Linear Algebra I

Skills, Concepts and Applications

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2 Euclidean Space and Vectors

Outcome:

2. Understand vectors and their algebra and geometry in $\mathbb{R}^n$. Understand the relationship of vectors to systems of equations.

   Performance Criteria:

   (a) Recognize the equation of a plane in $\mathbb{R}^3$ and determine where the plane intersects each of the three axes. Sketch a graph of the part of a plane in the first quadrant. Give the equation of a plane from a geometric description.

   (b) Find the vector from one point to another in $\mathbb{R}^n$. Find the length of a vector in $\mathbb{R}^n$.

   (c) Multiply vectors by scalars and add vectors, algebraically. Find linear combinations of vectors algebraically.

   (d) Illustrate the parallelogram method and tip-to-tail method for finding a linear combination of two vectors.

   (e) Find a linear combination of vectors equalling a given vector.

   (f) Give the linear combination form of a system of equations, give the system of linear equations equivalent to a given vector equation.

   (g) Sketch a picture illustrating the linear combination form of a system of equations of two equations in two unknowns.

   (h) Give an algebraic description of a set of a set of vectors that has been described geometrically, and vice-versa.

   (i) Determine whether a set of vectors is closed under vector addition; determine whether a set of vectors is closed under scalar multiplication. If it is, prove that it is; if it is not, give a counterexample.

   (j) Give the vector equation of a line through two points in $\mathbb{R}^2$ or $\mathbb{R}^3$ or the vector equation of a plane through three points in $\mathbb{R}^3$.

   (k) Write the solution to a system of equations in vector form and determine the geometric nature of the solution.

In the study of linear algebra we will be defining new concepts and developing corresponding notation. *The purpose of doing so is to develop more powerful machinery for investigating the concepts of interest.* We begin the development of notation with the following. The set of all real numbers is denoted by $\mathbb{R}$, and the set of all ordered pairs of real numbers is $\mathbb{R}^2$, spoken as "R-two." Geometrically, $\mathbb{R}^2$ is the familiar Cartesian coordinate plane. Similarly, the set of all ordered triples of real numbers is the three-dimensional space referred to as $\mathbb{R}^3$, "R-three." The set of all ordered ***n*-tuples** (lists of n real numbers in a particular order) is denoted by $\mathbb{R}^n$. Although difficult or impossible to visualize physically, $\mathbb{R}^n$ can be thought of as n-dimensional space. All of the $\mathbb{R}^n$s are what are called **Euclidean space**.

2.1 Euclidean Space

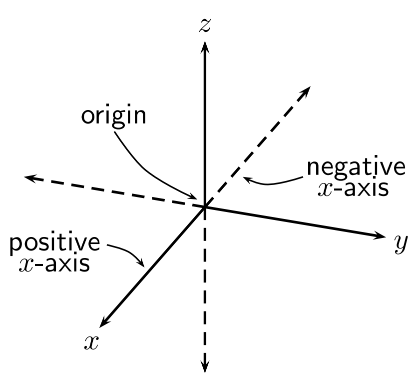
Performance Criteria:

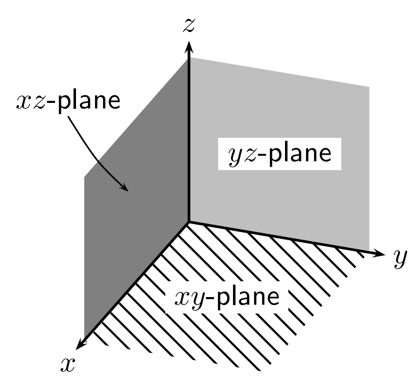
2. (a) Recognize the equation of a plane in $\mathbb{R}^3$ and determine where the plane intersects each of the three axes. Sketch a graph of the part of a plane in the first quadrant. Give the equation of a plane from a geometric description.

It is often taken for granted that everyone knows what we mean by the **real numbers**. To actually define the real numbers precisely is a bit of a chore and very technical. Suffice it to say that the real numbers include all numbers other than complex numbers (numbers containing $\sqrt{-1} = i$ or, for electrical engineers, j) that a scientist or engineer is likely to run into. The numbers 5, -31.2, π, $\sqrt{2}$, $\frac{2}{7}$, and e are all real numbers. We denote the set of all real numbers with the symbol $\mathbb{R}$, and the geometric representation of the real numbers is the familiar **real number line**, a horizontal line on which every real number has a place. This is possible because the real numbers are ordered: given any two real numbers, either they are equal to each other, one is less than the other, or vice-versa.

As mentioned previously, the set $\mathbb{R}^2$ is the set of all ordered pairs of real numbers. Geometrically, every such pair corresponds to a point in the **Cartesian plane**, which is the familiar xy-plane. $\mathbb{R}^3$ is the set of all ordered triples, each of which represents a point in three-dimensional space. We can continue on - $\mathbb{R}^4$ is the set of all ordered "4-tuples", and can be thought of geometrically as four dimensional space. Continuing further, an "n-tuple" is n real numbers, in a specific order; each n-tuple can be thought of as representing a point in n-dimensional space. These spaces are sometimes called "two-space," "three-space" and "n-space" for short.

Two-space is fairly simple, with the only major features being the two axes and the four quadrants that the axes divide the space into. Three-space is a bit more complicated. Obviously there are three coordinate axes instead of two. In addition to those axes, there are also three coordinate planes as well, the xy-plane, the xz-plane and the yz-plane. Finally the three coordinate planes divide the space into eight **octants**. The pictures below illustrate the coordinate axes and planes. The first octant is the one we are looking into, where all three coordinates are positive. It is not important that we know the numbering of the other octants.





Every plane in $\mathbb{R}^3$ (we will be discussing only $\mathbb{R}^3$ for now) consists of a set of points that behave in an orderly mathematical manner, described here:

Equation of a Plane in $\mathbb{R}^3$: The ordered triples corresponding to all the points in a plane satisfy an equation of the form

$$ax + by + cz = d, \tag{1}$$

where a, b, c and d are constants, and not all of a, b and c are zero.

Note that equation (1) is a linear equation! The equation of any line in $\mathbb{R}^2$ can be written in the form $ax + by = c$, which is also a linear equation. A line is a one-dimensional object and a plane is a two-dimensional object, in a sense that we will see later. $ax+by = c$ then describes a one-dimensional "straight" object in the two-dimensional space $\mathbb{R}^2$, and (1) describes a two-dimensional "flat" object in $\mathbb{R}^3$. The corresponding equation in $\mathbb{R}^n$ is

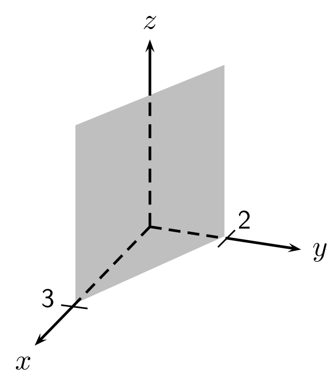
$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b, \tag{2}$$

where all of a_1, a_2, ..., a_n and b are numbers and x_1, x_2, ..., x_n are unknowns. (2) describes an $n-1$-dimensional object in n-dimensional space. When $n > 3$ we often call such an object a **hyperplane**.

Going back to (1) as describing a plane in $\mathbb{R}^3$, the xy-plane in $\mathbb{R}^3$ is the plane containing the x and y-axes. The only condition on points in that plane is that $z = 0$, so that is the equation of that plane. In that case the constants a, b and d are all zero, and $c = 1$. The plane $z = 5$ is a horizontal plane that is five units above the xy-plane, and $x = -2$ describes the vertical plane that is parallel to the yz-plane and passes through the x-axis at $x = -2$. What about an equation of the form $ax + by = c$ when we are in $\mathbb{R}^3$?

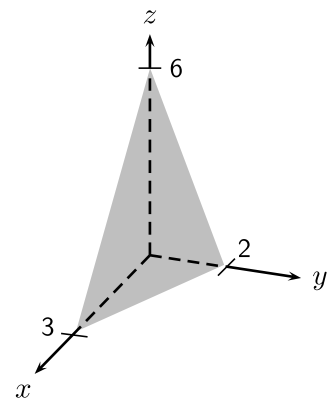
◇ **Example 2.1(a):** Graph the equation $2x + 3y = 6$ in the first octant. Indicate clearly where it intersects each of the coordinate axes, if it does.

Solution: Some points that satisfy the equation are $(3, 0, 0)$, $(6, -2, 5)$, and so on. Since z is not included in the equation, there are no restrictions on z; it can take any value. If we were to fix z at zero and plot all points that satisfy the equation, we would get a line in the xy-plane through the two points $(3, 0, 0)$ and $(0, 2, 0)$. These points are obtained by first letting y and z be zero, then by letting x and z be zero. Since z can be anything, the set of points satisfying $2x + 3y = 6$ is a vertical plane intersecting the xy-plane in that line. The plane is shown to the right.



Next we'll see how we can sometimes graphically represent a portion of a plane in 3-space.

◇ **Example 2.1(b):** Graph the equation $2x + 3y + z = 6$ in the first octant. Indicate clearly where it intersects each of the coordinate axes, if it does.



Solution: The set of points satisfying this equation is also a plane, but z is no longer free to take any value. The simplest way to "get a handle on" such a plane is to find where it intercepts the three axes. Note that every point on the x-axis has y- and z-coordinates of zero. So to find where the plane intersects the x-axis we put zero into the equation for y and z, then solve for x, getting $x = 3$. The plane then intersects the x-axis at $(3, 0, 0)$. A similar process gives us that the plane intersects the y and z axes at $(0, 2, 0)$ and $(0, 0, 6)$. From this information we get that the graph of the plane is that shown in the drawing above and to the right.

Consider now a system of equations like

$$\begin{aligned} x + 3y - 2z &= -4 \\ 3x + 7y + z &= 4 \\ -2x + y + 7z &= 7 \end{aligned},$$

which has solution $(3, -1, 2)$. We now know that each of the three equations represents a plane in $\mathbb{R}^3$. The point $(3, -1, 2)$ is where the three planes intersect! This is completely analogous to the interpretation of the solution of a system of two linear equations in two unknowns as the point where the two lines representing the equations cross. This is the first of three interpretations we'll have for the solution to a system of equations.

The only other basic geometric fact we need about three-space is this:

Distance Between Points: The distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbb{R}^3$ is given by

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

This is simply a three-dimensional version of the Pythagorean Theorem, and an equivalent formula is used in higher dimensions. Even though we cannot visualize the distance geometrically, this idea is both mathematically valid and useful in applications.

◇ **Example 2.1(c):** Find the distance in $\mathbb{R}^4$ between the points $(-4, 7, 1, -5)$ and $(13, 0, -6, 2)$.

Solution: Using the above formula with one more dimension we get

$$\begin{aligned} d &= \sqrt{(-4-13)^2 + (7-0)^2 + (1-(-6))^2 + (-5-2)^2} \\ &= \sqrt{(-17)^2 + 7^2 + 7^2 + (-7)^2} \;=\; \sqrt{436} \;\approx\; 20.9 \end{aligned}$$

Section 2.1 Exercises To Solutions

1. Determine whether each of the equations given describes a plane in $\mathbb{R}^3$. If not, say so. If it does describe a plane, give the points where it intersects each axis. If it doesn't intersect an axis, say so.

   (a) $-2x - y + 3z = -6$ (b) $x + 3z = 6$ (c) $y = -6$

   (d) $x + 3z^2 = 12$ (e) $x - 2y + 3z = -6$

2. (a) Give an equation of the plane in $\mathbb{R}^3$ that intersects the x-axis at 2, the y-axis at 5, and the z-axis at 4.

   (b) Give an equation of the plane in $\mathbb{R}^3$ that intersects the x-axis at -3, the y-axis at 1, and the z-axis at 7.

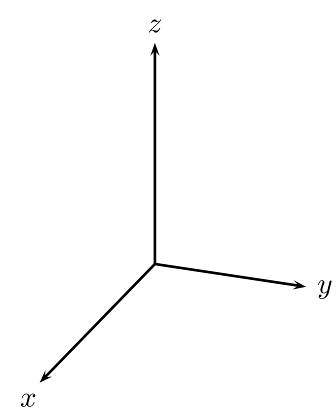
   (c) Give an equation of the plane in $\mathbb{R}^3$ that intersects the x-axis at 3, the z-axis at -2, and does not intersect the y-axis.

   (d) Give an equation of the plane in $\mathbb{R}^3$ that intersects the y-axis at 4 and does not intersect either of the other two axes.

   (e) Give an equation of the plane in $\mathbb{R}^3$ that intersects the y-axis at -4, the z-axis at 1, and does not intersect the x-axis.

   (f) Give an equation of the plane in $\mathbb{R}^3$ that intersects the z-axis at -2 and does not intersect either of the other two axes.

3. Each of the following equations describes a plane in $\mathbb{R}^3$. For each, sketch the graph of the part of the plane in the first octant, in the manner done in Examples 2.1(a) and (b). Begin with a sketch of the positive parts of the three coordinate axes, as shown to the right.



   (a) $2x + 2y + 5z = 10$ (b) $2y + 3z = 6$

   (c) $x + 3y = 6$ (d) $2x + 4y + z = 8$

   (e) $z = 3$ (f) $3x + y + 2z = 6$

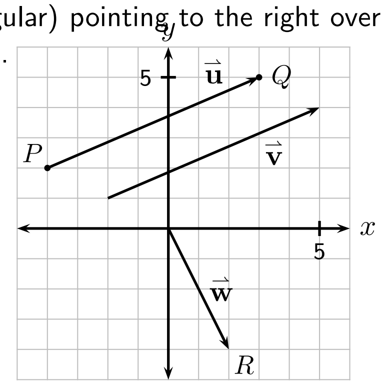
   (g) $2x + 4z = 8$ (h) $y = 5$ (i) $3x + y + 3z = 3$

2.2 Introduction to Vectors

Performance Criteria:

2. (b) Find the vector from one point to another in $\mathbb{R}^n$. Find the length of a vector in $\mathbb{R}^n$.

