the augmented matrix for a system of three equations in the unknowns x_1, x_2 and x_3. Determine the values of the unknowns.

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Solution: When using row-reduction to solve a system we first create the augmented matrix for the system, then row-reduce it, and then we go back to equations. The equations we would return to for the above matrix are

$$\begin{aligned} 1x_1 + 0x_2 + 0x_3 &= 3 \\ 0x_1 + 1x_2 + 0x_3 &= -7 \\ 0x_1 + 0x_2 + 1x_3 &= 4 \end{aligned}$$

and from these we can easily see the solution: $x_1 = 3$, $x_2 = -7$ and $x_3 = 4$.

In practice, very large systems are solved by row-reduction. Many issues arise when doing this. For example, coefficients are often obtained from some sorts of measurements that give rounded values. At every step of row-reduction more rounding needs to take place, resulting in rounding errors. Additionally, matrices used in practice can have entries that cause introduction of other errors in the process of row-reduction. We could spend an entire course examining such concerns, but instead we'll focus on less numerically oriented aspects of linear algebra.

That said, let's look at one thing that can come up in the process of row-reduction, illustrated in the following example.

◇ **Example 1.4(d):** Row-reduce the matrix $\begin{bmatrix} 1 & 3 & -2 & -4 \\ 2 & 6 & -1 & -13 \\ -1 & 4 & -8 & 3 \end{bmatrix}$.

Solution: We begin by adding negative two times the first row to the second, and put the result in the second row. Then we add two times the first row to the third, and place the result in the third. Using the notation R_n (not to be confused with $\mathbb{R}^n$!) to represent the nth row of the matrix, we can symbolize these two operations as shown in the middle below. The matrix to the right below is the result of those operations.

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 2 & 6 & -1 & -13 \\ -1 & 4 & -8 & 3 \end{bmatrix} \quad \begin{array}{c} -2R_1 + R_2 \to R_2 \\ \Longrightarrow \\ R_1 + R_3 \to R_3 \end{array} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & 0 & 3 & -5 \\ 0 & 7 & -10 & -1 \end{bmatrix}$$

We can see that the matrix would be in row-echelon form if we simply switched the second and third rows (which is equivalent to simply rearranging the order of our original equations), so that's what we do:

$$\begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & 0 & 3 & -5 \\ 0 & 7 & -10 & -1 \end{bmatrix} \quad \begin{array}{c} R_2 \longleftrightarrow R_3 \\ \Longrightarrow \end{array} \quad \begin{bmatrix} 1 & 3 & -2 & -4 \\ 0 & 7 & -10 & -1 \\ 0 & 0 & 3 & -5 \end{bmatrix}$$

The act of rearranging rows in a matrix is called **permuting** them. In general, a **permutation** of a set of objects is simply a rearrangement of them. When solving a system by row-reduction, permuting simply amounts to changing the order of the original equations, and doing so will not affect the solution to the system.

Row Reduction Using Technology

There are three main technologies that can be used to get an augmented matrix into reduced row-echelon form:

- Most or all graphing calculators (and the TI-36X Pro, a non-graphing calculator) will perform row reduction via the *rref* function. Do a search to find an article or video on how to *rref* with your particular model of calculator.
- There are numerous matrix calculators that can be found online - there is a link to one at the class web page.
- Various mathematical software programs, like MATLAB, will perform row-reduction.

Now that we know how to solve systems of linear equations we can complete an application.

◇ **Example 1.4(e):** Find the equation of the third degree polynomial containing the points $(-1,-7)$, $(0,1)$, $(1,5)$ and $(2,11)$.

Solution: A general third degree polynomial has an equation of the form $y = ax^3 + bx^2 + cx + d$; our goal is to find values of a, b, c and d so that the given points all satisfy the equation. Since the values $x = -1$, $y = -7$ must make the general equation true, we have $-7 =$

$a(-1)^3+b(-1)^2+c(-1)+d=-a+b-c+d$. Doing this with all four given ordered pairs and "flipping" each equation gives us the system

$$\begin{aligned} -a+b-c+d &= -7 \\ d &= 1 \\ a+b+c+d &= 5 \\ 8a+4b+2c+d &= 11 \end{aligned}$$

If we enter the augmented matrix for this system in our calculators and *rref* we get

$$\begin{bmatrix} -1 & 1 & -1 & 1 & -7 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 5 \\ 8 & 4 & 2 & 1 & 11 \end{bmatrix} \quad \stackrel{rref}{\Longrightarrow} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

So $a=1$, $b=-2$, $c=5$, $d=1$, and the desired polynomial equation is $y=x^3-2x^2+5x+1$.

Section 1.4 Exercises To Solutions

1. Give the coefficient matrix and augmented matrix for the system of equations

$$\begin{aligned} x + y - 3z &= 1 \\ -3x+2y - z &= 7 \\ 2x + y - 4z &= 0 \end{aligned} .$$

2. Determine which of the following matrices are in row-echelon form.

$$A=\begin{bmatrix} 3 & -7 & 5 & 0 & 2 & -4 \\ 0 & 0 & 0 & -2 & 5 & -1 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad C=\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 3 & 5 \end{bmatrix}$$

$$D=\begin{bmatrix} 0 & 0 & 4 & 4 \\ 0 & 1 & -3 & 2 \\ 6 & 1 & 3 & 5 \end{bmatrix} \qquad E=\begin{bmatrix} 1 & 3 & -5 & 10 & -7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad F=\begin{bmatrix} 1 & 3 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Determine which of the matrices in Exercise 2 are in reduced row-echelon form.

4. Perform the first two row operations for the augmented matrix from Exercise 1, to get zeros in the bottom two entries of the first column.

5. Fill in the blanks in the second matrix with the appropriate values after the first step of row-reduction. Fill in the long blanks with the row operations used.

   (a) $$\begin{bmatrix} 1 & 5 & -7 & 3 \\ -5 & 3 & -1 & 0 \\ 4 & 0 & 8 & -1 \end{bmatrix} \quad \begin{array}{c} \underline{\qquad\qquad\qquad} \\ \Longrightarrow \\ \underline{\qquad\qquad\qquad} \end{array} \quad \begin{bmatrix} \_\_ & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \\ 0 & \_\_ & \_\_ & \_\_ \end{bmatrix}$$

(b) $\begin{bmatrix} 2 & -8 & -1 & 5 \\ 0 & -2 & 0 & 0 \\ 0 & 6 & -5 & 2 \end{bmatrix} \quad \underset{\rule{4cm}{0.4pt}}{\Longrightarrow} \quad \begin{bmatrix} \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ 0 & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ 0 & 0 & \underline{\quad} & \underline{\quad} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 3 & 5 & -2 \\ 0 & 2 & -8 & 1 \end{bmatrix} \quad \underset{\rule{4cm}{0.4pt}}{\Longrightarrow} \quad \begin{bmatrix} \underline{\quad} & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ 0 & \underline{\quad} & \underline{\quad} & \underline{\quad} \\ 0 & 0 & \underline{\quad} & \underline{\quad} \end{bmatrix}$

6. Find x, y and z for the system of equations that reduces to each of the matrices shown.

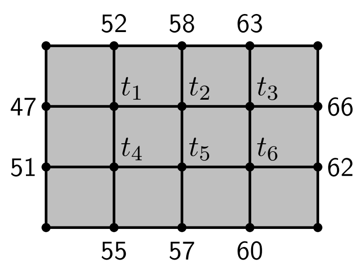
(a) $\begin{bmatrix} 1 & 6 & -2 & 7 \\ 0 & 8 & 1 & 0 \\ 0 & 0 & -2 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 6 & -2 & 7 \\ 0 & 2 & -5 & -13 \\ 0 & 0 & 3 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -4 & 8 \end{bmatrix}$

