Linear Algebra I

Skills, Concepts and Applications

Gregg Waterman
Oregon Institute of Technology

Contents

3 Matrices and Vectors

Outcome:

3. Understand matrices, their algebra, and their action on vectors. Use matrices to solve problems. Understand the algebra of matrices.

   Performance Criteria:

   (a) Give the dimensions of a matrix. Identify a given entry, row or column of a matrix.

   (b) Identify matrices as square, upper triangular, lower triangular, symmetric, diagonal. Give the transpose of a given matrix; know the notation for the transpose of a matrix.

   (c) Add or subtract matrices when possible. Multiply a matrix by a scalar.

   (d) Multiply a matrix times a vector, give the linear combination form of a matrix times a vector.

   (e) Give a specific matrix that is multiplied times an arbitrary vector to obtain a given resulting vector.

   (f) Express a system of equations as a coefficient matrix times a vector equalling another vector.

   (g) For a given matrix A and vector $\vec{\mathbf{b}}$, find a vector $\vec{\mathbf{x}}$ for which $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$,

   (h) Determine whether a matrix is a projection matrix, reflection matrix or rotation matrix, or none of these, by its action on a few vectors.

   (i) Determine whether a vector is an eigenvector of a matrix. If it is, give the corresponding eigenvalue.

   (j) Know when two matrices can be multiplied, and know that matrix multiplication is not necessarily commutative. Multiply two matrices "by hand."

   (k) Determine whether two matrices are inverses without finding the inverse of either.

   (l) Find the inverse of a 2×2 matrix using the formula. Find the inverse of a matrix using the Gauss-Jordan method. Describe the Gauss-Jordan method for finding the inverse of a matrix.

   (m) Solve a system of equations using an inverse matrix. Describe how to use an inverse matrix to solve a system of equations.

   (n) Find the determinant of a 2×2 or 3×3 matrix by hand. Use a calculator to find the determinant of an $n \times n$ matrix.

   (o) Use the determinant to determine whether a system of equations has a unique solution.

   (p) Determine whether a homogeneous system has more than one solution.

Continued on the next page.

Outcome:

3. Understand matrices, their algebra, and their action on vectors. Use matrices to solve problems. Understand the algebra of matrices.

   Performance Criteria:

   (q) Use Cramer's rule to solve a system of equations.

   (r) Give the geometric or algebraic representations of the inverse or square of a rotation. Demonstrate that the geometric and algebraic versions are the same

   (s) Give the incidence matrix of a graph or digraph. Given the incidence matrix of a graph or digraph, identify the vertices and edges using correct notation, and draw the graph.

   (t) Determine the number of k-paths from one vertex of a graph to another. Solve problems using incidence matrices.

3.1 Introduction to Matrices

Performance Criteria:

3. (a) Give the dimensions of a matrix. Identify a given entry, row or column of a matrix.
   (b) Identify matrices as square, upper triangular, lower triangular, symmetric, diagonal. Give the transpose of a given matrix; know the notation for the transpose of a matrix.
   (c) Add or subtract matrices when possible. Multiply a matrix by a scalar.

A **matrix** is simply an array of numbers arranged in rows and columns. Here are some examples:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix}$$

We will always denote matrices with italicized capital letters. There should be no need to define the **rows** and **columns** of a matrix. The number of rows and number of columns of a matrix are called its **dimensions**. The second matrix above, B, has dimensions 3×2, which we read as "three by two." The numbers in a matrix are called its **entries**. Each entry of a matrix is identified by its row, then column. For example, the $(3,2)$ entry of L is the entry in the 3rd row and second column, -2. In general, we will define the (i,j)th entry of a matrix to be the entry in the ith row and jth column.

There are a few special kinds of matrices that we will run into regularly:

- A matrix with the same number of rows and columns is called a **square matrix**. Matrices A, D and L above are square matrices.

- The entries that are in the same number row and column of a square matrix are called the **diagonal entries** of the matrix. For example, the diagonal entries of A are 1 and 3. All the diagonal entries taken together are called the **diagonal** of the matrix. (This *ALWAYS* refers to only the diagonal from upper left to lower right.)