There are a number of different ways of thinking about **vectors**; it is likely that you think of them as "arrows" in two- or three-dimensional space, which is the first concept of a vector that most people have. Each such arrow has a **length** (sometimes called **norm** or **magnitude**) and a direction. Physically, then, vectors represent quantities having both amount and direction. Examples would be things like force or velocity. Quantities having only amount, like temperature or pressure, are referred to as **scalar** quantities. We will represent scalar quantities with lower case italicized letters like $a, b, c, ..., x, y, z$ and we'll represent vectors with lower case boldface letters with "harpoon" arrows over them, like $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{x}}$, and so on.. In many texts vectors are denoted just by lower case boldface letters like $\mathbf{u}$, $\mathbf{v}$, $\mathbf{x}$, and so on. When writing by hand we'll just put a small arrow (harpoon or regular) pointing to the right over any letter denoting a vector, without trying to make the letter boldface.



Consider a vector represented by an arrow in $\mathbb{R}^2$. We will call the end with the arrowhead the **tip** of the vector, and the other end we'll call the **tail**. (The more formal terminology is **terminal point** and **initial point**.) The picture to the right shows three vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ in $\mathbb{R}^2$. It should be clear that a vector can be moved around in $\mathbb{R}^2$ in such a way that the direction and magnitude remain unchanged. Sometimes we say that two vectors related to each other this way are equivalent, but in this class we will simply say that they are the same vector. The vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are the same vector, just in different positions.

It is sometimes convenient to denote points with letters, and we use italicized capital letters for this. We commonly use P (for point!) Q and R, and the origin is denoted by O. (That's capital "oh," not zero.) Sometimes we follow the point immediately by its coordinates, like $P(-4, 2)$. The notation $\overrightarrow{PQ}$ denotes the vector that goes from point P to point Q, which in this case is vector $\vec{\mathbf{u}}$. Any vector $\overrightarrow{OR}$ with its tail at the origin is said to be in **standard position**, and is called a **position vector**; $\vec{\mathbf{w}}$ above is an example of such a vector. Note that for any point in $\mathbb{R}^2$ (or in $\mathbb{R}^n$), there is a corresponding vector that goes from the origin to that point. *In linear algebra we think of a point and its position vector as interchangeable.* In the next section you will see the advantage of thinking of a point as a position vector.

We will describe vectors with numbers - in $\mathbb{R}^2$ we give a vector as two numbers, the first telling how far to the right (positive) or left (negative) one must go to get from the tail to the tip of the vector, and the second telling how far up (positive) or down (negative) from tail to tip. These numbers are generally arranged in one of two ways. The first way is like an ordered pair, but with "square brackets" instead of parentheses. The vector $\vec{\mathbf{u}}$ above is then $\vec{\mathbf{u}} = [7, 3]$, and $\vec{\mathbf{w}} = [2, -4]$. The second way to write a vector is as a **column vector**: $\vec{\mathbf{u}} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$. This is, in fact, the form we will use more often. Observe that the vector from one point to another is obtained by subtracting the corresponding coordinates of the first point from those of the second point. So the vector $\overrightarrow{PQ}$ from $P(-4, 2)$ to $Q(3, 5)$ is $\overrightarrow{PQ} = \begin{bmatrix} 3-(-4) \\ 5-2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$.

The two numbers quantifying a vector in $\mathbb{R}^2$ are called the **components** of the vector. We generally use the same letter to denote the components of a vector as the one used to name the vector, but we distinguish them with subscripts. Of course the components are scalars, so we use italic letters for them. So we would have $\vec{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. The direction of a vector in $\mathbb{R}^2$ is given, in some sense, by the combination of the two components. The length is found using the Pythagorean theorem. For a vector $\vec{\mathbf{v}} = [v_1, v_2]$ we denote and define the length of the vector by $\| \vec{\mathbf{v}} \| = \sqrt{v_1^2 + v_2^2}$. Of course everything we have done so far applies to vectors in higher dimensions. A vector $\vec{\mathbf{x}}$ in $\mathbb{R}^n$ would be denoted by $\vec{\mathbf{x}} = [x_1, x_2, ..., x_n]$. This shows that, in some sense, a **vector** is just an ordered list of numbers, like an n-tuple but with differences you will see in the next section. The length of a vector in $\mathbb{R}^n$ is found as follows.

DEFINITION 2.2.1: **Magnitude of a Vector in $\mathbb{R}^n$**

The **magnitude**, or **length** of a vector $\vec{\mathbf{x}} = [x_1, x_2, ..., x_n]$ in $\mathbb{R}^n$ is given by

$$\| \vec{\mathbf{x}} \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

◇ **Example 2.2(a):** Find the vector $\vec{\mathbf{x}} = \overrightarrow{PQ}$ in $\mathbb{R}^3$ from the point $P(5, -3, 7)$ to $Q(-2, 6, 1)$, and find the length of the vector.

Solution: The components of $\vec{\mathbf{x}}$ are obtained by simply subtracting each coordinate of P from each coordinate of Q:

$$\vec{\mathbf{x}} = \overrightarrow{PQ} = \begin{bmatrix} -2-5 \\ 6-(-3) \\ 1-7 \end{bmatrix} = \begin{bmatrix} -7 \\ 9 \\ -6 \end{bmatrix}$$

The length of $\vec{\mathbf{x}}$ is

$$\| \vec{\mathbf{x}} \| = \sqrt{(-7)^2 + 9^2 + (-6)^2} = \sqrt{166} \approx 12.9$$

There will be times when we need a vector with length zero; this is the special vector we will call (surprise, surprise!) the **zero vector**. It is denoted by a boldface zero, $\vec{\mathbf{0}}$, to distinguish it from the scalar zero. This vector has no direction.

Let's finish with the following important note about how we will work with vectors in this class:

In this course, when working with vectors geometrically, we will almost always be thinking of them as position vectors. When working with vectors algebraically, we will always consider them to be column vectors.

Section 2.2 Exercises To Solutions

1. Find the magnitude of each vector - label each answer using correct notation.

$$\vec{\mathbf{u}} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \qquad \vec{\mathbf{x}} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}, \qquad \vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \qquad \vec{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

2. For each pair of points P and Q, find the vector $\overrightarrow{PQ}$ in the appropriate space. Then find the length of the vector.

   (a) $P(-4, 11, 7),\ Q(13, 5, -8)$

   (b) $P(-5, 1),\ Q(7, -2)$

   (c) $P(-3, 0, 6, 1),\ Q(7, -1, -1, 10)$

3. (a) The vector $\overrightarrow{PQ} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ in $\mathbb{R}^2$ has initial point $P(3, 5)$. What is the terminal point Q?

   (b) The vector $\overrightarrow{PQ} = [4, 0, -2, 1, 5]$ in $\mathbb{R}^5$ has initial point $P(-2, 7, 1, -8, 2)$. What is the terminal point Q?

   (c) The vector $\overrightarrow{PQ} = \begin{bmatrix} -4 \\ 6 \\ -1 \end{bmatrix}$ in $\mathbb{R}^3$ has terminal point $Q(5, -2, 4)$. What is the initial point P?

4. Consider the vector $\vec{\mathbf{u}} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ in $\mathbb{R}^3$.

   (a) Find the magnitude $\|\, \vec{\mathbf{u}} \,\|$, labeling it as such.

   (b) Later we will define what we mean by multiplication or division of a vector by a scalar. Divide each component of $\vec{\mathbf{u}}$ by your answer to (a). The result is the vector $\dfrac{\vec{\mathbf{u}}}{\|\, \vec{\mathbf{u}} \,\|}$, so label it that way.

   (c) Find $\left\| \dfrac{\vec{\mathbf{u}}}{\|\, \vec{\mathbf{u}} \,\|} \right\|$, the magnitude of $\dfrac{\vec{\mathbf{u}}}{\|\, \vec{\mathbf{u}} \,\|}$ from (b).

5. Repeat Exercise 4 for the vector $\vec{\mathbf{v}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ in $\mathbb{R}^2$. How does your final result compare with that of Exercise 4?

2.3 Addition and Scalar Multiplication of Vectors, Linear Combinations

Performance Criteria:

2. (c) Multiply vectors by scalars and add vectors, algebraically. Find linear combinations of vectors algebraically.

   (d) Illustrate the parallelogram method and tip-to-tail method for finding a linear combination of two vectors.

   (e) Find a linear combination of vectors equalling a given vector.

In the previous section a vector $\vec{\mathbf{x}} = [x_1, x_2, ..., x_n]$ in n dimensions was starting to look suspiciously like an n-tuple $(x_1, x_2, ..., x_n)$ and we established a correspondence between any point and the position vector with its tip at that point. One might wonder why we bother then with vectors at all! The reason is that we can perform algebraic operations with vectors that make sense physically, while such operations make no sense with n-tuples. The two most basic things we can do with vectors are add two of them or multiply one by a scalar, and both are done component-wise:

<u>Definition 2.3.1</u>: **Addition and Scalar Multiplication of Vectors**

Let $\vec{\mathbf{u}} = [u_1, u_2, ..., u_n]$ and $\vec{\mathbf{v}} = [v_1, v_2, ..., n_n]$, and let c be a scalar. Then we define the vectors $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ and $c\vec{\mathbf{u}}$ by

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} \quad \text{and} \quad c\vec{\mathbf{u}} = c\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix}$$

Note that result of adding two vectors or multiplying a vector by a scalar is also a vector. It clearly follows from these that we can get subtraction of vectors by first multiplying the second vector by the scalar -1, then adding the vectors. With just a little thought you will recognize that this is the same as just subtracting the corresponding components.

◇ **Example 2.3(a):** For $\vec{\mathbf{u}} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix}$, find $\vec{\mathbf{u}} + \vec{\mathbf{v}}$, $3\vec{\mathbf{u}}$ and $\vec{\mathbf{u}} - \vec{\mathbf{v}}$.

Solution:

$$\vec{\mathbf{u}} + \vec{\mathbf{v}} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 + (-4) \\ -1 + 9 \\ 2 + 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 8 \end{bmatrix}$$

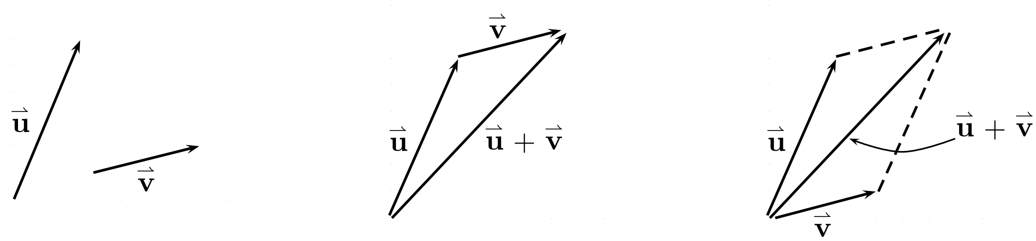
$$3\vec{\mathbf{u}} = 3\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3(5) \\ 3(-1) \\ 3(2) \end{bmatrix} = \begin{bmatrix} 15 \\ -3 \\ 6 \end{bmatrix}$$

$$\vec{\mathbf{u}} - \vec{\mathbf{v}} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 5-(-4) \\ -1-9 \\ 2-6 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ -4 \end{bmatrix}$$

Addition of vectors can be thought of geometrically in two ways, both of which are useful. The first way is what we will call the **tip-to-tail method**, and the second method is called the **parallelogram method**. You should become very familiar with both of these methods, as they each have their advantages; they are illustrated below.

◇ **Example 2.3(b):** Add the two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ shown below and to the left, first by the tip-to-tail method, and second by the parallelogram method.

Solution: To add using the tip-to-tail method, move the second vector so that its tail is at the tip of the first. (Be sure that its length and direction remain the same!) The vector $\vec{\mathbf{u}}+\vec{\mathbf{v}}$ goes from the tail of $\vec{\mathbf{u}}$ to the tip of $\vec{\mathbf{v}}$. See in the middle below.



tip-to-tail method parallelogram method

To add using the parallelogram method, put the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ together at their tails (again being sure to preserve their lengths and directions). Draw a dashed line from the tip of $\vec{\mathbf{u}}$, parallel to $\vec{\mathbf{v}}$, and draw another dashed line from the tip of $\vec{\mathbf{v}}$, parallel to $\vec{\mathbf{u}}$. $\vec{\mathbf{u}}+\vec{\mathbf{v}}$ goes from the tails of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ to the point where the two dashed lines cross. See to the right above. The reason for the name of this method is that the two vectors and the dashed lines create a parallelogram.

Each of these two methods has a natural physical interpretation. For the tip-to-tail method, imagine an object that gets *displaced* by the direction and amount shown by the vector $\vec{\mathbf{u}}$. Then suppose that it gets displaced by the direction and amount given by $\vec{\mathbf{v}}$ after that. Then the vector $\vec{\mathbf{u}}+\vec{\mathbf{v}}$ gives the *net* (total) displacement of the object. Now look at that picture for the parallelogram method, and imagine that there is an object at the tails of the two vectors. If we were then to have two forces acting on the object, one in the direction of $\vec{\mathbf{u}}$ and with an amount (magnitude) indicated by the length of $\vec{\mathbf{u}}$, and another with amount and direction indicated by $\vec{\mathbf{v}}$, then $\vec{\mathbf{u}}+\vec{\mathbf{v}}$ would represent the net force. (In a statics or physics course you might call this the **resultant** force.)

A very important concept in linear algebra is that of a **linear combination**. Let me say it again:

> Linear combinations are one of the most important concepts in linear algebra! You need to recognize them when you see them and learn how to create them. They will be central to almost everything that we will do from here on.

A linear combination of a set of vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_n$ (note that the subscripts now distinguish different *vectors*, not the components of a single vector) is obtained when each of the vectors is multiplied by a scalar, and the resulting vectors are added up. So if $c_1, c_2, ..., c_n$ are the scalars that $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_n$ are multiplied by, the resulting linear combination is the *single vector* $\vec{\mathbf{v}}$ given by

$$\vec{\mathbf{v}} = c_1 \vec{\mathbf{v}}_1 + c_2 \vec{\mathbf{v}}_2 + c_3 \vec{\mathbf{v}}_3 + \cdots + c_n \vec{\mathbf{v}}_n \,.$$

Emphasizing again the importance of this concept, let's provide a slightly more concise and formal definition:

DEFINITION 2.3.2: **Linear Combination**

A **linear combination** of the vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_n$, all in $\mathbb{R}^n$, is any vector $\vec{\mathbf{v}}$ of the form

$$\vec{\mathbf{v}} = c_1 \vec{\mathbf{v}}_1 + c_2 \vec{\mathbf{v}}_2 + c_3 \vec{\mathbf{v}}_3 + \cdots + c_n \vec{\mathbf{v}}_n,$$

where $c_1, c_2, ..., c_n$ are scalars.