7. Use row operations (by hand) on an augmented matrix to solve each system of equations.

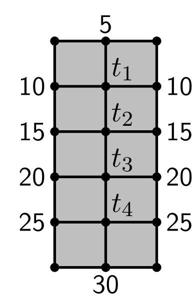
(a) $\begin{aligned} x - 2y - 3z &= -1 \\ 2x + y + z &= 6 \\ x + 3y - 2z &= 13 \end{aligned}$ (b) $\begin{aligned} -x - y + 2z &= 5 \\ 2x + 3y - z &= -3 \\ 5x - 2y + z &= -10 \end{aligned}$ (c) $\begin{aligned} x + 2y + 4z &= 7 \\ -x + y + 2z &= 5 \\ 2x + 3y + 3z &= 7 \end{aligned}$

8. Use the *rref* capability of your calculator to solve each of the systems from the previous exercise.

9. Temperatures at points along the edges of a rectangular plate are as shown below and to the left. Find the equilibrium temperature at each of the interior points, to the nearest tenth.



Exercise 9



Exercise 10

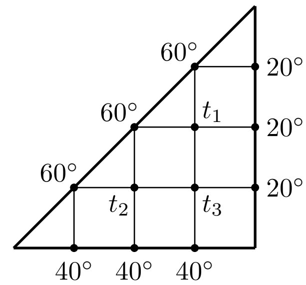
10. Consider the rectangular plate with boundary temperatures shown below and to the right of Exercise 9.

    (a) Intuitively, what do you think that the equilibrium temperatures t_1, t_2, t_3 and t_4 are?

    (b) Set up a system of equations and find the equilibrium temperatures. How was your intuition?

11. Look at your solutions *and the boundary temperatures* for Exercises 9 and 10. For each plate, look at where the maximum and minimum temperatures occur. What can we say in general about the locations of the maximum and minimum temperatures? Can you see how this is implied by the Mean Value Property?

12. For the diagram to the right, the mean value property still holds, even though the plate in this case is triangular. Find the interior equilibrium temperatures, rounded to the nearest tenth.



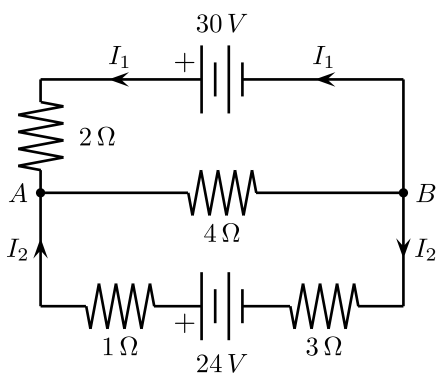
13. (a) Plot the points $(-4,0)$, $(-2,2)$, $(0,0)$, $(2,2)$ and $(3,0)$ neatly on an xy grid. Sketch the graph of a polynomial function with the fewest number of turning points ("humps") possible that goes through all the points. What is the degree of the polynomial function?

    (b) Find a fourth degree polynomial $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ that goes through the given points.

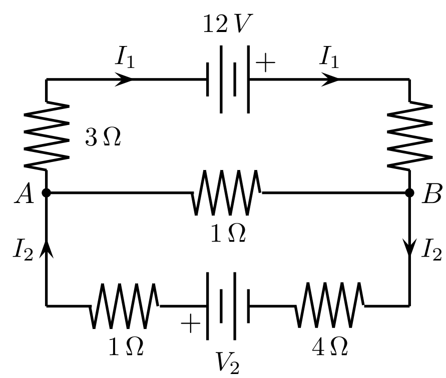
    (c) Graph your function from (b) on your calculator and sketch it, using a dashed line, on your graph from (a). Is the graph what you expected?

14. (a) Find the currents I_1 and I_2 in the circuit with the diagram shown below and to the left.

    (b) What is the value of the current through the 4 ohm resistor, and does it flow from A to B, or from B to A?



Exercise 14



Exercise 15

15. Consider the circuit shown to the right below Exercise 14.

    (a) Find the currents I_1, I_2 and I_3 when the voltage V_2 is 6 volts, rounded to the tenth's place.

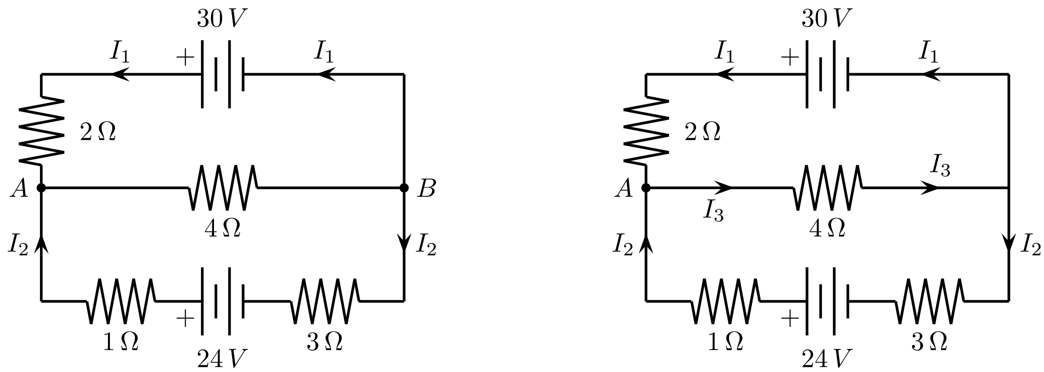
    (b) Does the current in the middle branch of the circuit flow from A to B, or from B to A?

    (c) Find the currents I_1, I_2 and I_3 when the voltage V_2 is 24 volts, rounded to the tenth's place.

    (d) Does the current in the middle branch of the circuit flow from A to B, or from B to A?

    (e) Determine the voltage needed for V_2 in order that no current flows through the middle branch. (You might wish to row reduce by hand for this...)

16. In this exercise you will find the currents in the circuit from Exercise 14 (shown to the left below) by a slightly different manner. Rather than working with just the currents I_1 and I_2 and then adding them to find the current from A to B, We will begin with another unknown current I_3, as shown in the diagram to the right below.



(a) Set up equations for both the upper and lower loops as before, but use I_3 as the current through the 4 ohm resistor, rather than $I_1 + I_2$. This will give you two equations with three unknowns in them, I_1, I_2 and I_3.

(b) We need one more equation, which we get as follows: The current into node A must equal the current out. Use this to write an equation, then get all of I_1, I_2 and I_3 on one side.

(c) Solve your system to get the three currents.

17. The equation of a non-vertical plane in $\mathbb{R}^3$ can always be written in the form $z = a + bx + cy$, where a, b and c are constants and (x, y, z) is any point on the plane. Use a method similar to the method for finding the equation of a polynomial through a given set of points to find the equation of the plane through the three points $P_1(-5, 0, 2)$, $P_2(4, 5, -1)$ and $P_3(2, 2, 2)$. Use your calculator's *rref* command to solve the system. Round a, b and c to the thousandth's place.

1.5 "When Things Go Wrong"

Performance Criteria:

1. (j) Given the row-echelon or reduced row-echelon form of an augmented matrix for a system of equations, determine the leading variables and free variables of the system.

   (k) Given the row-echelon or reduced row-echelon form for a system of equations:

   - Determine whether the system has a unique solution, and give the solution if it does.
   - If the system does not have a unique solution, determine whether it is inconsistent (no solution) or dependent (infinitely many solutions).
   - If the system is dependent, give the general form of a solution and give some particular solutions.