- A square matrix with zeros "above" the diagonal is called a **lower triangular matrix**; L is an example of a lower triangular matrix. Similarly, an **upper triangular matrix** is one whose entries below the diagonal are all zeros. (Note that the words "lower" and "upper" refer to the triangular parts of the matrices where the entries are *NOT* zero.)

- A square matrix all of whose entries above *AND* below the diagonal are zero is called a **diagonal matrix**. D is an example of a diagonal matrix. *Any diagonal matrix is also both upper and lower triangular.*

- A diagonal matrix with only ones on the diagonal is called "the" **identity matrix**. We use the word "the" because in a given size there is only one identity matrix. We will soon see why it is called the "identity."

- Given a matrix, we can create a new matrix whose rows are the columns of the original matrix. (This is equivalent to the columns of the new matrix being the rows of the original.) The new matrix is called the **transpose** of the original. The transposes of the matrices B and L above are denoted by B^T and L^T. They are the matrices

$$B^T = \begin{bmatrix} -5 & 0 & 2 \\ 1 & 4 & -3 \end{bmatrix} \qquad L^T = \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

If a matrix a is $m \times n$, then A^T is $n \times m$. Note that when a matrix is square, its transpose is obtained by "flipping the matrix over its diagonal."

- Notice that $A^T = A$. Such a matrix is called a **symmetric matrix**. One way of thinking of such a matrix is that the entries across the diagonal from each other are equal. Matrix D is also symmetric, as is the matrix

$$\begin{bmatrix} 1 & 5 & 0 & -2 \\ 5 & -4 & 7 & 3 \\ 0 & 7 & 0 & -6 \\ -2 & 3 & -6 & -3 \end{bmatrix}$$

When discussing an arbitrary matrix A with dimensions $m \times n$ we refer to each entry as a, but with a double subscript with each to indicate its position in the matrix. The first number in the subscript indicates the row of the entry and the second indicates the column of that entry:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2k} & \cdots & a_{2n} \\ \vdots & \vdots & & & \vdots & & \vdots \\ a_{j1} & a_{j2} & a_{j3} & \cdots & a_{jk} & \cdots & a_{jn} \\ \vdots & \vdots & & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mk} & \cdots & a_{mn} \end{bmatrix}$$

Under the right conditions it is possible to add, subtract and multiply two matrices. We'll save multiplication for a little, but we have the following:

DEFINITION 3.1.1: **Adding and Subtracting Matrices**

When two matrices have the same dimensions, they are added or subtracted by adding or subtracting their corresponding entries.

◇ **Example 3.1(a):** Determine which of the matrices below can be added, and add those that can be.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}, \qquad C = \begin{bmatrix} -7 & 4 \\ 1 & 5 \end{bmatrix}$$

Solution: B cannot be added to either A or C, but

$$A + C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -7 & 4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 3 & 8 \end{bmatrix}$$

It should be clear that A and C could be subtracted, and that $A+C=C+A$ but $A-C \neq C-A$.

Definition 3.1.2: Scalar Times a Matrix

The result of a scalar c times a matrix A is the matrix each of whose entries are c times the corresponding entry of A.

◇ **Example 3.1(b):** For the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 8 & 5 \\ 6 & -4 & -3 \end{bmatrix}$, find $3A$.

$$3A = 3\begin{bmatrix} 3 & 1 & -1 \\ -2 & 8 & 5 \\ 6 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & 3 & -3 \\ -6 & 24 & 15 \\ 18 & -12 & -9 \end{bmatrix}$$

Section 3.1 Exercises To Solutions

1. (a) Give the dimensions of matrices A, B and C in Exercise 3 below.

   (b) Give the entries b_{31} and c_{23} of the matrices B and C in Exercise 3 below.

2. Give the names of the matrices at the top of the next page that are

   (a) square (b) symmetric (c) diagonal

   (d) upper triangular (e) lower triangular

$$A = \begin{bmatrix} -4 & 0 & 0 \\ 2 & 6 & 0 \\ -1 & 5 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -4 & 3 \\ -4 & 5 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -4 \\ -4 & 5 \end{bmatrix} \qquad E = \begin{bmatrix} 2 & 5 & -3 & 1 \\ 0 & 4 & 0 & 2 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad F = \begin{bmatrix} 2 & 7 \\ -1 & 4 \\ 6 & 5 \end{bmatrix}$$

$$G = \begin{bmatrix} 3 & 4 & -5 & 0 \\ 4 & 1 & 7 & 0 \\ -5 & 7 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad H = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} \qquad J = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & -5 \end{bmatrix}$$