Note that when we create a linear combination of a set of vectors we are doing virtually everything possible algebraically with those vectors, which is just addition and scalar multiplication!

You have seen this idea before; every polynomial like $5x^3 - 7x^2 + \frac{1}{2}x - 1$ is a linear combination of $1, x, x^2, x^3, ...$ Those of you who have had a differential equations class have seen things like $\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} + 2y$, which is a linear combination of the second, first and "zeroth" derivatives of a function $y = y(t)$. Here is why linear combinations are so important: In many applications we seek to have a basic set of objects (vectors) from which all other objects can be built as linear combinations of objects from our basic set. A large part of our study will be centered around this idea. This may not make any sense to you now, but hopefully it will by the end of the course.

◇ **Example 2.3(c):** For the vectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix}$ and $\vec{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix}$, give the linear combination $2\vec{\mathbf{v}}_1 - 3\vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_3$ as one vector.

Solution:

$$\begin{aligned} 2\vec{\mathbf{v}}_1 - 3\vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_3 &= 2\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} - 3\begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix} \\ &= \begin{bmatrix} 10 \\ -2 \\ 4 \end{bmatrix} - \begin{bmatrix} -12 \\ 27 \\ 18 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ -26 \\ 30 \end{bmatrix} \end{aligned}$$

◇ **Example 2.3(d):** For the same vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_3$ as in the previous exercise and scalars c_1, c_2 and c_3, give the linear combination $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3$ as one vector.

Solution:

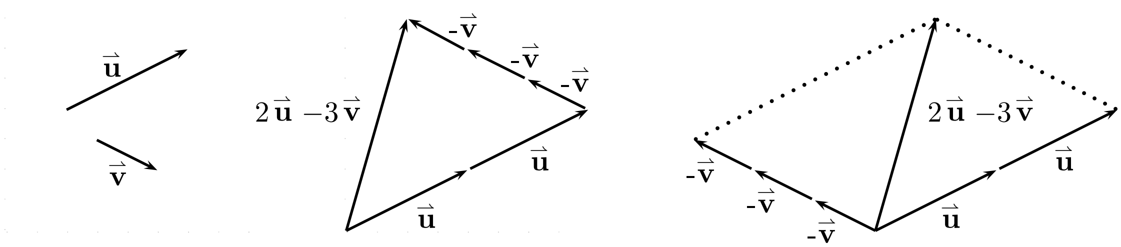
$$\begin{aligned}
c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3 &= c_1\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} -4 \\ 9 \\ 6 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} 5c_1 \\ -1c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} -4c_2 \\ 9c_2 \\ 6c_2 \end{bmatrix} + \begin{bmatrix} 0c_3 \\ 3c_3 \\ 8c_3 \end{bmatrix} \\
&= \begin{bmatrix} 5c_1 - 4c_2 + 0c_3 \\ -1c_1 + 9c_2 + 3c_3 \\ 2c_1 + 6c_2 + 8c_3 \end{bmatrix}
\end{aligned}$$

Note that the final result is a single vector with three components that look suspiciously like the left sides of a system of three equations in three unknowns!

In the previous two examples we found linear combinations algebraically - in the next example we find a linear combination geometrically.

◇ **Example 2.3(e):** In the space below and to the right, sketch the vector $2\vec{\mathbf{u}} - 3\vec{\mathbf{v}}$ for the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ shown below and to the left.

Solution: In the center below the linear combination is obtained by the tip-to-tail method, and to the right below it is obtained by the parallelogram method.



The final example is probably the most important in this section.

◇ **Example 2.3(f):** Find a linear combination of the vectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$ and $\vec{\mathbf{v}}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$ that equals the vector $\vec{\mathbf{w}} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$.

Solution: We are looking for two scalars c_1 and c_2 such that $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2 = \vec{\mathbf{w}}$. By the method of Example 2.3(d) we have

$$c_1\begin{bmatrix} 3 \\ -4 \end{bmatrix} + c_2\begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} 3c_1 \\ -4c_1 \end{bmatrix} + \begin{bmatrix} 7c_2 \\ -3c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

$$\begin{bmatrix} 3c_1 + 7c_2 \\ -4c_1 - 3c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

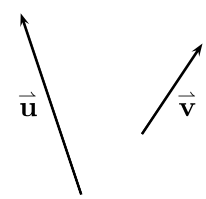
In the last line above we have two vectors that are equal. It should be intuitively obvious that this can only happen if the individual components of the two vectors are equal. This results in the system $\begin{array}{rcr} 3c_1 + 7c_2 & = & 1 \\ -4c_1 - 3c_2 & = & -14 \end{array}$ of two equations in the two unknowns c_1 and c_2. Solving, we arrive at $c_1 = 5,\ c_2 = -2$. It is easily verified that these are correct:

$$5\begin{bmatrix} 3 \\ -4 \end{bmatrix} - 2\begin{bmatrix} 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 15 \\ -20 \end{bmatrix} - \begin{bmatrix} 14 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

We now conclude with an important observation. Suppose that we consider all possible linear combinations of a single vector $\vec{\mathbf{v}}$. That is then the set of all vectors of the form $c\,\vec{\mathbf{v}}$ for some scalar c, which is just all scalar multiples of $\vec{\mathbf{v}}$. At the risk of being redundant, the set of all linear combinations of a single vector is all scalar multiples of that vector.

Section 2.3 Exercises To Solutions

1. Illustrate the tip-to-tail and parallelogram methods for finding $\vec{\mathbf{w}} = -\vec{\mathbf{u}} + 2\vec{\mathbf{v}}$ for the two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ shown to the right. Make it clear what portion of your diagram represents $\vec{\mathbf{w}}$ in each case.



2. For the vectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, $\vec{\mathbf{v}}_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$ and $\vec{\mathbf{v}}_4 = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$, give the linear combination $5\vec{\mathbf{v}}_1 + 2\vec{\mathbf{v}}_2 - 7\vec{\mathbf{v}}_3 + \vec{\mathbf{v}}_4$ as one vector.

3. For the vectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} -1 \\ 3 \\ -6 \end{bmatrix}$ and $\vec{\mathbf{v}}_2 = \begin{bmatrix} -8 \\ 1 \\ 4 \end{bmatrix}$, give the linear combination $c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2$ as one vector.

4. Give a linear combination of $\vec{\mathbf{u}} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$ that equals $\begin{bmatrix} 17 \\ -4 \\ -9 \end{bmatrix}$.
Demonstrate that your answer is correct by filling in the blanks:

$$\underline{\quad\quad}\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} + \underline{\quad\quad}\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} + \underline{\quad\quad}\begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} \underline{\quad\quad} \\ \underline{\quad\quad} \\ \underline{\quad\quad} \end{bmatrix} + \begin{bmatrix} \underline{\quad\quad} \\ \underline{\quad\quad} \\ \underline{\quad\quad} \end{bmatrix} + \begin{bmatrix} \underline{\quad\quad} \\ \underline{\quad\quad} \\ \underline{\quad\quad} \end{bmatrix} = \begin{bmatrix} 17 \\ -4 \\ -9 \end{bmatrix}$$

5. For each of the following, find a linear combination of the vectors $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$,...,$\vec{\mathbf{u}}_n$ that equals $\vec{\mathbf{v}}$. Conclude by giving the actual linear combination, not just some scalars.

(a) $\vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{\mathbf{u}}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(b) $\vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$

(c) $\vec{\mathbf{u}}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} -4 \\ 1 \\ 2 \end{bmatrix}$, $\vec{\mathbf{u}}_3 = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 6 \\ -18 \\ -7 \end{bmatrix}$

(d) $\vec{\mathbf{u}}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{\mathbf{u}}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$

(e) $\vec{\mathbf{u}}_1 = \begin{bmatrix} 7 \\ 1 \\ 3 \\ 0 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} -2 \\ 5 \\ 1 \\ -3 \end{bmatrix}$, $\vec{\mathbf{u}}_3 = \begin{bmatrix} 2 \\ 2 \\ -3 \\ 1 \end{bmatrix}$, $\vec{\mathbf{u}}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 19 \\ 10 \\ -12 \\ 12 \end{bmatrix}$

(f) $\vec{\mathbf{u}}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}$, $\vec{\mathbf{u}}_3 = \begin{bmatrix} 3 \\ 1 \\ -8 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$

(g) $\vec{\mathbf{u}}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}$, $\vec{\mathbf{u}}_3 = \begin{bmatrix} 3 \\ 1 \\ -8 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} -1 \\ 3 \\ -14 \end{bmatrix}$

6. (a) Consider the vectors $\vec{\mathbf{u}}_1 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{\mathbf{u}}_3 = \begin{bmatrix} -2 \\ 6 \\ 5 \end{bmatrix}$, $\vec{\mathbf{w}} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$.

If possible, find scalars a_1, a_2 and a_3 such that $a_1\vec{\mathbf{u}}_1 + a_2\vec{\mathbf{u}}_2 + a_3\vec{\mathbf{u}}_3 = \vec{\mathbf{w}}$.

(b) Consider the vectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{\mathbf{v}}_3 = \begin{bmatrix} -7 \\ 2 \\ 5 \end{bmatrix}$, $\vec{\mathbf{w}} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$.

If possible, find scalars b_1, b_2 and b_3 such that $b_1\vec{\mathbf{v}}_1 + b_2\vec{\mathbf{v}}_2 + b_3\vec{\mathbf{v}}_3 = \vec{\mathbf{w}}$.

(c) To do each of parts (a) and (b) you should have solved a system of equations. Let A be the coefficient matrix for the system in (a) and let B be the coefficient matrix for the system in part (b). Use your calculator to find $det(A)$ and $\det(B)$, the determinants of matrices A and B. You will probably find the command for the determinant in the same menu as *rref*.

2.4 Linear Combination Form of a System

Performance Criterion:

2. (f) Give the linear combination form of a system of equations, give the system of linear equations equivalent to a given vector equation.

   (g) Sketch a picture illustrating the linear combination form of a system of equations of two equations in two unknowns.

It should be clear that two vectors are equal if and only if their corresponding components are equal. That is,

$$\vec{\mathbf{u}} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \vec{\mathbf{v}} \qquad \text{if, and only if,} \qquad \begin{array}{c} u_1 = v_1, \\ u_2 = v_2, \\ \vdots \\ u_n = v_n \end{array}$$

The words "if, and only if" mean that the above works "both ways" in the following sense:

- If we have two vectors of length n that are equal, then their corresponding entries are all equal, resulting in n equations.
- If we have a set of n equations, we can create a two vectors, one of whose components are all the left hand sides of the equations and the other whose components are all the right hand sides of the equations, and the two vectors created this way are equal.

Using the second bullet above, we can take the system of equations below and to the left and turn them into the single vector equation shown below and to the right:

$$\begin{array}{rcr} x_1 + 3x_2 - 2x_3 & = & -4 \\ 3x_1 + 7x_2 + \ x_3 & = & 4 \\ -2x_1 + \ x_2 + 7x_3 & = & 7 \end{array} \qquad \Longrightarrow \qquad \begin{bmatrix} x_1 + 3x_2 - 2x_3 \\ 3x_1 + 7x_2 + \ x_3 \\ -2x_1 + \ x_2 + 7x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$$

We can take the vector on the left side of the equation and break it into three vectors to get

$$\begin{bmatrix} x_1 \\ 3x_1 \\ -2x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 7x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} -2x_3 \\ x_3 \\ 7x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$$

and then we can factor the scalar unknown out of each vector to get the *vector* equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} \qquad (1)$$

Our previous geometric interpretation of solving

$$\begin{array}{rcr} x_1 + 3x_2 - 2x_3 & = & -4 \\ 3x_1 + 7x_2 + \ x_3 & = & 4 \\ -2x_1 + \ x_2 + 7x_3 & = & 7 \end{array}$$

was that we were looking for the point (x_1, x_2, x_3) where the planes with equations $x_1+3x_2-2x_3 = -4$, $3x_1 + 7x_2 + \ x_3 = 4$ and $-2x_1 + \ x_2 + 7x_3 = 7$ intersect. (1) now gives us another interpretation - we are looking for the linear combination of the vectors

$$\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \text{ that equals } \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$$

The form (1) of a system of equations is quite important, so we give it a definition:

DEFINITION 2.4.1 **Linear Combination Form of a System**

A system

$$\begin{array}{rcl} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

of m linear equations in n unknowns can be written as a linear combination of vectors equalling another vector:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

We will refer to this as the **linear combination form of the system of equations**.

Thus the system of equations below and to the left can be rewritten in the linear combination form shown below and to the right.

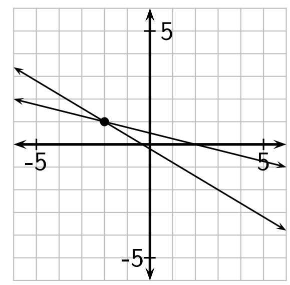
$$\begin{array}{rcr} x_1 + 3x_2 - 2x_3 & = & -4 \\ 3x_1 + 7x_2 + x_3 & = & 4 \\ -2x_1 + x_2 + 7x_3 & = & 7 \end{array} \qquad x_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$$

The question we originally asked for the system of linear equations was "Are there numbers x_1, x_2 and x_3 that make all three equations true?" Now we can see this is equivalent to a different question, "Is there a linear combination of the vectors $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$ that equals the vector $\begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix}$?"

◇ **Example 2.4(a):** Give the linear combination form of the system $\begin{array}{rcr} 3x_1 + 5x_2 & = & -1 \\ x_1 + 4x_2 & = & 2 \end{array}$ of linear equations.