Consider the three systems of equations

$$\begin{aligned} x-3y &= 6 \\ -2x+5y &= -5 \end{aligned} \qquad \begin{aligned} x-2y &= 3 \\ -2x+4y &= 1 \end{aligned} \qquad \begin{aligned} x-2y &= 3 \\ -2x+4y &= -6 \end{aligned}$$

For the first system, if we multiply the first equation by 2 and add it to the second, we get $-y=7$, so $y=-7$. This can be substituted into either equation to find $x=-15$, and the system is solved!

When attempting to solve the second and third systems, things do not "work out" in the same way. In both cases we would likely attempt to eliminate x by multiplying the first equation by two and adding it to the second. For the second system this results in $0=7$ and for the third the result is $0=0$. So what is happening? Let's keep the unknown value y in both equations: $0y=7$ and $0y=0$. There is no value of y that can make $0y=7$ true, so there is no solution to the second system of equations. We call a system of equations with no solution **inconsistent**.

The equation $0y=0$ is true for *any* value of y, so y can be anything in the third system of equations. Thus we will call y a **free variable**, meaning it is free to have any value. *In this sort of situation we will assign another unknown, usually t, to represent the value of the free variable.* (If there is another free variable we usually use s and t for the two free variables.) Once we have assigned the value t to y, we can substitute it into the first equation and solve for x to get $x=2t+3$.

What all this means is that any ordered pair of the form $(2t+3,t)$ will be a solution to the third system of equations above. For example, when $t=0$ we get the ordered pair $(3,0)$, when $t=-6$ we get $(-9,-6)$. You can verify that both of these are solutions, as are infinitely many other pairs. At this point you might note that we could have made x the free variable, then solved for y in terms of whatever variable we assigned to x. *It is standard convention, however, to start assigning free variables from the last variable, and you will be expected to follow that convention in this class.* A system like this, with infinitely many solutions, is called a **dependent** system.

The fundamental fact that should always be kept in mind is this.

Solutions to a System of Equations

Every system of linear equations has either

- one unique solution
- no solution (the system is inconsistent)
- infinitely many solutions (the system is dependent)

In the context of both linear algebra and differential equations, mathematicians are always concerned with "existence and uniqueness." What this means is that when attempting to solve a system of equations or a differential equation, one cares about

1) whether at least one solution exists and

2) if there is at least one solution, is there exactly one; that is, is the solution unique?

We'll now see if we can learn to recognize which of the above three situations is the case, based on the row-echelon or reduced row-echelon form of the augmented matrix of a system. If the three systems we have been discussing are put into augmented matrix form and row reduced we get

$$\begin{bmatrix} 1 & 0 & -15 \\ 0 & 1 & -7 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

It should be clear that the first matrix gives us the unique solution to that system. The second line of the second matrix "translates" back to the equation $0x + 0y = 7$, which clearly cannot be true for any values of x or y. So that system has no solution.

If the row reduced augmented matrix for a system has any row with entries all zeros EXCEPT the last one, the system has no solution. The system is said to be **inconsistent**.

We now consider the third row reduced matrix. The last line of it "translates" to $0x + 0y = 0$, which is true for *any* values of x and y. That means we are free to choose the value of either one but, as discussed before, it is customary to let y be the free variable. So we let $y = t$ and substitute that into the equation $x - 2y = 3$ represented by the first line of the reduced matrix. As before, that is solved for x to get $x = 2t + 3$. The solutions to the system are then $x = 2t + 3$, $y = t$ for all values of t.

Of the three cases (1) exactly one solution, (2) no solution, (3) infinitely many solutions, the third case is the most challenging to interpret in most situations. As an introduction, let's consider the system shown below and to the left; its augmented matrix reduces to the form shown below and to the right.

$$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= 7 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We now make the following definitions:

- The **leading variables** are the variables corresponding to the columns of the reduced matrix containing the first non-zero entries (always ones for reduced row-echelon form) in each row. For the above system the leading variables are x_1 and x_2.

- Any variables that are not leading variables are **free variables**, so x_3 is the free variable in the above system. This means it is free to take any value.

It is a bit difficult to explain how to solve systems with infinitely many solutions, and it is probably best seen by some examples. However, let me try to describe it. Start with the last variable and solve for it if it is a leading variable. If it is not, assign it a parameter, like t. If the next to last variable is a leading variable solve for it, either as a number or in terms of the parameter assigned to the last variable. Continue in this manner until all variables have been determined as numbers or in terms of parameters.

◇ **Example 1.5(a):** Solve the system
$$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= 7 \end{aligned}$$

Solution: The row-reduced form of the augmented matrix for this system is $\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

In this case the leading variables are x_1 and x_2. Any variables that are not leading variables are free variables, so x_3 is the free variable in this case. If we let $x_3 = t$, the last non-zero row gives the equation $x_2 + 2t = 1$, so $x_2 = -2t + 1$. The first row gives the equation $x_1 + 3x_3 = 4$, so $x_1 = -3t + 4$ and the final solution to the system is

$$x_1 = -3t + 4, \qquad x_2 = -2t + 1, \qquad x_3 = t$$

We can also think of the solution as being any ordered triple of the form $(-3t + 4, -2t + 1, t)$.

◇ **Example 1.5(b):** A system of three equations in the four variables x_1, x_2, x_3 and x_4 gives the row-reduced matrix

$$\begin{bmatrix} 1 & 0 & 3 & 0 & -1 \\ 0 & 1 & -5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Give the general solution to the system.

Solution: The leading variables are x_1, x_2 and x_4. Any variables that are not leading variables are the free variables, so x_3 is the free variable in this case. We can see that the last row gives us $x_4 = 4$. Letting $x_3 = t$, the second equation from the row-reduced matrix is $x_2 - 5t = 2$, so $x_2 = 5t + 2$. The first equation is $x_1 + 3t = -1$, giving $x_1 = -3t - 1$. The final solution to the system is then

$$x_1 = -3t - 1, \qquad x_2 = 5t + 2, \qquad x_3 = t, \qquad x_4 = 4,$$

or $(-3t - 1, 5t + 2, t, 4)$.

The solutions given in the previous two examples are called **general solutions**, because they tell us what any solution to the system looks like in the cases where there are infinitely many solutions. We can also produce some specific numbers that are solutions as well, which we will call **particular solutions**. These are obtained by simply letting any parameters take on whatever values we want.

◇ **Example 1.5(c):** Give three particular solutions to the system in Example 1.5(a).

Solution: If we take the easiest choice for t, zero, we get the particular solution $(4, 1, 0)$. Letting t equal negative one and one gives us the particular solutions $(7, 3, -1)$ and $(1, -1, 1)$.

The following examples show a situation in which there are two free variables, and one in which there is no solution.

◇ **Example 1.5(d):** A system of equations in the four variables x_1, x_2, x_3 and x_4 that has the row-reduced matrix

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Give the general solution and four particular solutions.

Solution: In this case the leading variables are x_1 and x_3, and the free variables are x_2 and x_4. We begin by letting $x_4 = t$; we have the equation $x_3 - 2t = 3$, giving us $x_3 = 2t + 3$. Since x_2 is a free variable, we call it something else. t has already been used, so let's say $x_2 = s$. The first equation indicated by the row-reduced matrix is then $x_1 + 2s - t = 2$, giving us $x_1 = -2s + t + 2$. The solution to the corresponding system is

$$x_1 = -2s + t + 2, \qquad x_2 = s, \qquad x_3 = 2t + 3, \qquad x_4 = t$$

If we let $s = 0$ and $t = 0$ we get the solution $(2, 0, 3, 0)$, and if we let $s = 2$ and $t = -1$ we get $(-3, 2, 1, -1)$. Letting $s = 0$ and $t = 1$ gives the particular solution $(3, 0, 5, 1)$ and letting $s = 1$ and $t = 0$ gives the particular solution $(0, 1, 3, 0)$.