3. Give examples of each of the following types of matrices.

   (a) lower triangular (b) diagonal

   (c) symmetric (d) identity

   (e) upper triangular but not diagonal (f) symmetric but without any zero entries

   (g) symmetric but not diagonal (h) diagonal but not a multiple of an identity

4. Give the transpose of each matrix. Use the correct notation to denote the transpose.

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -3 & 1 & -2 \\ 4 & 7 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 4 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & -1 & 3 \\ -3 & 1 & 2 & 0 \\ 4 & 7 & 0 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & -3 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$$

5. Give all possible sums and differences of matrices from Exercise 4.

6. Consider the matrix below and to the left.

   (a) What are some ways we could describe the matrix? Give all you can think of.

   (b) Below and to the right the dotted lines indicate how the matrix can be broken into four **blocks**, each of which is a 3×3 matrix. Give all ways you can think of to describe the matrix consisting of only the block in the upper left.

   (c) Give all ways you can think of to describe the matrix consisting of the block in the lower right.

   (d) Give all ways you can think of to describe the matrices in the upper right and lower left. There is one more word to describe them that was not given in the section. What do you think it is?

$$\begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \qquad \left[\begin{array}{ccc:ccc} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{array}\right]$$

7. (a) For the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find the matrix $B = A + A^T$.

   (b) What kind of matrix is B?

3.2 Matrix Times a Vector

Performance Criteria:

3. (d) Multiply a matrix times a vector, give the linear combination form of a matrix times a vector.
   (e) Give a specific matrix that is multiplied times an arbitrary vector to obtain a given resulting vector.
   (f) Express a system of equations as a coefficient matrix times a vector equalling another vector.
   (g) For a given matrix A and vector $\vec{\mathbf{b}}$, find a vector $\vec{\mathbf{x}}$ for which $A\,\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

Matrix Times a Vector

Multiplying a matrix times a vector is in some sense *THE* foundational operation of linear algebra. When we study algebra, trigonometry and calculus, what we are really interested in how functions act on numbers. *In linear algebra we are interested in how matrices act on vectors*. This action is usually referred to as multiplication of a matrix times a vector, but we will be well served by remembering that it is truly an *action* of a matrix on a vector.

Before getting into how to do this, we need to devise a useful notation. Consider the matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Each column of A, taken by itself, is a vector. We'll refer to the first column as the vector $\vec{\mathbf{a}}_{*1}$, with the asterisk * indicating that the row index will range through all values, and the 1 indicating that the values all come out of column one. Of course $\vec{\mathbf{a}}_{*2}$ denotes the second column, and so on. Similarly, $\vec{\mathbf{a}}_{1*}$ will denote the first row, $\vec{\mathbf{a}}_{2*}$ the second row, etc. Technically speaking, the rows are not vectors, but we'll call them **row vectors** and we'll call the columns **column vectors**. If we use just the word *vector*, we will mean a column vector.

◇ **Example 3.2(a):** Give $\vec{\mathbf{a}}_{2*}$ and $\vec{\mathbf{a}}_{*3}$ for the matrix $A = \begin{bmatrix} -5 & 3 & 4 & -1 \\ 7 & 5 & 2 & 4 \\ 2 & -1 & -6 & 0 \end{bmatrix}$

$$\vec{\mathbf{a}}_{2*} = \begin{bmatrix} 7 & 5 & 2 & 4 \end{bmatrix} \qquad \text{and} \qquad \vec{\mathbf{a}}_{*3} = \begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix}$$

A row vector can be multiplied times a column vector (*only in that order*) if they have the same number of components. To do this, we simply multiply each entry of the row vector times each entry of the column vector and add all the results. *The result of a row vector times a column vector is then a single number.* This is shown in the following example.

◇ **Example 3.2(b):** Multiply $\vec{\mathbf{u}} = \begin{bmatrix} 5 & -3 & 2 \end{bmatrix}$ times $\vec{\mathbf{v}} = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}$.