Solution: The linear combination form of the system is $x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

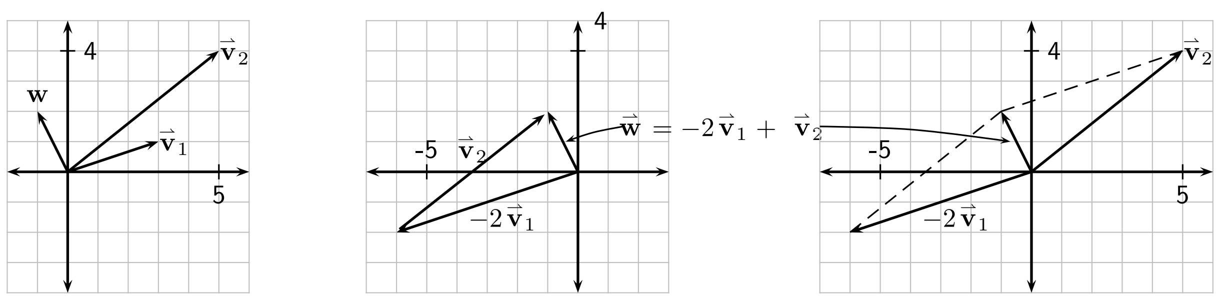
Let's consider the system from this last example a bit more. The goal is to solve the system of equations $\begin{array}{rcr} 3x_1 + 5x_2 & = & -1 \\ x_1 + 4x_2 & = & 2 \end{array}$. In the past our geometric interpretation has been this: The set of solutions to the first equation is a line in $\mathbb{R}^2$, and the set of solutions to the second equation is another line. The solution to the system happens to be $x_1 = -2,\ x_2 = 1$, and the point $(-2, 1)$ in $\mathbb{R}^2$ is the point where the two lines cross. This is shown in the picture to the right.



Now consider the linear combination form $x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ of the system. Let $\vec{\mathbf{v}}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. These vectors are shown in the diagram to the left at the top of the next page. The solution $x_1 = -2,\ x_2 = 1$ to the system is the scalars that we can use for a linear combination of the vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ to get the vector $\vec{\mathbf{w}}$. That is,

$$-2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

This is shown in the middle diagram below by the tip-to-tail method, and in the diagram below and to the right by the parallelogram method.



tip-to-tail method

parallelogram method

Section 2.4 Exercises

To Solutions

1. Give the linear combination form of each system:

(a)
$$\begin{aligned} x + y - 3z &= 1 \\ -3x + 2y - z &= 7 \\ 2x + y - 4z &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 5x_1 \qquad + x_3 &= -1 \\ 2x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 - 4x_3 &= 2 \end{aligned}$$

(c)
$$\begin{aligned} b + 0.5m &= 8.1 \\ b + 1.0m &= 6.9 \\ b + 1.5m &= 6.2 \\ b + 2.0m &= 5.3 \\ b + 2.5m &= 4.5 \\ b + 3.0m &= 3.8 \\ b + 3.5m &= 3.0 \end{aligned}$$

(d)
$$\begin{aligned} x_1 - 4x_2 + x_3 + 2x_4 &= -1 \\ 3x_1 + 2x_2 - x_3 - 7x_4 &= 0 \\ -2x_1 + x_2 - 4x_3 + x_4 &= 2 \end{aligned}$$

2. Give the system of equations that is equivalent to each vector equation.

(a) $x_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

(b) $x_1 \begin{bmatrix} 5 \\ 1 \\ -4 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 2 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 5 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ -4 \\ 6 \\ 7 \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ 5 \\ 4 \end{bmatrix}$

(c) $x_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -7 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ -4 \end{bmatrix}$

3. The system of equations $\begin{aligned} 2x - 3y &= -6 \\ 3x - y &= 5 \end{aligned}$ has solution $x = 3$, $y = 4$. Write the system in linear combination form, then replace x and y with their values. Finally, sketch a picture illustrating the resulting vector equation. See the explanation after Example 2.4(a) if you have no idea what I am talking about.

2.5 Sets of Vectors

Performance Criterion:

2. (h) Give an algebraic description of a set of a set of vectors that has been described geometrically, and vice-versa.

   (i) Determine whether a set of vectors is closed under vector addition; determine whether a set of vectors is closed under scalar multiplication. If it is, prove that it is; if it is not, give a counterexample.

One of the most fundamental concepts of mathematics is that of **sets**, collections of objects called **elements**. You have probably encountered various sets of numbers, like the **whole numbers**

$$\{0, 1, 2, 3, ...\}$$

and the integers

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}.$$

As shown above, when we describe a set by listing all or some of its elements, we enclose them with "curly brackets." Sets are usually named by upper case letters. The above two sets are **infinite sets**.

Another kind of infinite set is an interval of the number line, like all real numbers between 1 and 5, including 1 and 5. You have likely seen the interval notation $[1, 5]$ for such sets. This set is also infinite, but in a different sense than the whole numbers and integers. That difference is not of concern to us here, but some of you may encounter that idea again. Of course, there are also finite sets like $\{2, 4, 6, 8\}$.

There will be a time soon when we will be very interested in sets of vectors, both finite and infinite. For example we might be interested in the finite set

$$A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\},$$

or the infinite set of all vectors of the form $\begin{bmatrix} a \\ 2a \end{bmatrix}$, where a is any real number. Let's examine this set a bit more.

◇ **Example 2.5(a):** Let B be the set of all vectors of the form $\begin{bmatrix} a \\ 2a \end{bmatrix}$, where a is any real number. Are the vectors $\vec{\mathbf{u}} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$ in B?

Solution: Because $2(3) \neq 10$, $\vec{\mathbf{u}}$ is not in B. But $2(-2) = -4$, so $\vec{\mathbf{v}}$ is in B.

◇ **Example 2.5(b):** Let C be the set of all vectors of the form $\begin{bmatrix} a \\ a+1 \end{bmatrix}$, where a is any real number. Are the vectors $\vec{\mathbf{u}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ in C?

Solution: $3 + 1 = 4$ and $-2 + 1 = -1$, so both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are in C.

In the future we will also be very interested in this question: Given an infinite set of vectors, is the sum of any two vectors a vector that is also in the set? When faced with such a question we should do one of two things:

- Give two *specific* vectors that are in the set, and show that their sum is not.
- Give two *arbitray* (general) vectors in the set and show that their sum is also in the set.

The following examples illustrate these.

◇ **Example 2.5(c):** Let B be the set of all vectors of the form $\begin{bmatrix} a \\ 2a \end{bmatrix}$, where a is any real number. Determine whether the sum of any two vectors in B is also in B.

Solution: The vectors $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -4 \end{bmatrix}$ are both in B, and their sum $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is as well. We may have just gotten lucky, though, and maybe the sum of *any* two vectors in B is not necessarily in B. Let's see if the sum of two *arbitrary* vectors in B is in B. For any numbers a and b, the vectors $\begin{bmatrix} a \\ 2a \end{bmatrix}$ and $\begin{bmatrix} b \\ 2b \end{bmatrix}$ are in B. We then compute their sum to get

$$\begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} b \\ 2b \end{bmatrix} = \begin{bmatrix} a+b \\ 2a+2b \end{bmatrix} = \begin{bmatrix} a+b \\ 2(a+b) \end{bmatrix} = \begin{bmatrix} c \\ 2c \end{bmatrix},$$

where $c = a + b$. Therefore the sum of *any* two vectors in B is a vector in B.

◇ **Example 2.5(d):** Let C be the set of all vectors of the form $\begin{bmatrix} a \\ a+1 \end{bmatrix}$, where a is any real number. Is the sum of any two vectors in C also a vector in C?

Solution: The vectors $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ are both in C, as shown in Example 2.5(b). Their sum is the vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, which is not in C because $1 + 1 \neq 3$. Thus the sum of any two vectors in C is not necessarily another vector in C.

◇ **Example 2.5(e):** Let D be the set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x \geq 0$ and $y \geq 0$. Is the sum of any two vectors in D also a vector in D?

Solution: Suppose that $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ are both in D, so all of x_1, y_1, x_2, y_2 are greater than or equal to zero. Clearly $x_1 + x_2 \geq 0$ and $y_1 + y_2 \geq 0$, so their sum $\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$ is in D.

Given a set of vectors, we are also interested in whether a scalar multiple of a vector in the set is in the set as well. In the next example we determine whether that is the case for the set D from the previous example.

◇ **Example 2.5(f):** Let D be the set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x \geq 0$ and $y \geq 0$. Is *any* scalar times any vector in D also a vector in D?

Solution: Suppose that $\begin{bmatrix} x \\ y \end{bmatrix}$ is in D, so both $x \geq 0$ and $y \geq 0$. Is it possible that for some scalar a, the vector $a\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ ay \end{bmatrix}$ is *NOT* in D? That would only be the case if either $ax < 0$ or $ay < 0$, which would happen if a was negative and at least one of x or y was positive. To give a specific example, for the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $a = -3$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in D but $(-3)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \end{bmatrix}$ is not.

It is very important that we note the difference between the approaches of Examples 2.5(e) and (f). When trying to show that something is true in general about a set, we must demonstrate it for arbitrary elements in the set, as done in Example 2.5(e). When trying to show something is not true all that is needed is one specific example that fails to be true, which is called a **counterexample**. The vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and scalar -3 are a specific counterexample for Example 2.5(f). We *ALWAYS* use *specific* counterexamples to show that something is not true, but a general example to show that something *IS* true.

Section 2.5 Exercises To Solutions

Do each of the following for each of Exercises 1 - 10.

(a) Give several vectors in the set.

(b) Determine whether the set is closed under addition.

(c) Determine whether the set is closed under scalar multiplication.

1. $\mathcal{S} = \left\{ \begin{bmatrix} a \\ a^2 \end{bmatrix} \in \mathbb{R}^2 \right\}$

2. $\mathcal{S} = \left\{ \begin{bmatrix} a \\ b \\ a+b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$

3. $\mathcal{S} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid xy \geq 0 \right\}$

4. $\mathcal{S} = \left\{ t \begin{bmatrix} 1 \\ -2 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$

5. $\mathcal{S} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$

6. $\mathcal{S} = \left\{ s \begin{bmatrix} 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}$

7. $\mathcal{S} = \left\{ \begin{bmatrix} -4 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$

8. $\mathcal{S} = \left\{ \begin{bmatrix} a \\ b \\ |a| \end{bmatrix} \in \mathbb{R}^3 \right\}$

9. $\mathcal{S} = \left\{ \begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}$

10. $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \,\middle|\, s, t \in \mathbb{R} \right\}$

11. On separate graphs, plot the points corresponding to each of the sets of vectors (taken as position vectors) in Exercises 1 - 4 above.

12. (a) The sets in Exercises 5 and 7 look very similar, but one is closed under both addition and scalar multiplication, and the other is not. What do you notice about the vectors used to describe the set that *IS* closed under both addition and scalar multiplication?

    (b) Graph the sets from Exercises 5 and 7 on separate graphs. How are the two graphs alike? How are they different?

13. Determine what the set described in Exercise 10 is, geometrically.

14. You might find it surprising that the vectors $\begin{bmatrix} 17 \\ -22 \end{bmatrix}$ and $\begin{bmatrix} -10 \\ -10 \end{bmatrix}$ are in the set $\mathcal{S}$ described in Exercise 6 above. For each of those two vectors, determine the values of s and t that give them, to the nearest hundredth.

15. Determine what the set described in Exercise 6 is, geometrically.

2.6 Vector Equations of Lines and Planes

Performance Criterion:

2. (j) Give the vector equation of a line through two points in $\mathbb{R}^2$ or $\mathbb{R}^3$ or the vector equation of a plane through three points in $\mathbb{R}^3$.

The idea of a linear combination does more for us than just give another way to interpret a system of equations. The set of points in $\mathbb{R}^2$ satisfying an equation of the form $y = mx + b$ is a line; any such equation can be rearranged into the form $ax + by = c$. (The values of b in the two equations are clearly not the same.) But if we add one more term to get $ax + by + cz = d$, with the (x, y, z) representing the coordinates of a point in $\mathbb{R}^3$, *we get the equation of a plane, not a line!* In fact, we cannot represent a line in $\mathbb{R}^3$ with a single *scalar* equation. The object of this section is to show how we can represent lines, planes and higher dimensional objects called **hyperplanes** using linear combinations of vectors.

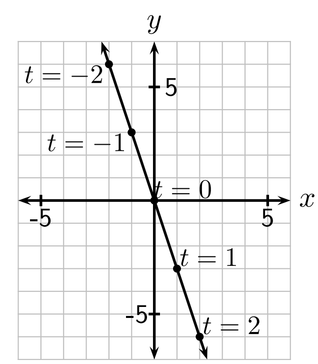
For the bulk of this course, we will think of most vectors as position vectors. (Remember, this means their tails are at the origin.) We will also think of each position vector as corresponding to the point at its tip, so the coordinates of the point will be the same as the components of the vector. Thus, for example, in $\mathbb{R}^2$ the vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ corresponds to the ordered pair $(x_1, x_2) = (1, -3)$.

◇ **Example 2.6(a):** Graph the set of points corresponding to all vectors $\vec{\mathbf{x}}$ of the form $\vec{\mathbf{x}} = t\begin{bmatrix} 1 \\ -3 \end{bmatrix}$, where t represents any real number.

Solution: We already know that when $t = 1$ the the vector x corresponds to the point $(1, -3)$. We then let $t = -2, -1, 0, 2$ and determine the corresponding vectors $\vec{\mathbf{x}}$:

$$t = -2 \;\Rightarrow\; x = \begin{bmatrix} -2 \\ 6 \end{bmatrix}, \qquad t = -1 \;\Rightarrow\; x = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$t = 0 \;\Rightarrow\; x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad t = 2 \;\Rightarrow\; x = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$



These vectors correspond to the points with ordered pairs $(-2, 6)$, $(-1, 3)$, $(0, 0)$ and $(2, -6)$, which lie on a line through the origin. If we were to continue plotting more such points for all possible values of t we get the line shown above and to the right.

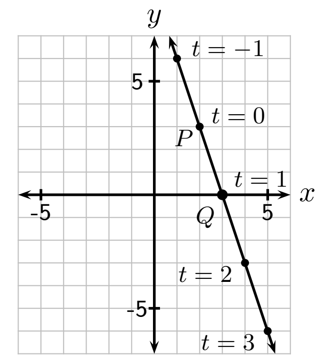
It should be clear from the above example that we could create a line through the origin in any direction by simply replacing the vector $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ with a vector in the direction of the desired line. The next example illustrates how we get a line not through the origin using vectors.