The values used for the parameters in Examples 1.5(c) and (d) were chosen arbitrarily; any values can be used for s and t.

◇ **Example 1.5(e):** A system of equations in the four variables x_1, x_2, x_3 and x_4 has the row-reduced matrix

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Solve the system.

Solution: Since the last row is equivalent to the equation $0x_1 + 0x_2 + 0x_3 + 0x_4 = 5$, which has no solution, the system itself has no solution.

Section 1.5 Exercises To Solutions

1. Consider the system of equations $\begin{array}{rcl} 2x-4y\,-\,z &=& -4 \\ 4x-8y-\,z &=& -4 \\ -3x+6y+\,z &=& 4 \end{array}$.

   (a) Determine which of the following ordered triples are solutions to the system of equations:

   $$(6,3,4) \qquad (3,-1,4) \qquad (0,0,4) \qquad (-2,-1,4) \qquad (5,2,0) \qquad (2,1,4)$$

   Look for a pattern in the ordered triples that *ARE* solutions. Try to guess another solution, and test your guess by checking it in all three equations. How did you do?

   (b) When you tried to solve the system using your calculator, you should have gotten the reduced echelon matrix as

   $$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   Give the system of equations that this matrix represents. Which variable can you determine?

   (c) It is not possible to determine y, so we simply let it equal some arbitrary value, which we will call t. So at this point, $z=4$ and $y=t$. Substitute these into the first equation and solve for x. Your answer will be in terms of t. Write the ordered triple solution to the system.

   NOTE: The system of equations you obtained in part (b) and solved in part (c) has infinitely many solutions, but we do know that every one of them has the form $(2t,t,4)$. Note how this explains the results of part (a).

2. The reduced echelon form of the matrix for the system

   $$\begin{array}{rcl} 3x-2y\,+\,z &=& -7 \\ 2x+\,y-4z &=& 0 \\ x+\,y-3z &=& 1 \end{array} \qquad \text{is} \qquad \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) Give the free variable(s) and leading variable(s).

   (b) In this case, z cannot be determined, so we let $z=t$. Now solve for y, in terms of t. Then solve for x in terms of t.

   (c) Pick a specific value for t and substitute it into your general form of a solution triple for the system. Check it by substituting it into all three equations in the original system.

   (d) Repeat (b) for a different value of t.

3. The reduced echelon forms of some systems are given below.

   - If the system has a unique solution, give it. If the system has no solution, say so.
   - If the system has infinitely many solutions, give the general solution in terms of parameters s, t, etc., then give two particular solutions.

(a) $\begin{bmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 1 & 2 & 0 & -5 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -3 & 0 & 1 & -4 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & -2 & 1 & 6 \\ 0 & 1 & 3 & 5 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (h) $\begin{bmatrix} 1 & 4 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 5 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

4. Give four particular solutions from the general solution of Example 1.5(b), which had general solution $(-3t-1, 5t+2, t, 4)$.

5. For the systems whose augmented matrices row reduce to the forms shown below, do one of the following:

   - If the system has a unique solution, give it. If the system has no solution, say so.
   - If the system has infinitely many solutions, give the general solution in terms of parameters s, t, etc., then give two particular solutions.

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -1 & 0 & 5 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1.6 Back to Applications

Performance Criterion:

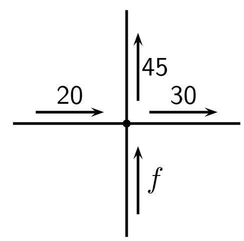
1. (l) Use systems of equations to solve network problems.

The concept of a network was introduced in Exercise 7 of Section 1.1. To review, a network is a set of junctions, which we'll call **nodes**, connected by what could be called pipes, or wires, but which we'll call **directed edges**. The word "directed" is used to mean that we'll assign a direction of flow to each edge. There will also be directed edges coming into or leaving the network. It is probably easiest to just think of a network of plumbing, with water coming in at perhaps several places, and leaving at several others.

Our study of networks will be based on one simple idea, known as **conservation of flow**:

At each node of a network, the flow into the node must equal the flow out.

◇ **Example 1.6(a):** A one-node network is shown to the right. Find the unknown flow f.

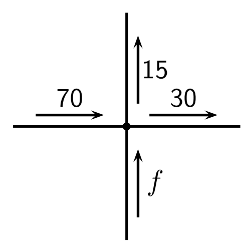


Solution: The flow in is $20 + f$ and the flow out is $45 + 30$, so we have

$$20 + f = 45 + 30.$$

Solving, we find that $f = 55$.

◇ **Example 1.6(b):** Another one-node network is shown to the right. Find the unknown flow f.



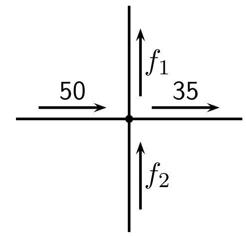
Solution: The flow in is $70 + f$ and the flow out is $15 + 30$, so we have

$$70 + f = 15 + 30.$$

Solving, we find that $f = -25$, so the flow at the arrow labeled f is actually in the direction opposite to the arrow.

When setting up a network we must commit to a direction of flow for any edges in which the flow is unknown, but when solving the system we may find that the flow is in the opposite direction from the way the edge was directed initially, as we just saw. We may also have less information than we did in the previous two examples, as shown by the next example.

◇ **Example 1.6(c):** For the one-node network is shown to the right, find the unknown flow f_1 in terms of the flow f_2.



Solution: By conservation of flow,

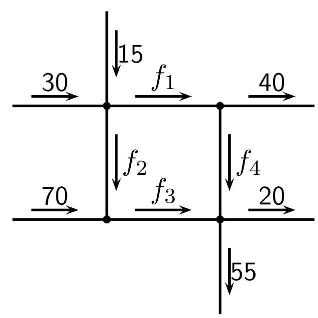
$$50 + f_2 = f_1 + 35.$$

Solving for f_1 gives us $f_1 = f_2 + 15$. Thus if f_2 was 10, f_1 would be 25 (look at the diagram and think about that), if f_2 was 45, f_1 would be 60, and so on.

The previous example represents, in an applied setting, the idea of a **free variable**. In this example either variable can be taken as free, but if we know the value of one of them, we'll "automatically" know the value of the other. The way the example was worded, we were taking f_2 to be the free variable, with the value of f_1 then depending on the value of f_2.

The systems in these first three examples have been very simple; let's now look at a more complex system.

◇ **Example 1.6(d):** Determine the flows f_1, f_2, f_3 and f_4 in the network shown to the right.



Solution: Utilizing conservation of flow at each node, we get the equations

$$30 + 15 = f_1 + f_2, \qquad 70 + f_2 = f_3,$$

$$f_1 = 40 + f_4, \qquad f_3 + f_4 = 20 + 55$$

Rearranging these give us the system of equations shown below and to the left. The augmented matrix for this system reduces to the matrix shown below and to the right.

$$\begin{array}{rcr} f_1 + f_2 \qquad\qquad & = & 45 \\ f_2 - f_3 \qquad & = & -70 \\ f_1 \qquad\qquad - f_4 & = & 40 \\ f_3 + f_4 & = & 75 \end{array} \implies \begin{bmatrix} 1 & 1 & 0 & 0 & 45 \\ 0 & 1 & -1 & 0 & -70 \\ 1 & 0 & 0 & -1 & 40 \\ 0 & 0 & 1 & 1 & 75 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 & -1 & 40 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 75 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we can see that f_4 is a free variable, so lets say it has value t. The solution to the network is then

$$f_1 = 40 + t, \qquad f_2 = 5 - t, \qquad f_3 = 75 - t, \qquad f_4 = t,$$

where t is the flow f_4.