$\vec{\mathbf{u}}\vec{\mathbf{v}} = (5)(-4) + (-3)(2) + (2)(1) = -20 + (-6) + 2 = -24.$ Note that if $\vec{\mathbf{u}}$ was a true (column) vector, this would be $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$.

Technically, the result -24 should be the 1×1 matrix $[-24]$, but we are really only interested in the above operation as it relates to taking a matrix times a vector. Once we are able to do that, there is no need to do the above operation except as a part of the process of taking a matrix times a vector. We now define how to multiply a matrix times a vector.

Definition 3.2.1: **Matrix Times a Vector**

An $m \times n$ matrix A can be multiplied times a vector $\vec{\mathbf{x}}$ with n components. The result is a vector with m components, the ith component being the product of the ith row of A with $\vec{\mathbf{x}}$, as shown below.

$$A\,\vec{\mathbf{x}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{a}}_{1*}\vec{\mathbf{x}} \\ \vec{\mathbf{a}}_{2*}\vec{\mathbf{x}} \\ \vdots \\ \vec{\mathbf{a}}_{m*}\vec{\mathbf{x}} \end{bmatrix}$$

◇ **Example 3.2(c):** Multiply $\begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \\ 1 & -6 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix}$.

$$\begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \\ 1 & -6 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix} = \begin{bmatrix} (3)(2) + (0)(1) + (-1)(-7) \\ (-5)(2) + (2)(1) + (4)(-7) \\ (1)(2) + (-6)(1) + (0)(-7) \end{bmatrix} = \begin{bmatrix} 13 \\ -36 \\ -4 \end{bmatrix}$$

There is no need for the matrix multiplying a vector to be square, but when it is not, the resulting vector is not the same length as the original vector:

◇ **Example 3.2(d):** Find $A\vec{\mathbf{x}}$ for $A = \begin{bmatrix} 7 & -4 & 2 \\ -1 & 0 & 6 \end{bmatrix}$ and $\vec{\mathbf{x}} = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$.

$$A\vec{\mathbf{x}} = \begin{bmatrix} 7 & -4 & 2 \\ -1 & 0 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} (7)(3) + (-4)(-5) + (2)(1) \\ (-1)(3) + (0)(-5) + (6)(1) \end{bmatrix} = \begin{bmatrix} 43 \\ 3 \end{bmatrix}$$

We sometimes describe what we see in the above example by saying that the matrix A *transforms* the vector $\vec{\mathbf{x}} = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$ into the vector $\begin{bmatrix} 43 \\ 3 \end{bmatrix}$. This emphasizes the fact that we are interested in a matrix times a vector as more than just an algebraic computation.

Note that, if a matrix A is $1 \times n$ and $\vec{\mathbf{x}}$ is a vector with n components, the multiplication $\vec{\mathbf{x}}A$ can be performed, but such multiplication do not arise naturally in applications. We will *always* multiply a matrix times a vector, in that order, so we always write $A\vec{\mathbf{x}}$ rather than $\vec{\mathbf{x}}A$.

Matrix Times a Vector Form of a System

We now see an example that has important implications.

◇ **Example 3.2(e):** Multiply $A = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 4 & -2 \\ 2 & 1 & 5 \end{bmatrix}$ times the vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$A\vec{\mathbf{x}} = \begin{bmatrix} 1 & -1 & 2 \\ -3 & 4 & -2 \\ 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -3x_1 + 4x_2 - 2x_3 \\ 2x_1 + x_2 + 5x_3 \end{bmatrix}$$

Consider now the system shown below and to the left:

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 5 \\ -3x_1 + 4x_2 - 2x_3 &= -1 \\ 2x_1 + x_2 + 5x_3 &= 2 \end{aligned} \qquad\qquad \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -3x_1 + 4x_2 - 2x_3 \\ 2x_1 + x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$$

Because two vectors are equal if their corresponding components are equal, we can rewrite the system in the form shown above and to the right and, as a result of Example 3.2(e), this leads to

$$\begin{bmatrix} 1 & -1 & 2 \\ -3 & 4 & -2 \\ 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}.$$

This is what we will refer to as the **matrix times a vector form** of a system of equations.