◇ **Example 2.6(b):** Graph the set of points corresponding to all vectors $\vec{\mathbf{x}}$ of the form $\vec{\mathbf{x}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, where t represents any real number.

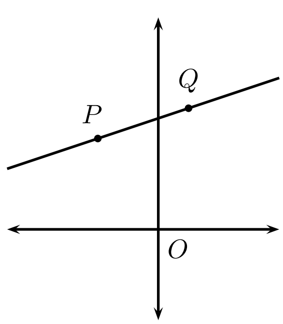
Solution: Performing the scalar multiplication by t and adding the two vectors, we get

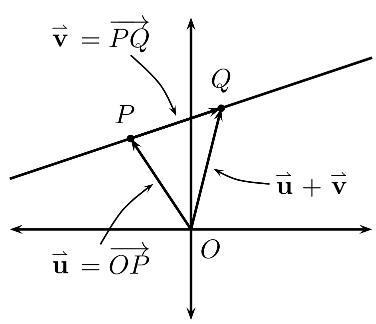
$$\vec{\mathbf{x}} = \begin{bmatrix} 2-3t \\ 3+\ t \end{bmatrix}.$$

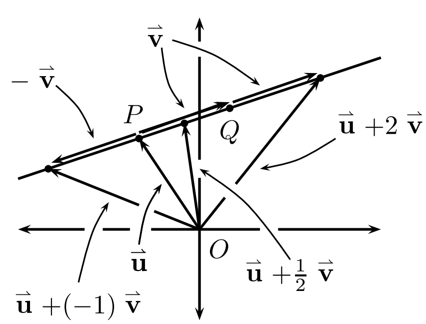
These vectors then correspond to all points of the form $(2-3t, 3+t)$. When $t=0$ this is the point $(2,3)$ so our line clearly passes through that point. Plotting the points obtained when we let $t=-1,\ 1,\ 2$ and 3, we see that we will get the line shown to the right.



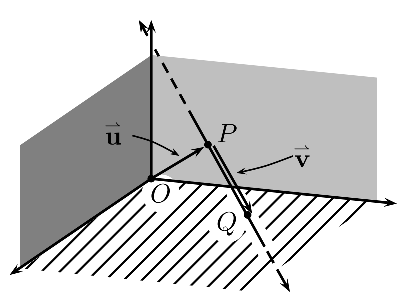
Now let's make two observations about the set of points represented by the set of all vectors $\vec{\mathbf{x}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, where t again represents any real number. These vectors correspond to the ordered pairs of the form $(2-3t, 3+t)$. Plotting these results in the line through the point $(2,3)$ and in the direction of the vector $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. This is not a coincidence. Consider the line shown below and to the left, containing the points P and Q. If we let $\vec{\mathbf{u}} = \overrightarrow{OP}$ and $\vec{\mathbf{v}} = \overrightarrow{PQ}$, then the points P and Q correspond to the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{u}} + \vec{\mathbf{v}}$ (in standard position, which you should assume we mean from here on), as shown in the second picture. From this you should be able to see that if we consider all the vectors $\vec{\mathbf{x}}$ defined by $\vec{\mathbf{x}} = \vec{\mathbf{u}} + t\,\vec{\mathbf{v}}$ as t ranges over all real numbers, the resulting set of points is our line! This is shown in the third picture, where t is given the values $-1,\ 0,\ \frac{1}{2}$ and 2.







This may seem like an overly complicated way to describe a line, but with a little thought you should see that the idea translates directly to three (and more!) dimensions, as shown in the picture to the right. This is all summarized by the following.



Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$

The **vector equation of a line** through two points P and Q in $\mathbb{R}^2$ and $\mathbb{R}^3$ (and even higher dimensions) is

$$\vec{\mathbf{x}} = \overrightarrow{OP} + t\,\overrightarrow{PQ}.$$

By this we mean that the line consists of all the points corresponding to the position vectors $\vec{\mathbf{x}}$ as t varies over all real numbers. The vector $\overrightarrow{PQ}$ is called the **direction vector** of the line.

◇ **Example 2.6(c):** Give the vector equation of the line in $\mathbb{R}^2$ through the points $P(-4,1)$ and $Q(5,3)$.

Solution: We need two vectors, one from the origin out to the line, and one in the direction of the line. For the first we will use $\overrightarrow{OP}$, and for the second we will use $\overrightarrow{PQ} = [9,2]$. We then have

$$\vec{\mathbf{x}} = \overrightarrow{OP} + t\overrightarrow{PQ} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} + t\begin{bmatrix} 9 \\ 2 \end{bmatrix},$$

where $\vec{\mathbf{x}} = [x_1, x_2]$ is the position vector corresponding to any point (x_1, x_2) on the line.

◇ **Example 2.6(d):** Give a vector equation of the line in $\mathbb{R}^3$ through the points $(-5,1,2)$ and $(4,6,-3)$.

Solution: Letting P be the point $(-5,1,2)$ and Q be the point $(4,6,-3)$, we get $\overrightarrow{PQ} = \langle 9,5,-5\rangle$. The vector equation of the line is then

$$\vec{\mathbf{x}} = \overrightarrow{OP} + t\overrightarrow{PQ} = \begin{bmatrix} -5 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 9 \\ 5 \\ -5 \end{bmatrix},$$

where $\vec{\mathbf{x}} = [x_1, x_2, x_3]$ is the position vector corresponding to any point (x_1, x_2, x_3) on the line.

The vector equation of a line is not unique! The first vector can be any point on the line, so it could be the vector $\overrightarrow{OQ} = [4,6,-3]$ instead of $[-5,1,2]$, for example. The second vector is simply a direction vector, so can be any scalar multiple of $\overrightarrow{PQ} = [9,5,-5]$, including $\overrightarrow{QP} = [-9,-5,5]$.

The same general idea can be used to describe a plane in $\mathbb{R}^3$. Before seeing how that works, let's define something and look at a situation in $\mathbb{R}^2$. We say that two vectors are **parallel** if one is a scalar multiple of the other. Now suppose that $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ are two nonzero vectors in $\mathbb{R}^2$ that are *not* parallel, as shown in Figure 2.6(a) on the next page, and let P be the randomly chosen point in $\mathbb{R}^2$ shown in the same picture. Figure 2.6(b) shows that a linear combination of $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ can be formed that gives us a vector $s\vec{\mathbf{v}} + t\vec{\mathbf{w}}$ corresponding to the point P. In this case the scalar s is positive and less than one, and t is positive and greater than one. Figures 2.6(c) and 2.6(d) show the same thing for another point Q, with s being negative and t positive in that case. It should now be clear that *any* point in $\mathbb{R}^2$ can be obtained in this manner.

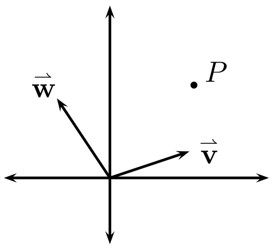


Figure 2.6(a)

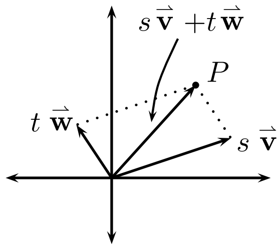


Figure 2.6(b)

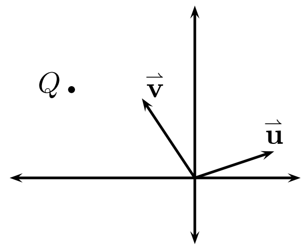


Figure 2.6(c)

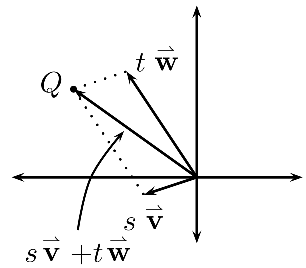
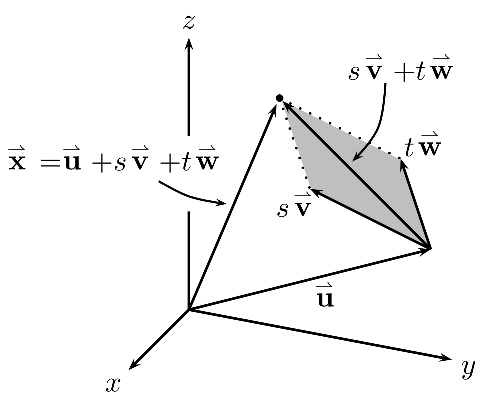
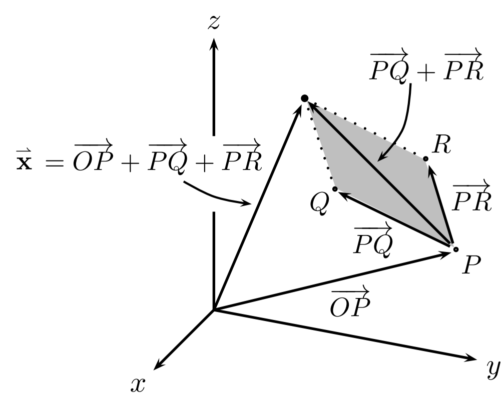


Figure 2.6(d)

Now let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ be three vectors in $\mathbb{R}^3$, and consider the vector $\vec{\mathbf{x}} = \vec{\mathbf{u}} + s\vec{\mathbf{v}} + t\vec{\mathbf{w}}$, where s and t are scalars that are allowed to take all real numbers as values. The vectors $s\vec{\mathbf{v}} + t\vec{\mathbf{w}}$ all lie in the plane containing $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$. Adding $\vec{\mathbf{u}}$ "moves the plane off the origin" to where it passes through the tip of $\vec{\mathbf{u}}$ (again, in standard position). This is probably best visualized by thinking of adding $s\,\vec{\mathbf{v}}$ and $t\,\vec{\mathbf{w}}$ with the parallelogram method, then adding the result to $\vec{\mathbf{u}}$ with the tip-to-tail method. I have attempted to illustrate this to the left at the top of the next page, with the gray parallelogram being part of the plane created by all the points corresponding to the vectors $\vec{\mathbf{x}}$.





The same diagram above and to the right shows how all of the previous discussion relates to the plane through three points P, Q and R in $\mathbb{R}^3$. This leads us to the description of a plane in $\mathbb{R}^3$ given at the top of the next page.

Planes in $\mathbb{R}^3$

The **vector equation of a plane** through three points P, Q and R in $\mathbb{R}^3$ (or higher dimensions) is

$$\vec{\mathbf{x}} = \overrightarrow{OP} + s\,\overrightarrow{PQ} + t\,\overrightarrow{PR}.$$

By this we mean that the plane consists of all the points corresponding to the position vectors $\vec{\mathbf{x}}$ as s and t vary over all real numbers.

◇ **Example 2.6(e):** Give a vector equation of the plane in $\mathbb{R}^3$ through the points $(2,-1,3)$, $(-5,1,2)$ and $(4,6,-3)$. What values of s and t give the point R?

Solution: Letting P be the point $(2,-1,3)$, Q be $(-5,1,2)$ and R be $(4,6,-3)$, we get

$\overrightarrow{PQ} = [-7, 2, -1]$ and $\overrightarrow{PR} = [2, 7, -6]$. The vector equation of the plane is then

$$\vec{\mathbf{x}} = \overrightarrow{OP} + s\overrightarrow{PQ} + t\overrightarrow{PR} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + s \begin{bmatrix} -7 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 7 \\ -6 \end{bmatrix},$$

where $\vec{\mathbf{x}} = [x_1, x_2, x_3]$ is the position vector corresponding to any point (x_1, x_2, x_3) on the plane. It should be clear that there are other possibilities for this. The first vector in the equation could be any of the three position vectors for P, Q or R. The other two vectors could be any two vectors from one of the points to another.

The vector corresponding to point R is $\overrightarrow{OR}$, which is equal to $\vec{\mathbf{x}} = \overrightarrow{OP} + \overrightarrow{PR}$ (think about that), so $s = 0$ and $t = 1$.

We now summarize all of the ideas from this section.

Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$, Planes in $\mathbb{R}^3$

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ with $\vec{\mathbf{v}} \neq \mathbf{0}$. Then the set of points corresponding to the vector $\vec{\mathbf{x}} = \vec{\mathbf{u}} + t\,\vec{\mathbf{v}}$ as t ranges over all real numbers is a line through the point corresponding to $\vec{\mathbf{u}}$ and in the direction of $\vec{\mathbf{v}}$. (So if $\vec{\mathbf{u}} = \vec{\mathbf{0}}$ the line passes through the origin.)

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ be vectors $\mathbb{R}^3$, with $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$ being nonzero and not parallel. (That is, not scalar multiples of each other.) Then the set of points corresponding to the vector $\vec{\mathbf{x}} = \vec{\mathbf{u}} + s\,\vec{\mathbf{v}} + t\,\vec{\mathbf{w}}$ as s and t range over all real numbers is a plane through the point corresponding to $\vec{\mathbf{u}}$ and containing the vectors $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$. (If $\vec{\mathbf{u}} = \vec{\mathbf{0}}$ the plane passes through the origin.)

Section 2.6 Exercises To Solutions

1. For each of the following, give the vector equation of the line or plane described.

   (a) The line in $\mathbb{R}^2$ through the points $P(3, -1)$ and $Q(6, 0)$.

   (b) The plane in $\mathbb{R}^3$ through the points $P(3, -1, 4)$, $Q(2, 6, 0)$ and $R(-1, 0, 3)$.

   (c) The line in $\mathbb{R}^3$ through the two points $P(3, -1, 4)$ and $Q(2, 6, 0)$.

   (d) The line in $\mathbb{R}^2$ through the points $(-1, 4)$ and $(2, 5)$.

   (e) The line in $\mathbb{R}^2$ through $(-4, 3)$ and the origin. (**Hint:** Use the method of Example 2.6(c), taking P to be $(0, 0)$ and Q to be $(-4, 3)$.)

   (f) The plane in $\mathbb{R}^3$ through $(-5, 1, 3)$, $(2, 0, 4)$ and $(1, -2, 3)$.