Underdetermined and Overdetermined Systems

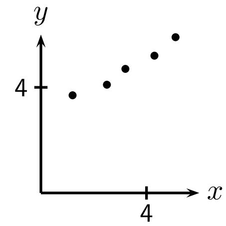
Let's think a bit more about this last example. Suppose that $f_4 = t = 0$. The equations given as the solution to the network then give us $f_1 = 40$, $f_2 = 5$, $f_3 = 75$. We can see this without even solving the system of equations. Looking at the node in the lower right, if $f_4 = 0$ one can easily see that f_3 must be 75 in order for the flow in to equal the flow out. Knowing f_3, we can go to the node in the lower left and see that $f_2 = 5$. Finally, $f_2 = 5$ gives is $f_1 = 40$. The information given originally was not sufficient to determine the values of the flows f_1, f_2, f_3 and f_4. In such a case, we sometimes say that the system is **undetermined**, meaning that there is too little information to guarantee a single solution. We just saw that with one more piece of information, the value of f_4, all of the remaining flows were then determined by that value.

Now consider the situation described in Exercise 6 of Section 1.1. Given the points

$$(1.2, 3.7) \qquad (2.5, 4.1) \qquad (3.2, 4.7) \qquad (4.3, 5.2) \qquad (5.1, 5.9)$$

we wish to find the equation $y = mx + b$ of a line containing them. Substituting each pair into $y = mx + b$ gives us the system shown below and to the left.

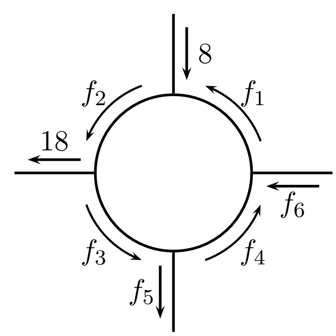
$$\begin{array}{rcl} 1.2m + b & = & 3.7 \\ 2.5m + b & = & 4.1 \\ 3.2m + b & = & 4.7 \\ 4.3m + b & = & 5.2 \\ 5.1m + b & = & 5.9 \end{array} \quad \Longrightarrow \quad \begin{bmatrix} 1.2 & 1 & 3.7 \\ 2.5 & 1 & 4.1 \\ 3.2 & 1 & 4.7 \\ 4.3 & 1 & 5.2 \\ 5.1 & 1 & 5.9 \end{bmatrix} \quad \overset{rref}{\Longrightarrow} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



When we row-reduce the augmented matrix for this system, we get the last matrix above, indicating that the system has no solution. The five points are plotted above and to the right, and we can see that they are not on a line, which is why we were not able to solve the system. In this case the system is **overdetermined**, meaning that there is too much information to allow a solution to the system.

You might think "Well, why not just use less data, so that the resulting system has a solution?" Well the additional data gives us some redundancy that can give us better results *if we know how to deal with it.* The way out of this problem is a method called **least-squares**, which we'll see later. It is a method for dealing with systems that don't have solutions. What it allows us to do is obtain values that are in some sense the "closest" values there are to an actual solution. Again, more on this later.

Section 1.6 Exercises To Solutions



1. The network to the right represents a traffic circle. The numbers next to each of the paths leading into or out of the circle are the net flows in the directions of the arrows, in vehicles per minute, during the first part of lunch hour.

   (a) Suppose that $f_3 = 7$ and $f_5 = 4$. You should be able to work your way around the circle, eventually figuring out what each flow is. Do this.

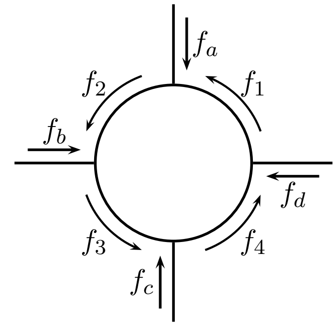
   (b) Now assume that the only flows you know are the ones shown in the diagram. When you set up a system of equations, based on the flows in and out of each junction, how many equations will you have? How many unknowns?

   (c) Go ahead and set up the system of equations. Give the augmented matrix and the reduced matrix (obtained with your calculator), and then give the general solution to the system.

   (d) Choose the value(s) of the parameter(s) that make $f_3 = 7$ and $f_5 = 4$, then find the resulting particular solution. *If your answers do not match what you got for part (a), go back and check your work for (c).*

   (e) What restriction(s) is(are) there on the parameter(s), in order that all flows go in the directions indicated. (Allow a flow of zero for each flow as well.)

2. For another traffic circle, a student uses the diagram shown to the right and obtains the flows given below, in vehicles per minute.



$$f_1 = t - 8, \qquad f_2 = t + 3, \qquad f_3 = t - 5, \qquad f_4 = t$$

(a) Determine the minimum value of t that makes each of f_1 through f_4 zero or greater. Give the minimum allowable values for each flow, in the form $f_i \geq a$, assuming that no vehicles ever go the wrong way around a portion of the circle. Remember that setting a value for any flow determines all the other flows. You may neglect units.

(b) Give each of the flows f_1 through f_4 when the flow in the northeast quarter (f_1) is 12 vehicles per minute. You may neglect units.

(c) Determine each of the flows f_a through f_d, still for $f_1 = 12$. You should be able to do this based only on the four equations given. At least one of them will be negative, indicating that the corresponding arrow(s) should be reversed.

1.7 Chapter 1 Exercises

1. Consider the system of equations below and to the right. Solve the system by Gaussian elimination (get in row-echelon form, then perform back substitution), **by hand** (no calculator). Show all steps, including what operation was performed for each step. **Hint:** You may find it useful to put the equations in a different order before forming the augmented matrix.

$$\begin{aligned} 5x - y + 2z &= 17 \\ x + 3y - z &= -4 \\ 2x + 4y - 3z &= -9 \end{aligned}$$

2. Find the equation of the parabola through the points $(0, 3)$, $(1, 4)$ and $(3, 18)$.

3. Consider the points $(1, 5)$, $(2, 2)$, $(4, 3)$ and $(5, 4)$.

   (a) What is the smallest degree polynomial whose graph will contain all of these points?

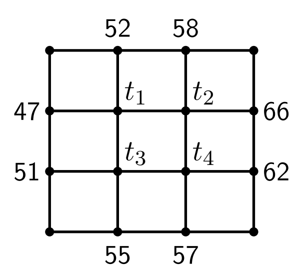
   (b) Find the polynomial whose graph contains all the points.

   (c) Check by graphing on your calculator.

4. Why would we not be able to find the equation of a line through $(0, 6)$, $(2, 3)$ and $(6, 1)$? We will see later what this means in terms of systems of equations, and we will resolve the problem in a reasonable way.