Definition 3.2.2 Matrix Times a Vector Form of a System

A system

$$\begin{array}{rcl} a_{11}x_1 + \ a_{12}x_2 + \cdots + \ a_{1n}x_n & = & b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \cdots + \ a_{2n}x_n & = & b_2 \\ \vdots & & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

of m linear equations in n unknowns can be written as a matrix times a vector equalling another vector:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

We will refer to this as the **matrix times a vector form of the system of equations**, and we express it compactly as $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

◇ **Example 3.2(f):** Give the matrix times vector form of the system $\begin{array}{rcr} 3x_1 + 5x_2 & = & -1 \\ x_1 + 4x_2 & = & 2 \end{array}$.

Solution: The matrix times vector form of the system is $\begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Now we can solve the "inverse problem" of a matrix times a vector:

◇ **Example 3.2(g):** Let $A = \begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$. Find a vector $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for which $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

Solution: We are trying to solve the matrix-vector equation $\begin{bmatrix} 3 & 5 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$. This is the matrix times a vector form of the system

$$\begin{array}{rcr} 3x_1 + \ x_2 & = & 4 \\ x_1 + 4x_2 & = & -1 \end{array},$$

which we can solve by the addition method or row-reduction to obtain $\vec{\mathbf{x}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Linearity of the Action of a Matrix on a Vector

Multiplication of vectors by matrices has the following important properties, which are easily verified.

> **Theorem 3.2.3**
>
> Let A and B be matrices, $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ be vectors, and c be any scalar. Assuming that all the indicated operations below are defined (possible), then
>
> (a) $A(\vec{\mathbf{x}}+\vec{\mathbf{y}}) = A\vec{\mathbf{x}} + A\vec{\mathbf{y}}$ (b) $A(c\vec{\mathbf{x}}) = c(A\vec{\mathbf{x}})$
>
> (c) $(A+B)\vec{\mathbf{x}} = A\vec{\mathbf{x}} + B\vec{\mathbf{x}}$

We now come to a very important idea that depends on the first two properties of Theorem 3.2.3. When we act on a mathematical object with another object, the object doing the "acting on" is often called an **operator**. Some operators you are familiar with are the derivative operator and the antiderivative operator (indefinite integral), which act on functions to create other functions. Note that the derivative operator has the following two properties, for any functions f and g and real number c:

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}, \qquad\qquad \frac{d}{dx}(cf) = c\frac{df}{dx}$$

These are the same as the first two properties above for multiplication of a vector by a matrix. A matrix can be thought of as an operator that operates on vectors (through multiplication). The first two properties of multiplication of a vector by a matrix, as well as the corresponding properties of the derivative, are called the **linearity properties**. Both the derivative operator and the matrix multiplication operator are then called **linear operators**. This is why this subject is called *linear* algebra!

Linear Combination Form of a Matrix Times a Vector

There is another way to compute a matrix times a vector. It is not as efficient to do by hand as the method implied by Definition 3.2.1, but it will be very important conceptually quite soon. Using our earlier definition of a matrix A times a vector $\vec{\mathbf{x}}$, we see that

$$A\vec{\mathbf{x}} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{21}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix}$$

$$= x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2\begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Let's think about what the above shows. It gives us the result below, which is illustrated in Examples 3.2(h) and (i).

Linear Combination Form of a Matrix Times a Vector

The product of a matrix A and a vector $\vec{\mathbf{x}}$ is a linear combination of the columns of A, with the scalars being the corresponding components of $\vec{\mathbf{x}}$.

◇ **Example 3.2(h):** Give the linear combination form of $\begin{bmatrix} 7 & -4 & 2 \\ -1 & 0 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$.

$$\begin{bmatrix} 7 & -4 & 2 \\ -1 & 0 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 7 \\ -1 \end{bmatrix} - 5\begin{bmatrix} -4 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

◇ **Example 3.2(i):** Give the linear combination form of $\begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$$

Section 3.2 Exercises

To Solutions

1. Multiply $\begin{bmatrix} 1 & 0 & -1 \\ -3 & 1 & 2 \\ 4 & 7 & -2 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 & -4 & 0 \\ -1 & 5 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$ **by hand**.