   (g) The line in $\mathbb{R}^3$ through $(2, 0, -1)$ and $(1, 1, 3)$.

   (h) The plane in $\mathbb{R}^3$ through $(-4, 5, 1)$, $(2, 2, 2)$ and the origin.

   (i) The line in $\mathbb{R}^3$ through $(1, 2, 3)$ and the origin.

2. Each of the lines or planes in the previous exercise constitutes a set of points in $\mathbb{R}^2$ or $\mathbb{R}^3$. Which are sets closed under addition and scalar multiplication?

3. Give the equation of the three-dimensional hyperplane in $\mathbb{R}^4$ containing the points $(-2,1,1,5)$, $(3,7,-5,2)$, $(4,0,5,-2)$ and $(3,-5,4,7)$.

4. Consider the two points $P(5,-1,4)$ and $Q(1,1,2)$ in $\mathbb{R}^3$.

   (a) Give the vector equation of the line for which the value $t=0$ for the parameter gives the point Q and $t=1$ gives the point P.

   (b) Give the vector equation of the line for which the value $t=0$ for the parameter gives the point Q and $t=-1$ gives the point P.

   (c) Give the vector equation of the line for which the value $t=0$ for the parameter gives the point P and $t=2$ gives the point Q.

5. Consider the two points $P(-3,1)$ and $Q(2,5)$ in $\mathbb{R}^2$.

   (a) Give the "next" point with integer coordinates as one goes from P to Q.

   (b) Give the vector equation for the line for which the parameter value $t=0$ gives the point P and $t=1$ gives Q.

   (c) What value of t in your equation from (b) gives the point you found in part (a)?

   (d) Give the "next" point with integer coordinates as one goes from Q to P. What value of the parameter t in your equation from (b) gives this point?

6. Consider the three points $P(1,6,-2)$, $Q(-5,7,4)$ and $R(3,0,-1)$ in $\mathbb{R}^3$. Give the vector equation of plane in the form $\vec{\mathbf{x}} = \vec{\mathbf{u}} + s\vec{\mathbf{v}} + t\vec{\mathbf{w}}$ for which $s=0$ and $t=0$ gives the point Q, $s=0$ and $t=1$ gives R, and $s=1, t=0$ gives P.

7. Give the "next" three points on the line in $\mathbb{R}^3$ containing $P(5,-1,4)$ and $Q(1,1,2)$, traveling in the direction from P to Q.

8. Find another point in the plane containing $P_1(-2,1,5)$, $P_2(3,2,1)$ and $P_3(4,-2,-3)$. **Show clearly how you do it**. (**Hint:** Find and use the vector equation of the plane.)

9. "Usually" a vector equation of the form $\vec{\mathbf{x}} = \vec{\mathbf{p}} + s\,\vec{\mathbf{u}} + t\,\vec{\mathbf{v}}$ gives the equation of a plane in $\mathbf{R}^3$. Answer the following first allowing any of $\vec{\mathbf{p}}$, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ to be the zero vector, then give answers assuming that none of them are zero.

   (a) Under what conditions on $\vec{\mathbf{p}}$ and/or $\vec{\mathbf{u}}$ and/or $\vec{\mathbf{v}}$ would this be the equation of a line?

   (b) Under what conditions on $\vec{\mathbf{p}}$ and/or $\vec{\mathbf{u}}$ and/or $\vec{\mathbf{v}}$ would this be the equation of a plane through the origin?

2.7 Interpreting Solutions to Systems of Linear Equations

Performance Criterion:

2. (k) Write the solution to a system of equations in vector form and determine the geometric nature of the solution.

We begin this section by considering the following two systems of equations.

$$\begin{aligned} 3x_1 - 3x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= -3 \end{aligned} \qquad\qquad \begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ -3x_1 + 3x_2 - 6x_3 &= -3 \\ 2x_1 - 2x_2 + 4x_3 &= 2 \end{aligned}$$

The augmented matrices for these two systems reduce to the following matrices, respectively.

$$\begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let's look at the first system. x_3 is a free variable, and x_1 and x_2 are leading variables. The general solution is $x_1 = -3t+4$, $x_2 = -2t+1$, $x_3 = t$. Algebraically, x_1, x_2 and x_3 are just numbers, but we can think of (x_1, x_2, x_3) as a point in $\mathbb{R}^3$. The corresponding position vector is

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4-3t \\ 1-2t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3t \\ -2t \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

We will call this the **vector form of the solution** to the system of equations. The beauty of expressing the solutions to a system of equations in vector form is that we can see what the set of all solutions looks like. In this case, the set of solutions is the set of all points in $\mathbb{R}^3$ on the line through $(4,1,0)$ and with direction vector $[-3,-2,1]$.

◇ **Example 2.7(a):** The general solution to the second system of equations is $x_1 = 1+s-2t$, $x_2 = s$, $x_3 = t$. Express the solution in vector form and determine the geometric nature of the solution set in $\mathbb{R}^3$.

Solution: A process like the one just carried out leads to the general solution with position vector

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(Check to make sure that you understand how this was arrived at.) Here the set of solutions is the set of all points in $\mathbb{R}^3$ on the plane through $(1,0,0)$ with direction vectors $[1,1,0]$ and $[-2,0,1]$.

Now recall that the three equations from this last example,

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ -3x_1 + 3x_2 - 6x_3 &= -3 \\ 2x_1 - 2x_2 + 4x_3 &= 2 \end{aligned}$$

represent three planes in $\mathbb{R}^3$, and when we solve the system we are looking for all points in $\mathbb{R}^3$ that are solutions to all three equations. Our results tell us that the set of solution points in this case is itself a plane, which can only happen if all three equations represent the same plane. If you look at them carefully you can see that the second and third equations are multiples of the first, so the points satisfying them also satisfy the first equation.

◇ **Example 2.7(b):** The general solution to the second system of equations is $x_1 = 1 + s - 2t$, $x_2 = s$, $x_3 = t$. Express the solution in vector form and determine the geometric nature of the solution set in $\mathbb{R}^3$.

Solution: A process like the one just carried out leads to the general solution with position vector

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(Check to make sure that you understand how this was arrived at.) Here the set of solutions is the set of all points in $\mathbb{R}^3$ on the plane through $(1,0,0)$ with direction vectors $[1,1,0]$ and $[-2,0,1]$.

◇ **Example 2.7(c):** Give the vector form of the solution to the system

$$\begin{aligned} 3x_2 - 6x_3 - 4x_4 - 3x_5 &= -5 \\ x_1 - 3x_2 + 10x_3 + 4x_4 + 4x_5 &= 2 \\ 2x_1 - 6x_2 + 20x_3 + 2x_4 + 8x_5 &= -8 \end{aligned}$$

Solution: The augmented matrix of the system reduces to

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 1 & -3 \\ 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$

We can see that x_3 and x_5 are free variables, and we can also see that $x_4 = 2$. Letting $x_5 = t$ and $x_3 = s$, $x_2 = 1 + 2s + t$ and $x_1 = -3 - 4s - t$. Therefore

$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

How do we interpret this result geometrically? The set of points $(x_1, x_2, x_3, x_4, x_5)$ represents a two-dimensional plane in five-dimensional space. We could also have ended up with one, three or four dimensional "plane", often called a **hyperplane**, in five-dimensional space.

Section 2.7 Exercises To Solutions

1. For each of the following, a student correctly finds the given the general solution (x_1, x_2, x_3) to a system of three equations in three unknowns. Give the vector form of the solution, then tell whether the set of all particular solutions is a point, line or plane.

   (a) $x_1 = s - t + 5, \quad x_2 = s, \quad x_3 = t$

   (b) $x_1 = 2t + 5, \quad x_2 = t, \quad x_3 = -1$

   (c) $x_1 = s - 2t + 5, \quad x_2 = s, \quad x_3 = t$

2. In each of the following, the vector form of the solution to a system of linear equations is given. Give the dimension of the solution, and the dimension of the space it is in. For example you might answer "three-dimensional plane in five-dimensional space."

   (a)
$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -1 \\ 4 \end{bmatrix} + r\begin{bmatrix} 3 \\ 7 \\ 1 \\ -4 \end{bmatrix} + s\begin{bmatrix} 6 \\ 3 \\ -10 \\ 8 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

   (b)
$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

   (c)
$$\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + t_1\begin{bmatrix} 5 \\ -3 \\ -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 1 \\ 5 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t_3\begin{bmatrix} 2 \\ -5 \\ 1 \\ 0 \\ 3 \\ 9 \\ 3 \end{bmatrix} + t_4\begin{bmatrix} 5 \\ 7 \\ -4 \\ -2 \\ 1 \\ -4 \\ 5 \end{bmatrix} + t_5\begin{bmatrix} 3 \\ 1 \\ -1 \\ 6 \\ 10 \\ -4 \\ 1 \end{bmatrix}$$

   (d)
$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 3 \\ 6 \\ -1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ -8 \\ -1 \\ 4 \end{bmatrix}$$

(e)
$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ 7 \\ 1 \\ -4 \end{bmatrix} + s \begin{bmatrix} 6 \\ 3 \\ -10 \\ 8 \end{bmatrix}$$

(f)
$$\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 5 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ -5 \\ 1 \\ 0 \\ 3 \\ 9 \\ 3 \end{bmatrix} + t_3 \begin{bmatrix} 5 \\ 7 \\ -4 \\ -2 \\ 1 \\ -4 \\ 5 \end{bmatrix} + t_4 \begin{bmatrix} 3 \\ 1 \\ -1 \\ 6 \\ 10 \\ -4 \\ 1 \end{bmatrix}$$

2.8 The Dot Product of Vectors, Projections

Performance Criteria:

3. (g) Find the dot product of two vectors, determine the length of a single vector.
   (h) Determine whether two vectors are orthogonal (perpendicular).
   (i) Find the projection of one vector onto another, graphically or algebraically.

The Dot Product and Orthogonality

There are two ways to "multiply" vectors, both of which you have likely seen before. One is called the **cross product**, and only applies to vectors in $\mathbb{R}^3$. It is quite useful and meaningful in certain physical situations, but it will be of no use to us here. More useful is the other method, called the **dot product**, which is valid in all dimensions.

DEFINITION 2.8.1: **Dot Product**

Let $\vec{\mathbf{u}}= [u_1, u_2, ..., u_n]$ and $\vec{\mathbf{v}}= [v_1, v_2, ..., v_n]$. The **dot product** of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, denoted by $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$, is given by

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}= u_1v_1 + u_2v_2 + u_3v_3 + \cdots + u_nv_n$$

The dot product is useful for a variety of things. Recall that the length of a vector $\vec{\mathbf{v}}= [v_1, v_2, ..., v_n]$ is given by $\| \vec{\mathbf{v}} \| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}$. Note also that $v_1^2 + v_2^2 + \cdots + v_n^2 =\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$, which implies that $\| \vec{\mathbf{v}} \| = \sqrt{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}$. Perhaps the most important thing about the dot product is that the dot product of two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ is zero if, and only if, the two vectors are perpendicular. In general, we make the following definition.

DEFINITION 2.8.2: **Orthogonal Vectors**

Two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in $\mathbb{R}^n$ are said to be **orthogonal** if, and only if, $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}= 0$.

◇ **Example 2.8(a):** For the three vectors $\mathbf{u} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$,

find $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$, $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$ and $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$. Are any of the vectors orthogonal to each other?

Solution: We find that

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}= (5)(-1) + (-1)(3) + (2)(4) = -5 + (-3) + 8 = 0,$$

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} = (5)(2) + (-1)(-1) + (2)(-3) = 10 + 1 + (-6) = 5,$$
$$\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = (-1)(2) + (3)(-1) + (4)(-3) = -2 + (-3) + (-12) = -17$$

From the first computation we can see that $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal.

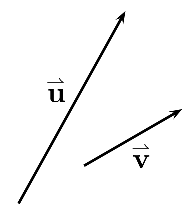
Projections

Given two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, we can create a new vector $\vec{\mathbf{w}}$ called the **projection of $\vec{\mathbf{u}}$ onto $\vec{\mathbf{v}}$**, denoted by $\text{proj}_{\mathbf{v}}\,\vec{\mathbf{u}}$. This is a very useful idea, in many ways. Geometrically, we can find $\text{proj}_{\mathbf{v}}\,\vec{\mathbf{u}}$ as follows:

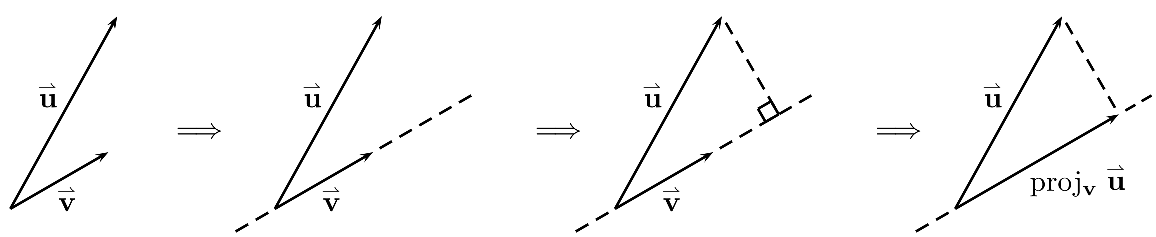
- Bring $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ together tail-to-tail.
- Sketch in the line containing $\vec{\mathbf{v}}$, as a dashed line.
- Sketch in a dashed line segment from the tip of $\vec{\mathbf{u}}$ to the dashed line containing $\vec{\mathbf{v}}$, *perpendicular to that line.*
- Draw the vector $\text{proj}_{\mathbf{v}}\,\vec{\mathbf{u}}$ from the point at the tails of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ to the point where the dashed line segment meets $\vec{\mathbf{v}}$ or the dashed line containing $\vec{\mathbf{v}}$.

Note that $\text{proj}_{\mathbf{v}}\,\vec{\mathbf{u}}$ is parallel to $\vec{\mathbf{v}}$; if we were to find $\text{proj}_{\mathbf{u}}\,\vec{\mathbf{v}}$ instead, the result would be parallel to $\vec{\mathbf{u}}$ in that case. The above steps are illustrated in the following example.

◇ **Example 2.8(b):** For the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ shown to the right, find the projection $\text{proj}_{\mathbf{v}}\,\vec{\mathbf{u}}$.