5. Find the equation of the plane through the three points $P_1(4, 1, -3)$, $P_2(0, -5, 1)$ and $P_3(3, 3, 2)$.

6. (a) A student is attempting to find the equilibrium temperatures at points t_1, t_2, t_3 and t_4 on a plate with a grid and boundary temperatures shown below and to the left. They get $t_1 = 50.3$, $t_2 = 67.4$, $t_3 = 53.6$, $t_4 = 60.5$. Explain in one *complete sentence* why their answer must be incorrect, *without finding the solution*.



$$\begin{bmatrix} 4 & -1 & 0 & 0 & -1 & 0 & 103 \\ -1 & 4 & -1 & 0 & 0 & -1 & 92 \\ 0 & -1 & 4 & -1 & -1 & 0 & 110 \\ 0 & 0 & -1 & 4 & 0 & 0 & 98 \\ -1 & 0 & -1 & 0 & 4 & 0 & 105 \\ 0 & -1 & 0 & 0 & -1 & 4 & 107 \end{bmatrix}$$

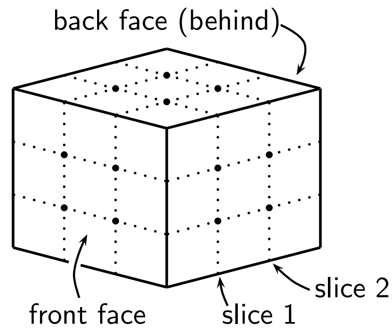
   (b) A different student is trying to solve another such problem, and their augmented matrix is shown above and to the right. How do we know that one of their equations is incorrect, *without setting up the equations ourselves?*

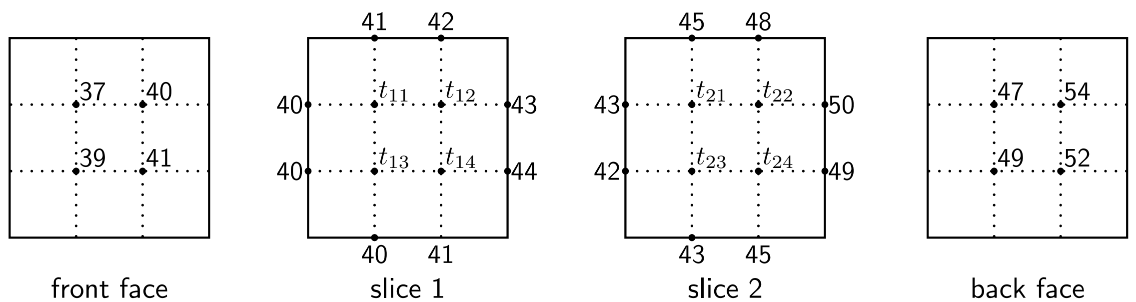
7. Suppose we are solving a system of three equations in the three unknowns x_1, x_2 and x_3, *with the unknowns showing up in the equations in that order*. It is possible to do row reduction in such a way as to obtain the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 3 & -2 & 0 & 7 \\ -1 & 5 & 2 & -3 \end{bmatrix}$$

   Determine x_1, x_2 and x_3 *without row-reducing this matrix!* you should be able to simply set up equations and find values for the unknowns.

8. Given a cube of some solid material, it is possible to put a three-dimensional grid into the solid, in the same way that we put a two-dimensional grid on a rectangular plate. Given temperatures at all nodes on the exterior faces of the cube, we can find equilibrium temperatures at each interior node using a system of equations. Once again the key is the mean-value property. In this three dimensional case this property tells us that the equilibrium temperature at each interior node is equal to the average of all the temperatures at nodes of the grid that are immediately adjacent to the point in question. To the right I have shown a cube that has eight interior grid points. The word "slice" is used here to mean a cross section through the cube. The grids below show temperatures, known or unknown, at all nodes on the front face, each of the two slices, and the back face. Above and to the right I have "exploded" the cube to show the temperatures on the front and back faces, and the two slices. Of course each node on any slice is connected to the corresponding node on the adjacent slice or face.





(a) Using the Mean Value Property in three dimensions, the temperature at each interior point will *NOT* be the average of four temperatures, like it was on a plate. How many temperatures will be averaged in this case?

(b) Set up a system of equations to solve for the interior temperatures, and find each to the nearest tenth.

9. Do any of your observations from Exercise 7 change in the three dimensional case?

10. Do one of the following for each of the systems whose augmented matrices row reduce to the forms shown below. **Assume that the unknowns are** x_1**,** x_2**, ...**

- If the system has a unique solution, give it. If the system has no solution, say so.
- If the system has infinitely many solutions, give the general solution in terms of parameters s, t, etc., then give two particular solutions.

(a) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & -1 & 0 & -4 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

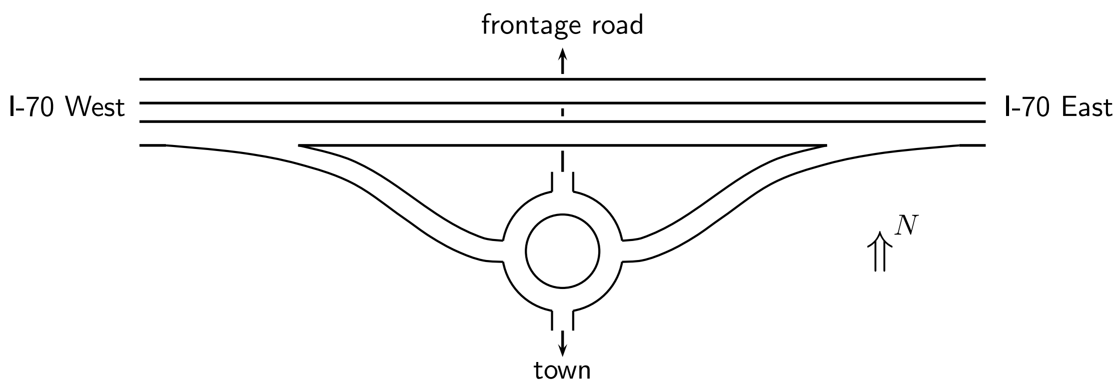
11. Consider the row-echelon augmented matrix $\begin{bmatrix} 1 & -1 & 3 & -2 & 4 \\ 0 & 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 2 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   (a) Give the general solution to the system of equations that this arose from.

   (b) Give three specific solutions to the system.

   (c) Change one entry in the matrix so that the system of equations would have no solution.

12. Vail, Colorado recently put in traffic "round-a-bouts" at all of its exits off Interstate 70. Each of these consists of a circle in which traffic is only allowed to flow counter-clockwise (do that all turns are right turns), and four points at which the circle can be entered or exited. See the diagram below.



It is known that at 7:30 AM the following is occurring:

- 22 vehicles per minute are entering the roundabout from the west. (These are the workers who cannot afford to live in Vail, and commute on I-70 from towns 30 and 40 miles west.)
- 4 vehicles per minute are exiting the roundabout to go east on I-70. (These are the tourists headed to the airport in Denver.)
- 7 vehicles per minute are exiting the roundabout toward town and 11 per minute are exiting toward the frontage road.

Solve the system and answer the following:

   (a) What is the minimum number of cars per minute passing through the southeast quarter of the roundabout?

   (b) If 18 vehicles per minute are passing through the southeast (SE) quarter of the roundabout per minute, how many are passing through each of the other quarters (NW, NE, SW)?

13. Consider the system
$$\begin{aligned} x_1 - x_2 - 4x_3 &= 6 \\ 5x_1 + x_2 - 2x_3 &= 18 \\ 2x_1 + 4x_2 + 10x_3 &= 0 \end{aligned}.$$

   (a) Use your calculator or an online tool to reduce the matrix to reduced row echelon form. Write the system of two equations represented by the first two rows of the reduced matrix. (The last equation is of no use, so don't bother writing it.)

(b) The second equation contains x_2 and x_3. Suppose that $x_3 = 1$ and compute x_2 using that equation. Then use the values you have for x_2 and x_3 in the first equation to find x_1.

(c) Verify that the values you obtained in (b) are in fact a solution to the original system given.