2. Multiply each of the following by hand, when possible. Some are on the next page.

(a) $\begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -5 & 2 \\ 6 & 3 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -5 & 2 \\ 6 & 3 & -4 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 6 \\ -5 & 3 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

(e) $\begin{bmatrix} 7 & -2 & 0 & 4 \\ 1 & 5 & 3 & -3 \\ 2 & 1 & -1 & 5 \\ -3 & 7 & 2 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 1 \\ 3 \\ 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & -5 \\ 2 & 2 & 3 \\ -4 & 7 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

3. Give each of the products from Exercise 2 in linear combination form.

4. Give the product $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ as

   (a) a single vector

   (b) a linear combination of vectors

5. (a) Find a matrix A such that $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 - 5x_2 \\ x_1 + x_2 \end{bmatrix}$.

   (b) Find a matrix B such that $B \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 - x_2 \\ 5x_1 + 4x_2 \end{bmatrix}$.

   (c) Find a matrix C such that $C \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 4x_2 - x_3 \\ -5x_1 + x_2 + 2x_3 \\ x_1 + 3x_2 \end{bmatrix}$.

   (d) Find a matrix D such that $D \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_1 - x_2 - x_3 \end{bmatrix}$.

6. Give the matrix times a vector form of each system:

   (a)
$$\begin{aligned} x + y - 3z &= 1 \\ -3x + 2y - z &= 7 \\ 2x + y - 4z &= 0 \end{aligned}$$

   (b)
$$\begin{aligned} 5x_1 + x_3 &= -1 \\ 2x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 - 4x_3 &= 2 \end{aligned}$$

   (c)
$$\begin{aligned} b + 0.5m &= 8.1 \\ b + 1.0m &= 6.9 \\ b + 1.5m &= 6.2 \\ b + 2.0m &= 5.3 \\ b + 2.5m &= 4.5 \\ b + 3.0m &= 3.8 \\ b + 3.5m &= 3.0 \end{aligned}$$

   (d)
$$\begin{aligned} x_1 - 4x_2 + x_3 + 2x_4 &= -1 \\ 3x_1 + 2x_2 - x_3 - 7x_4 &= 0 \\ -2x_1 + x_2 - 4x_3 + x_4 &= 2 \end{aligned}$$

7. For each part of this exercise, find a vector $\vec{\mathbf{x}}$ for which $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ for the A and $\vec{\mathbf{b}}$ given.

   (a)
$$A = \begin{bmatrix} 3 & -1 & 5 \\ 2 & 0 & 2 \\ -1 & 4 & -3 \end{bmatrix}, \quad \vec{\mathbf{b}} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

   (b)
$$A = \begin{bmatrix} 1 & -3 \\ 5 & -2 \end{bmatrix}, \quad \vec{\mathbf{b}} = \begin{bmatrix} 24 \\ 29 \end{bmatrix}$$

   (c)
$$A = \begin{bmatrix} 1 & 2.1 \\ 1 & 2.7 \\ 1 & 3.2 \\ 1 & 3.9 \end{bmatrix}, \quad \vec{\mathbf{b}} = \begin{bmatrix} 2.54 \\ 2.78 \\ 2.98 \\ 3.26 \end{bmatrix}$$

   (d)
$$A = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 3 & 1 \end{bmatrix}, \quad \vec{\mathbf{b}} = \begin{bmatrix} 13 \\ 19 \\ 11 \\ 7 \end{bmatrix}$$

8. The relationship between stress and strain in a small cube of solid material can be expressed by the matrix equation

$$\begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} \quad \text{where} \quad \begin{array}{l} E \text{ is elastic modulus} \\ G \text{ is shear modulus} \\ \nu \text{ is Poisson's ratio} \\ \epsilon \text{ is normal strain} \\ \gamma \text{ is shear strain} \\ \sigma \text{ is normal stress} \\ \tau \text{ is shear stress} \end{array}$$

and the subscripts indicate faces of the cube and directions of forces. The vector being multiplied is the stress vector, and the right hand side is the strain vector.

(a) Give the relationship expressed by the product of the second row of the matrix times the stress vector.

(b) Give an expression for the ϵ_{zz} strain in terms of the three stresses σ_{xx}, σ_{yy}, σ_{zz} and the parameters E and ν.

(c) Give the relationship between τ_{zx} and γ_{zx}.

(d) The coefficient matrix is what we call a **block matrix** made up of blocks, each of which is a smaller matrix. In this case there are four 3×3 matrices, two of which are zero matrices. Each of the nonzero blocks is multiplied by only a portion of the stress vector and gives only a portion of the strain vector. Write the two matrix equations for the nonzero blocks.