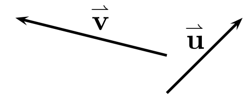


Solution: Following the above steps we get

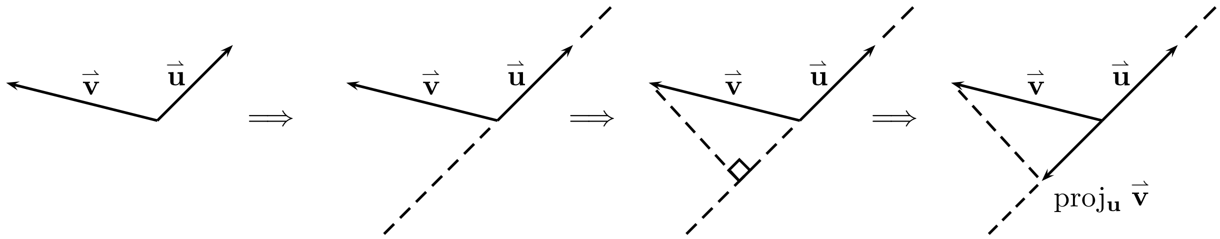


Projections are a bit less intuitive when the angle between the two vectors is obtuse, as seen in the next example.

◇ **Example 2.8(c):** For the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ shown to the right, find the projection $\mathrm{proj}_{\mathbf{u}}\ \vec{\mathbf{v}}$.



Solution: We follow the steps again, noting that this time we are projecting $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$:



Here we see that $\mathrm{proj}_{\mathrm{u}}\ \vec{\mathbf{v}}$ is in the direction opposite $\vec{\mathbf{u}}$.

We will also want to know how to find projections algebraically:

> Definition 2.8.3: **The Projection of One Vector on Another**
>
> For two vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, the vector $\mathrm{proj}_{\mathrm{v}}\ \vec{\mathbf{u}}$ is given by
>
> $$\mathrm{proj}_{\mathrm{v}}\ \vec{\mathbf{u}} = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}\ \vec{\mathbf{v}}$$

Note that since both $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$ and $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$ are scalars, so is $\dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}}$. That scalar is then multiplied times $\vec{\mathbf{v}}$, resulting in a vector parallel $\vec{\mathbf{v}}$. if the scalar is positive the projection is in the direction of $\vec{\mathbf{v}}$, as shown in Example 2.8(b); when the scalar is negative the projection is in the direction opposite the vector being projected onto, as shown in Example 2.8(c).

◇ **Example 2.8(d):** For the vectors $\mathbf{u} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$, find $\mathrm{proj}_{\mathbf{u}}\ \vec{\mathbf{v}}$.

Note that here we are projecting $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$. first we find

$$\vec{\mathbf{v}} \cdot \vec{\mathbf{u}} = (2)(5) + (-1)(-1) + (-3)(2) = 5 \quad \text{and} \quad \vec{\mathbf{u}} \cdot \vec{\mathbf{u}} = 5^2 + (-1)^2 + 2^2 = 30$$

Then

$$\mathrm{proj}_{\mathbf{u}}\ \vec{\mathbf{v}} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{u}}}{\vec{\mathbf{u}} \cdot \vec{\mathbf{u}}}\ \vec{\mathbf{u}} = \frac{5}{30}\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ -\frac{1}{6} \\ \frac{1}{3} \end{bmatrix}$$

As stated before, the idea of projection is extremely important in mathematics, and arises in situations that do not appear to have anything to do with geometry and vectors as we are thinking of them now. You will see a clever geometric use of vectors in one of the exercises.

Section 2.8 Exercises To Solutions

1. Consider the vectors $\vec{\mathbf{v}} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

   (a) Draw a *neat and accurate* graph of $\vec{\mathbf{v}}$ and $\vec{\mathbf{b}}$, with their tails at the origin, labeling each.

   (b) Use the formula to find $\text{proj}_{\mathbf{b}}\,\vec{\mathbf{v}}$, with its components rounded to the nearest tenth.

   (c) Add $\text{proj}_{\mathbf{b}}\,\vec{\mathbf{v}}$ to your graph. Does it look correct?

2. For each pair of vectors **v** and **b** below, do each of the following

   i) Sketch **v** and **b** with the same initial point.

   ii) Find $\text{proj}_{\mathbf{b}}\vec{\mathbf{v}}$ algebraically, using the formula for projections.

   iii) On the same diagram, sketch the $\text{proj}_{\mathbf{b}}\vec{\mathbf{v}}$ you obtained in part (ii). If it does not look the way it should, find your error.

   iv) Find $\text{proj}_{\mathbf{b}}\vec{\mathbf{v}}$, and sketch it as a new sketch. Compare with your previous sketch.

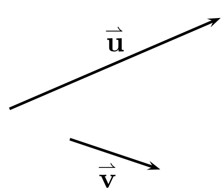
   (a) $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

   (b) $\mathbf{v} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

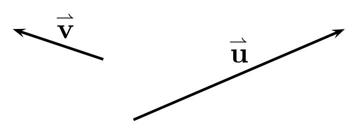
   (c) $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

3. For each pair of vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, sketch $\text{proj}_{\mathbf{v}}\,\vec{\mathbf{u}}$. Indicate any right angles with the standard symbol.

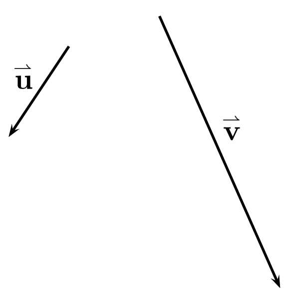
(a)



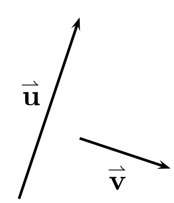
(b)



(c)



(d)



($\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal)

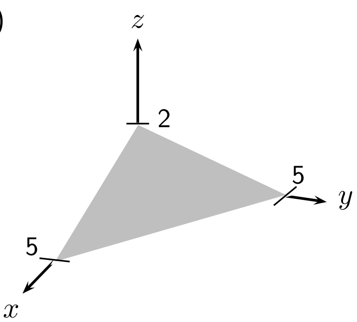
B Solutions to Exercises

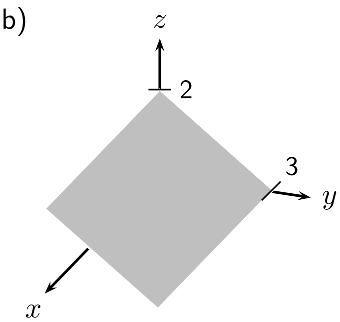
B.2 Chapter 2 Solutions

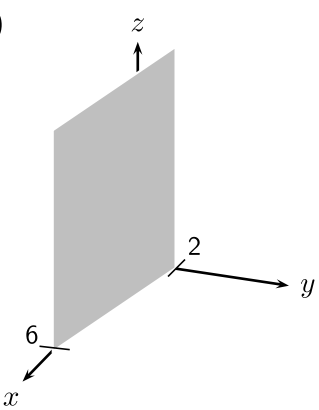
Section 2.1 Solutions Back to 2.1 Exercises

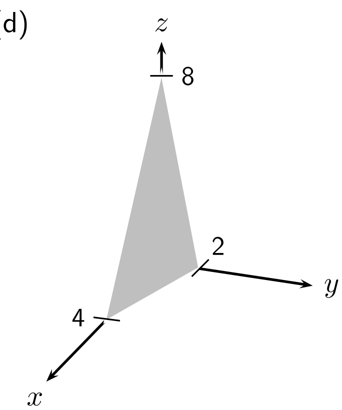
1. (a) Plane, intersects the x, y and z-axes at $(3, 0, 0)$, $(0, 6, 0)$ and $(0, 0, -2)$, respectively.
   (b) Plane, intersects the x and z-axes at $(6, 0, 0)$ and $(0, 0, 2)$, respectively. Does not intersect the y-axis.
   (c) Plane, intersects only the y-axis, at $(0, -6, 0)$.
   (d) Not a plane.
   (e) Plane, intersects the x, y and z-axes at $(-6, 0, 0)$, $(0, 3, 0)$ and $(0, 0, -2)$, respectively.

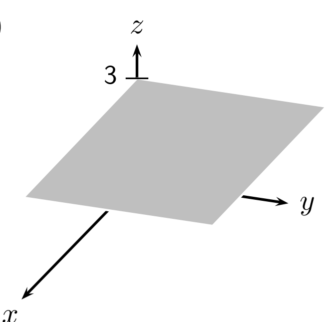
2. (a) $10x + 4y + 5z = 20$ (b) $-7x + 21y + 3z = 21$
   (c) $2x - 3z = 6$ (d) $y = 4$ (e) $-y + 4z = 4$ (f) $z = -2$

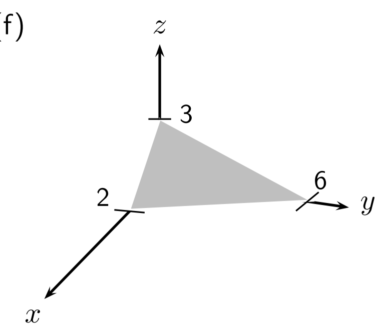
3. (a)


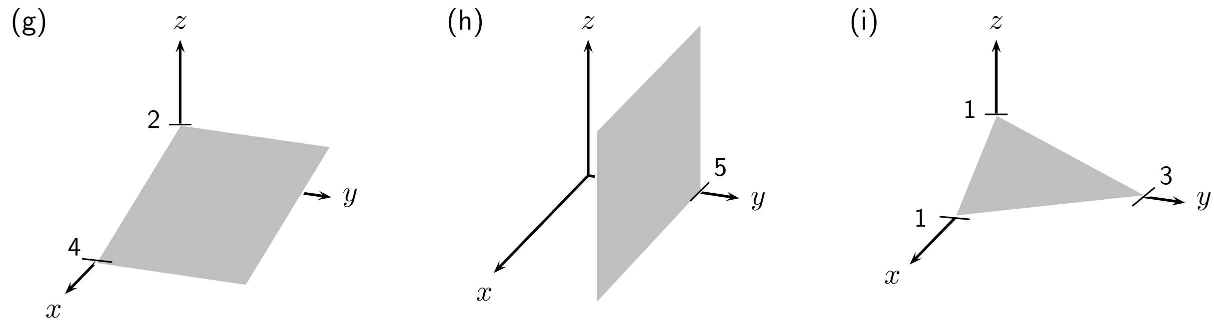
(b)


(c)


(d)


(e)


(f)




Section 2.2 Solutions Back to 2.2 Exercises

1. $\|\vec{\mathbf{u}}\| = \sqrt{14}$, $\|\vec{\mathbf{x}}\| = \sqrt{74}$, $\|\vec{\mathbf{v}}\| = \sqrt{30}$, $\|\vec{\mathbf{b}}\| = \sqrt{b_1^2 + b_2^2 + b_3^2 + b_4^2}$

2. (a) $\overrightarrow{PQ} = \begin{bmatrix} 17 \\ -6 \\ -15 \end{bmatrix}$, $\|\overrightarrow{PQ}\| = \sqrt{550} = 23.4$

   (b) $\overrightarrow{PQ} = \begin{bmatrix} 12 \\ -3 \end{bmatrix}$, $\|\overrightarrow{PQ}\| = \sqrt{153} = 12.4$

   (c) $\overrightarrow{PQ} = \begin{bmatrix} 10 \\ -1 \\ -7 \\ 9 \end{bmatrix}$, $\|\overrightarrow{PQ}\| = \sqrt{231} = 15.2$

3. (a) $Q(5,4)$ (b) $Q(6,7,-1,-7,7)$ (c) $P(9,-8,5)$

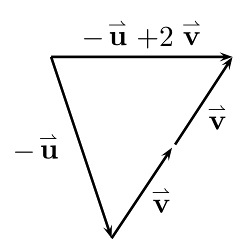
4. (a) $\|\vec{\mathbf{u}}\| = 3$ (b) $\dfrac{\vec{\mathbf{u}}}{\|\vec{\mathbf{u}}\|} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$ (c) $\left\| \dfrac{\vec{\mathbf{u}}}{\|\vec{\mathbf{u}}\|} \right\| = 1$

5. (a) $\|\vec{\mathbf{v}}\| = 5$ (b) $\dfrac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$ (c) $\left\| \dfrac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} \right\| = 1$

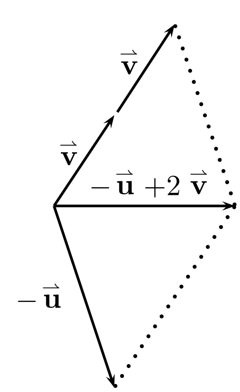
   For any vector $\vec{\mathbf{v}}$, the magnitude of $\dfrac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}$ is always one.

Section 2.3 Solutions Back to 2.3 Exercises

1. Tip-to-tail:



Parallelogram:



2. $\begin{bmatrix} -45 \\ 30 \end{bmatrix}$

3. $\begin{bmatrix} -c_1 - 8c_2 \\ 3c_1 + c_2 \\ -6c_1 + 4c_2 \end{bmatrix}$

5. (a) $0\vec{\mathbf{u}}_1 + 2\vec{\mathbf{u}}_2 + 1\vec{\mathbf{u}}_3 = \vec{\mathbf{v}}$ (b) $2\vec{\mathbf{u}}_1 - 3\vec{\mathbf{u}}_2 = \vec{\mathbf{v}}$

(c) $-5\vec{\mathbf{u}}_1 + \vec{\mathbf{u}}_2 + 3\vec{\mathbf{u}}_3 = \vec{\mathbf{v}}$

(d) Any vector of the form $(-\frac{1}{2}t + \frac{7}{2})\vec{\mathbf{u}}_1 + (\frac{1}{2}t - \frac{5}{2})\vec{\mathbf{u}}_2 + t\,\vec{\mathbf{u}}_3$ equals $\vec{\mathbf{v}}$

(e) $2\vec{\mathbf{u}}_1 - \vec{\mathbf{u}}_2 + 4\vec{\mathbf{u}}_3 - 5\vec{\mathbf{u}}_4 = \vec{\mathbf{v}}$

(f) There is no linear combination of $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$ and $\vec{\mathbf{u}}_3$ equalling $\vec{\mathbf{v}}$.