(d) Now let $x_3 = 0$ and repeat the process from (b) to obtain another solution. Verify that solution as well.

(e) Let $x_3 = 2$ to find yet another solution.

(f) Because there is no equation allowing us to determine x_3, we say that it is a **free variable**. What we will usually do in situations like this is let x_3 equal some **parameter** (number) that we will denote by t. That is, we set $x_3 = t$, which is really just renaming it. Substitute t into the second equation from (a) and solve for x_2 in terms of t. Then substitute that result into the first equation for x_2, along with t for x_3, and solve for x_1 in terms of t. Summarize by giving each of x_1, x_2 and x_3 in terms of t, all in one place.

(g) Substitute the number one for t into your answer to (f) and check to see that it matches what you got for (b). If doesn't, you've gone wrong somewhere - find the error and fix it.

14. Solve each of the following systems of equations that have solutions. Do/show the following things:

- Enter the augmented matrix for the system in your calculator.
- Get the row-reduced echelon form of the matrix using the *rref* command. **Write down the resulting matrix**.
- Write the system of equations that is equivalent to the row-reduced echelon matrix.
- Find the solutions, if there are any. *Use the letters that were used in the original system for the unknowns!* For those with multiple solutions, give them in terms of a parameter t or, when necessary, two parameters s and t.

(a) $$\begin{aligned} x_1 - x_2 + 3x_3 &= -4 \\ -2x_1 + 3x_2 - 8x_3 &= 13 \\ 5x_1 - 3x_2 + 11x_3 &= -10 \end{aligned}$$

(b) $$\begin{aligned} c_1 + 3c_2 + 5c_3 &= 3 \\ 2c_1 + 7c_2 + 9c_3 &= 5 \\ 2c_1 + 6c_2 + 11c_3 &= 7 \end{aligned}$$

(c) $$\begin{aligned} x_1 + 3x_2 - 2x_3 &= -1 \\ -7x_1 - 21x_2 + 14x_3 &= 7 \\ 2x_1 + 6x_2 - 4x_3 &= -2 \end{aligned}$$

(d) $$\begin{aligned} c_1 - c_2 + 3c_3 &= -4 \\ -2c_1 + 3c_2 - 8c_3 &= 13 \\ 5c_1 - 3c_2 + 11c_3 &= 4 \end{aligned}$$

(e) $$\begin{aligned} x - 3y + 7z &= 4 \\ 5x - 14y + 42z &= 29 \\ -2x + 5y - 20z &= -16 \end{aligned}$$

(f) $$\begin{aligned} x + 3y \phantom{{}+z} &= 2 \\ 4x + 12y + z &= 1 \\ -x - 3y - 2z &= 12 \end{aligned}$$

- Give three *specific* solutions to the system from part (a) above.
- Give three *specific* solutions to the system from part (c) above.
- Solve the system from part (b) above by hand, showing all steps of the row reduction and indicating what you did at each step.

15. Give the reduced row echelon form of an augmented matrix for a system of four equations in four unknowns x_1, x_2, x_3 and x_4 for which

- $x_4 = 7$
- x_2 and only x_2, is a free variable

16. (Erdman) Consider the system $\begin{array}{rcl} x + ky & = & 1 \\ kx + \ y & = & 1 \end{array}$, where k is some constant.

    (a) Set up the augmented matrix and use a row operation to get a zero in the lower left corner.

    (b) For what value or values of k would the system have infinitely many solutions? What is the form of the general solution?

    (c) For what value or values of k would the system have no solution?

    (d) For all remaining values of k the system has a unique solution (that depends on the choice of k). What is the solution in that case? Your answer will contain the parameter k.

17. (Erdman) Consider the system $\begin{array}{rcl} x - \ y - 3z & = & 3 \\ 2x + \qquad\quad z & = & 0 \\ 2y + 7z & = & c \end{array}$, where c is some constant.

    (a) Set up the augmented matrix and use a row operation to get a zero in the first entry of the second row.

    (b) Look at the second and third rows. For what value or values of c can the system be solved? Give the solution if there is a unique solution. Give the general solution if there are infinitely many solutions.

18. In a previous exercise, you may have attempted to find the equation of a parabola through the three points $(-1, -6)$, $(0, -4)$ and $(1, -1)$. You set up a system to find values of a, b and c in the parabola equation $y = ax^2 + bx + c$. There was a unique solution, meaning that there is only one parabola containing those three points. Expect the following to not work out as "neatly."

    (a) Use a system of equations to find the equation of a parabola that goes through just the two points $(-1, -6)$ and $(1, -1)$. Explain your results.

    (b) Use a system of equations to find the equation $y = mx + b$ of a line through the four points $(1.3, 1.5)$, $(0.8, 0.4)$, $(2.6, 3.0)$ and $(2.0, 2.0)$.

    (c) Plot the four points from (b) on a neat and accurate graph, and use what you see to explain your answer to (b). **You should be able to give your explanation in one or two complete sentences.**

B Solutions to Exercises

B.1 Chapter 1 Solutions

Section 1.1 Solutions Back to 1.1 Exercises

1. (b), (c) and (d) are linear equations

2. The first and second systems are linear, the third is not.

3. (a) $(-2,-3,2)$ and $(7,3,1)$ are solutions, $(5,-2,4)$ is not.

   (b) Only $(-1,3,2)$ is a solution.

   (c) $(2,1)$ and $(-1,-2)$ are solutions, $(3,5)$ is not.

4. (a) $5 = -8a + 4b - 2c + d$, this is a linear equation

   (b) $y = 7x^3 - 2x^2 - 5x + 1$, this is not a linear equation

5.
$$\begin{aligned} -8a + 4b - 2c + d &= 5 \\ -a + b - c + d &= 2 \\ a + b + c + d &= 3 \\ 27a + 9b + 3c + d &= 0 \end{aligned}$$

6.
$$\begin{aligned} 1.2m + b &= 3.7 \\ 2.5m + b &= 4.1 \\ 3.2m + b &= 4.7 \\ 4.3m + b &= 5.2 \\ 5.1m + b &= 5.9 \end{aligned}$$

7. (a)
$$\begin{aligned} f_1 - f_2 &= 5 \\ f_2 - f_3 &= -1 \\ f_3 - f_4 &= 4 \\ -f_1 + f_4 &= -8 \end{aligned}$$

7. (b) $(12,7,8,4)$, $(7,2,3,-1)$ and $(10,5,6,2)$ are solutions, $(9,4,3,1)$ is not

   (c) The solution $(7,2,3,-1)$ indicates a net flow of one vehicle *in the wrong direction* between nodes n_3 and n_4.