(g) Any vector of the form $(1-2t)\vec{\mathbf{u}}_1 + (4-3t)\vec{\mathbf{u}}_2 + t\,\vec{\mathbf{u}}_3$ equals $\vec{\mathbf{v}}$

6. (a) $a_1 = \frac{3}{2}, \quad a_2 = 4, \quad a_3 = -\frac{1}{2}$ (b) There are no such b_1, b_2 and b_3

(c) $\det(A) = -42, \quad \det(B) = 0$

Section 2.4 Solutions

Back to 2.4 Exercises

1. (a) $$x\begin{bmatrix}1\\-3\\2\end{bmatrix} + y\begin{bmatrix}1\\2\\1\end{bmatrix} + z\begin{bmatrix}-3\\-1\\-4\end{bmatrix} = \begin{bmatrix}1\\7\\0\end{bmatrix}$$

(b) $$x_1\begin{bmatrix}5\\0\\2\end{bmatrix} + x_2\begin{bmatrix}0\\2\\1\end{bmatrix} + x_3\begin{bmatrix}1\\3\\-4\end{bmatrix} = \begin{bmatrix}-1\\0\\2\end{bmatrix}$$

(c) $$b\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\end{bmatrix} + m\begin{bmatrix}0.5\\1.0\\1.5\\2.0\\2.5\\3.0\\3.5\end{bmatrix} = \begin{bmatrix}8.1\\6.9\\6.2\\5.3\\4.5\\3.8\\3.0\end{bmatrix}$$

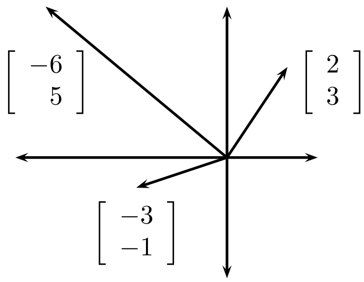
(d) $$x_1\begin{bmatrix}1\\3\\-2\end{bmatrix} + x_2\begin{bmatrix}-4\\2\\1\end{bmatrix} + x_3\begin{bmatrix}1\\-1\\-4\end{bmatrix} + x_4\begin{bmatrix}2\\-7\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\2\end{bmatrix}$$

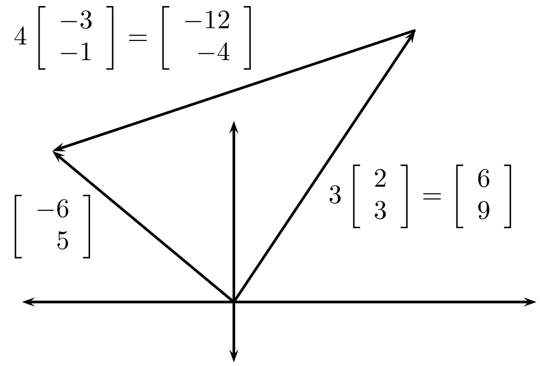
2. (a) $$\begin{aligned} -3x_1 + x_2 &= 5\\ x_1 + x_2 &= -2 \end{aligned}$$

(b) $$\begin{aligned} 5x_1 - 3x_2 + 2x_3 + 7x_4 &= -8\\ x_1 + 2x_2 - 4x_4 &= 1\\ -4x_1 + x_2 + 5x_3 + 6x_4 &= 5\\ -3x_1 - x_2 + 4x_3 + 7x_4 &= 4 \end{aligned}$$

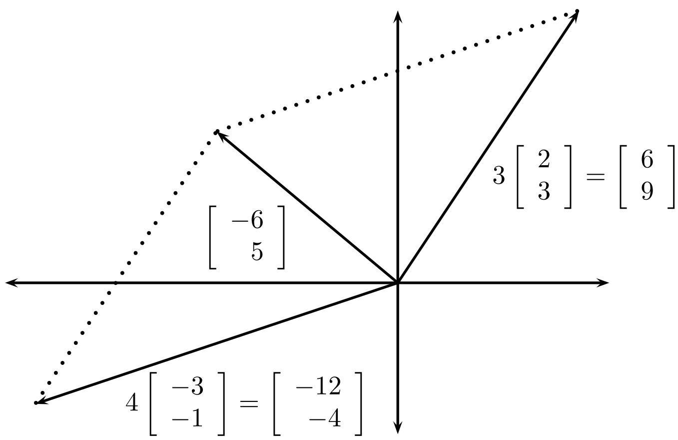
(c) $$\begin{aligned} 3x_1 + 4x_2 + x_3 &= -5\\ 2x_1 - 7x_2 + x_3 &= 3\\ x_1 + 5x_2 + x_3 &= -4 \end{aligned}$$

3. $x\begin{bmatrix}2\\3\end{bmatrix}+y\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-6\\5\end{bmatrix} \quad\Longrightarrow\quad 3\begin{bmatrix}2\\3\end{bmatrix}+4\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-6\\5\end{bmatrix}$





tip-to-tail method



parallelogram method

Section 2.5 Solutions

Back to 2.5 Exercises

1. (a) $\begin{bmatrix}0\\0\end{bmatrix},\ \begin{bmatrix}1\\1\end{bmatrix},\ \begin{bmatrix}3\\9\end{bmatrix},\ \begin{bmatrix}-2\\4\end{bmatrix},\ \begin{bmatrix}\pi\\\pi^2\end{bmatrix},\ \dots$

   (b) The set is not closed under addition. (c) $\mathcal{S}$ is not closed under scalar multiplication.

2. (a) $\begin{bmatrix}0\\0\\0\end{bmatrix},\ \begin{bmatrix}1\\1\\2\end{bmatrix},\ \begin{bmatrix}5\\-7\\-2\end{bmatrix},\ \begin{bmatrix}-3\\-1\\-4\end{bmatrix},\ \begin{bmatrix}\pi\\e\\\pi+e\end{bmatrix},\ \dots$

   (b) The set is closed under addition. (c) The set is closed under scalar multiplication.

3. (a) $\begin{bmatrix}0\\0\end{bmatrix},\ \begin{bmatrix}1\\1\end{bmatrix},\ \begin{bmatrix}5\\0\end{bmatrix},\ \begin{bmatrix}2\\3\end{bmatrix},\ \begin{bmatrix}-2\\-3\end{bmatrix},\ \begin{bmatrix}0\\-5\end{bmatrix},\ \begin{bmatrix}-1\\-1\end{bmatrix},\ \dots$

   (b) $\mathcal{S}$ is not closed under addition. (c) $\mathcal{S}$ is closed under scalar multiplication.

4. (a) $\begin{bmatrix}0\\0\end{bmatrix},\ \begin{bmatrix}1\\-2\end{bmatrix},\ \begin{bmatrix}2\\-4\end{bmatrix},\ \begin{bmatrix}3\\-6\end{bmatrix},\ \begin{bmatrix}-3\\6\end{bmatrix},\ \begin{bmatrix}-1\\2\end{bmatrix},\ \dots$

(b) $\mathcal{S}$ is closed under scalar multiplication. (c) The set is closed under addition.

5. (a) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \dots$

   (b) $\mathcal{S}$ is not closed under addition. (c) $\mathcal{S}$ is not closed under scalar multiplication.

6. (a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 17 \\ -22 \end{bmatrix}, \begin{bmatrix} -10 \\ -10 \end{bmatrix}, \dots$

   (b) The set is closed under addition. (c) $\mathcal{S}$ is closed under scalar multiplication.

7. (a) $\begin{bmatrix} -4 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \end{bmatrix}, \dots$

   (b) The set is closed under addition. (c) $\mathcal{S}$ is closed under scalar multiplication.

8. (a) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -8 \\ 0 \\ 8 \end{bmatrix}, \dots$

   (b) The set is not closed under addition. (c) The set is not closed under scalar multiplication.

9. (a) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} \pi \\ 2\pi \\ 3\pi \end{bmatrix}, \dots$

   (b) $\mathcal{S}$ is closed under addition. (c) $\mathcal{S}$ is closed under scalar multiplication.

10. (a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 200 \\ -300 \end{bmatrix}, \begin{bmatrix} 1 \\ \pi \\ e \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 1 \end{bmatrix}, \dots$

    (b) The set is not closed under addition. (c) The set is not closed under scalar multiplication.

Section 2.6 Solutions

Back to 2.6 Exercises

1. Each of the following is just one possibility - each line or plane has more than one equation possible.

   (a) $\vec{\mathbf{x}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

   (b) $\vec{\mathbf{x}} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + s \begin{bmatrix} -1 \\ 7 \\ -4 \end{bmatrix} + t \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix}$

   (c) $\vec{\mathbf{x}} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + t \begin{bmatrix} -1 \\ 7 \\ -4 \end{bmatrix}$

   (d) $\vec{\mathbf{x}} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

(e) $\vec{\mathbf{x}} = t\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

(f) $\vec{\mathbf{x}} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix} + s\begin{bmatrix} 7 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 6 \\ -3 \\ -1 \end{bmatrix}$

(g) $\vec{\mathbf{x}} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$

(h) $\vec{\mathbf{x}} = s\begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

(i) $\vec{\mathbf{x}} = t\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2. The sets described in parts (e), (h) and (i) are closed under addition and scalar multiplication.

3. $\vec{\mathbf{x}} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 5 \end{bmatrix} + r\begin{bmatrix} 5 \\ 6 \\ -6 \\ -3 \end{bmatrix} + s\begin{bmatrix} 6 \\ -1 \\ 4 \\ -7 \end{bmatrix} + t\begin{bmatrix} 5 \\ -6 \\ 3 \\ 2 \end{bmatrix}$

4. (a) $\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$ (b) $\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -4 \\ 2 \\ -2 \end{bmatrix}$

(c) $\vec{\mathbf{x}} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$

5. (a) $(7, 9)$ (b) $\vec{\mathbf{x}} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} + t\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ (c) $t = 2$ (d) $(-8, -3),\ t = -1$

6. $\vec{\mathbf{x}} = \begin{bmatrix} -5 \\ 7 \\ 4 \end{bmatrix} + s\begin{bmatrix} 6 \\ -1 \\ -6 \end{bmatrix} + t\begin{bmatrix} 8 \\ -7 \\ -5 \end{bmatrix}$

7. $(-3, 3, 0),\ (-7, 5, -2),\ (-11, 7, -4)$

8. The vector equation of the plane is $\vec{\mathbf{x}} = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} + s\begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix} + t\begin{bmatrix} 6 \\ -3 \\ -8 \end{bmatrix}$. Letting $s = 1$ and $t = 1$ gives the point $(9, -1, -7)$. (This is just one possibility - we can get other points by choosing other values of s and t.)

9. (a) If either, but not both, of $\vec{\mathbf{u}}$ or $\vec{\mathbf{v}}$ are the zero vector, then $\vec{\mathbf{x}} = \vec{\mathbf{p}} + s\vec{\mathbf{u}} + t\vec{\mathbf{v}}$ will be the equation of a line. If $\vec{\mathbf{u}} \neq \vec{\mathbf{0}}$ and $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$ are scalar multiples of each other, then $\vec{\mathbf{x}} = \vec{\mathbf{p}} + s\vec{\mathbf{u}} + t\vec{\mathbf{v}}$ will be the equation of a line.

   (b) If $\vec{\mathbf{p}}$ is the zero vector, then $\vec{\mathbf{x}} = \vec{\mathbf{p}} + s\vec{\mathbf{u}} + t\vec{\mathbf{v}}$ will be the equation of a plane through the origin. If $\vec{\mathbf{p}} \neq \vec{\mathbf{0}}$ is a scalar multiple of either $\vec{\mathbf{u}} \neq \vec{\mathbf{0}}$ or $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$ and $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are *NOT* scalar multiples of each other, then $\vec{\mathbf{x}} = \vec{\mathbf{p}} + s\vec{\mathbf{u}} + t\vec{\mathbf{v}}$ will be the equation of a plane through the origin.

Section 2.7 Solutions

Back to 2.7 Exercises

1. (a) $\vec{\mathbf{x}} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (b) $\vec{\mathbf{x}} = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

   (c) $\vec{\mathbf{x}} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

2. (a) three-dimensional plane in four-dimensional space
   (b) three-dimensional plane in five-dimensional space
   (c) five-dimensional plane in seven-dimensional space
   (d) one-dimensional plane (line) in five-dimensional space
   (e) two-dimensional plane in four-dimensional space
   (f) four-dimensional plane in seven-dimensional space

Section 2.8 Solutions

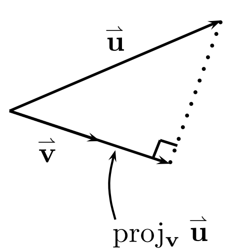
Back to 2.8 Exercises

2. (a) $\text{proj}_{\mathbf{b}}\,\vec{\mathbf{v}} = \dfrac{15-2}{25+4}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \dfrac{13}{29}\begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{65}{29} \\ -\frac{26}{29} \end{bmatrix}$

   (b) $\text{proj}_{\mathbf{b}}\,\vec{\mathbf{v}} = \dfrac{10-0}{4+1}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

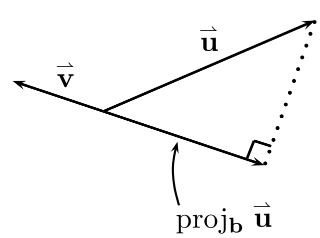
   (c) $\text{proj}_{\mathbf{b}}\,\vec{\mathbf{v}} = \dfrac{-6-4}{16+4}\begin{bmatrix} -2 \\ -4 \end{bmatrix} = -\dfrac{1}{2}\begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3. (a)

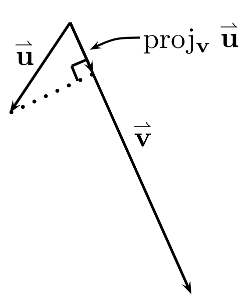


(tail is where $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ meet)

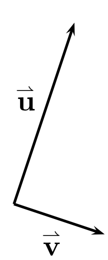
   (b)



   (c)



   (d)



because $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal, $\text{proj}_{\mathbf{v}}\,\vec{\mathbf{u}}$ is the zero vector