   (d) $f_4 = 5$, $f_1 = 13$ and $f_2 = 8$

Section 1.2 Solutions Back to 1.2 Exercises

1. (a) $a - b + c - d = 3$

   (b)
$$\begin{aligned} a + b + c + d &= 5 \\ a + 2b + 4c + 8d &= 4 \\ a + 4b + 16c + 64d &= -1 \end{aligned}$$

2.
$$\begin{aligned} a - b + c &= -4 \\ a + b + c &= 1 \\ 9a + 3b + c &= 0 \end{aligned}$$

3. (a) $4t_1 - t_2 - t_3 = 129$

   (b)
$$\begin{aligned} t_2 &= \frac{t_1 + 65 + 59 + t4}{4} &\implies&& -t_1 + 4t_2 - t_4 &= 129 \\ t_3 &= \frac{t_1 + t_4 + 53 + 55}{4} &\implies&& -t_1 + 4t_3 - t_4 &= 108 \\ t_4 &= \frac{t_2 + 52 + 50 + t_3}{4} &\implies&& -t_2 - t_3 + 4t_4 &= 102 \end{aligned}$$

4. (a)
$$\begin{aligned} 10I_1 + 2(I_1 - I_2) + 8(I_1 - I_3) &= 20 &\implies&& 20I_1 - 2I_2 - 8I_3 &= 20 \\ 6I_2 + 1(I_2 - I_3) + 2(I_2 - I_1) &= 0 &\implies&& -2I_1 + 9I_2 - I_3 &= 0 \\ 5I_3 + 8(I_3 - I_2) + 1(I_3 - I_2) &= 0 &\implies&& -8I_1 - I_2 + 14I_3 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 12I_1+2(I_1-I_2)+5(I_1-I_3) &= 12 & \Longrightarrow && 19I_1 -2I_2 -5I_3 &= 12\\ 8I_2+3(I_2-I_3)+2(I_2-I_1) &= 8 & \Longrightarrow && -2I_1+13I_2 -3I_3 &= 8\\ 4I_3+5(I_3-I_2)+3(I_3-I_2) &= 0 & \Longrightarrow && -5I_1 -3I_2+12I_3 &= 0 \end{aligned}$$

(c) The current from A to C is I_1-I_2 because I_1 is from A to C and I_2 is from C to A. Thus the current from A to C is $1.36-0.39=0.97$ amperes. The current flows from A to C because this value is positive.

(d) The current from B to C is I_2-I_3 because I_2 is from B to C and I_3 is from C to B. Thus the current from B to C is $0.39-0.81=-0.42$ amperes. The current flows from C to B because this value is negative.

Section 1.3 Solutions Back to 1.3 Exercises

1. (a) $(-2,1)$ (b) $(3,4)$ (c) $(3,2)$
   (d) $(1,-1)$ (e) $(-1,4)$ (f) $(-2,\frac{1}{3})$
2. (a) $(4,1)$ (b) $(0,-3)$ (c) $(2,2)$

Section 1.4 Solutions Back to 1.4 Exercises

1. The coefficient matrix is $\begin{bmatrix} 1 & 1 & -3\\ -3 & 2 & -1\\ 2 & 1 & -4 \end{bmatrix}$ and the augmented matrix is $\begin{bmatrix} 1 & 1 & -3 & 1\\ -3 & 2 & -1 & 7\\ 2 & 1 & -4 & 0 \end{bmatrix}$

2. All the matrices but D are in row-echelon form.

3. Matrices B and F are in reduced row-echelon form.

4. $\begin{bmatrix} 1 & 1 & -3 & 1\\ -3 & 2 & -1 & 7\\ 2 & 1 & -4 & 0 \end{bmatrix} \begin{matrix} 3R_1+R_2\to R_2\\ \Longrightarrow\\ -2R_1+R_3\to R_3 \end{matrix} \begin{bmatrix} 1 & 1 & -3 & 1\\ 0 & 5 & -10 & 10\\ 0 & -1 & 2 & -2 \end{bmatrix}$

5. (a) $\begin{bmatrix} 1 & 5 & -7 & 3\\ -5 & 3 & -1 & 0\\ 4 & 0 & 8 & -1 \end{bmatrix} \begin{matrix} 5R_1+R_2\to R_2\\ \Longrightarrow\\ -4R_1+R_3\to R_3 \end{matrix} \begin{bmatrix} 1 & 5 & -7 & 3\\ 0 & 28 & -36 & 15\\ 0 & -20 & 36 & -13 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & -8 & -1 & 5\\ 0 & -2 & 0 & 0\\ 0 & 6 & -5 & 2 \end{bmatrix} \begin{matrix} \Longrightarrow\\ 3R_2+R_3\to R_3 \end{matrix} \begin{bmatrix} 2 & -8 & -1 & 5\\ 0 & -2 & 0 & 0\\ 0 & 0 & -5 & 2 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & -2 & 4 & 1\\ 0 & 3 & 5 & -2\\ 0 & 2 & -8 & 1 \end{bmatrix} \begin{matrix} \Longrightarrow\\ -\frac{2}{3}R_2+R_3\to R_3 \end{matrix} \begin{bmatrix} 1 & -2 & 4 & 1\\ 0 & 3 & 5 & -2\\ 0 & 0 & -\frac{34}{3} & \frac{7}{3} \end{bmatrix}$

6. (a) $(-4,\frac{1}{2},-4)$ (b) $(33,-4,1)$ (c) $(7,0,-2)$

7. (a) $(2,3,-1)$ (b) $(-2,1,2)$ (c) $(-1,2,1)$

8. Same as Exercise 7.

9. $t_1=52.6^\circ$, $t_2=57.3^\circ$, $t_3=61.6^\circ$, $t_4=53.9^\circ$, $t_5=57.1^\circ$, $t_6=60.2^\circ$

12. $t_1 = 44.6°$, $t_2 = 49.6°$, $t_3 = 38.6°$

13. (a) fourth degree (b) $y = \frac{2}{5}x + \frac{23}{30}x^2 - \frac{1}{10}x^3 - \frac{1}{15}x^4$ or $y = 0.4x + 0.77x^2 - 0.1x^3 - 0.07x^4$

14. (a) $I_1 = 4.5$ amperes, $I_2 = 0.75$ amperes (b) $I_3 = 5.25$ amperes

17. $z = 3.765 + 0.353x - 1.235y$

Section 1.5 Solutions

Back to 1.5 Exercises

1. (a) $(6,3,4)$, $(0,0,4)$, $(-2,-1,4)$, $(2,1,4)$
   (b) $x - 2y = 0$, $z = 4$, we can determine z (c) $(2t, t, 4)$

2. (a) z is a free variable, x and y are leading variables
   (b) $y = 2t + 2$, $x = t - 1$

3. (a) $(4+t, -5-2t, t, 3)$, $(4,-5,0,3)$, $(5,-7,1,3)$, $(6,-9,2,3)$, $(3,-3,-1,3), ...$
   (b) $(5-3t, t, 1, -4)$, $(5,0,1,-4)$, $(2,1,1,-4)$, $(-1,2,1,-4)$, $(8,-1,1,-4), ...$
   (c) $(-4+3s-t, s, 5+2t, t)$, $(-4,0,5,0)$, $(-1,1,5,0)$, $(-5,0,7,1)$, $(-2,1,7,1), ...$
   (d) no solution
   (e) $(6+2s-t, -3-3s-5t, s, t)$, $(6,-3,0,0)$, $(6,-3,0,0)$, $(8,-6,1,0)$, $(5,-8,0,1)$, $(7,-11,1,1), ...$
   (f) $(-1,2,0)$
   (g) $(1-2s+t, s, t)$, $(1,0,0)$, $(-1,1,0)$, $(2,0,1)$, $(0,1,1), ...$
   (h) no solution
   (i) $(2-5t, t, 1, -4)$, $(2,0,1,-4)$, $(-3,1,1,-4)$, $(-8,2,1,-4)$, $(7,-1,1,-4), ...$

4. $t=-2$: $(5,-8,-2,4)$, $t=-1$: $(2,-3,-1,4)$, $t=0$: $(-1,2,0,4)$, $t=1$: $(-4,7,1,4)$,
   $t=2$: $(-7,12,2,4)$

5. There is no solution to the system with the first reduced matrix given. The system with the second reduced matrix has general solution $(-2s+t+5, s, t, -4)$ and some particular solutions of

$$s = t = 0 : (5,0,0,-4), \qquad s = 1,\ t = 0 : (3,1,0,-4), \qquad s = 0,\ t = 1 : (6,0,1,-4)$$