3.3 Actions of Matrices on Vectors: Transformations in $\mathbb{R}^2$

Performance Criteria:

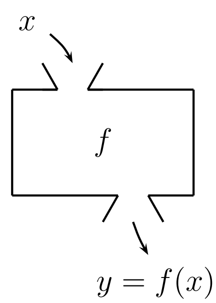
3. (h) Determine whether a matrix is a projection matrix, reflection matrix or rotation matrix, or none of these, by its action on a few vectors.

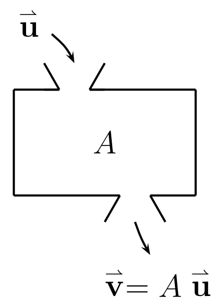
   (i) Determine whether a vector is an eigenvector of a matrix. If it is, give the corresponding eigenvalue.

Most of the mathematics that you have studied revolves around the idea of a function, which is simply a rule that assigns to every number in the *domain* of the function another number. There need not be any logic to how this is done, but usually there is, and that logic is seen in the fact that the function is given as some sort of specific relationship, like

$$f(x) = x^2 - 5x + 2\,, \qquad y = \sqrt{x+3}\,, \qquad g(t) = 3\sin(2t-5)\,, \qquad h(x) = \ln x.$$

(The last two of these are sort of redundant, in the sense that they are built out of other functions, sine and the natural logarithm.) For those who are visually or mechanically inclined, a function can also be thought of as a "machine," commonly named f, whose input is a number x and output is some other number y. We write $y = f(x)$ and say "y equals f of x" to indicate that y is the result of the function f acting on x. This is shown in the picture below and to the left.





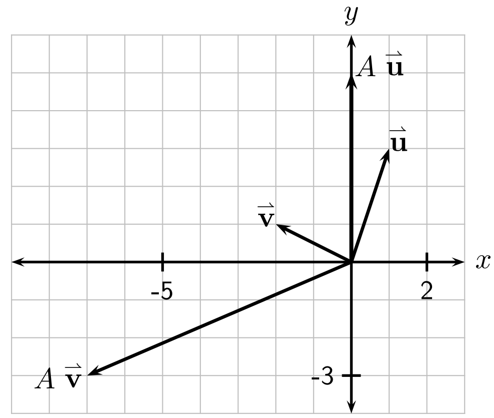
When we multiply a vector by a matrix, the result is another vector - *this is essentially the same idea as a function, but with vectors playing the role of numbers and a matrix taking the place of the function*. This is shown in the picture above and to the right. We should really think of a matrix times a vector as the matrix acting on the vector to create another vector. We sometimes say that the matrix A *transforms* the original vector $\vec{\mathbf{u}}$ to the new one $\vec{\mathbf{v}} = A\,\vec{\mathbf{u}}$. This happens by the purely computational means that you learned in the previous section.

One powerful tool in the study of any function is its graphical representation. Unfortunately, it is difficult to graphically represent the action of a matrix on all vectors - we instead must picture the action of a matrix on vectors one or two vectors at a time. Even that is difficult with vectors in $\mathbb{R}^3$ and impossible in higher dimensions. Let's see how we do it in $\mathbb{R}^2$. Suppose that we have the matrix and vectors

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}, \quad \vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \vec{\mathbf{v}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

for which

$$A\,\vec{\mathbf{u}} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \qquad A\,\vec{\mathbf{v}} = \begin{bmatrix} -7 \\ -3 \end{bmatrix}.$$



The graph to the right shows the vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $A\,\vec{\mathbf{u}}$ and $A\,\vec{\mathbf{v}}$, and we can see the result of A acting on $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. The two attributes of any vector are of course magnitude and direction, and we can see that A altered the magnitude and the direction of both vectors, but apparently not in any special way.

In general, when a matrix acts on a vector the resulting vector will have a different magnitude and direction than the original vector, with the change in magnitude and direction being different for most vectors. There are a few notable exceptions to this:

- The matrix that acts on a vector without actually changing it at all is called the **identity matrix**. Clearly, then, when the identity matrix acts on a vector, neither the direction or magnitude is changed.
- A matrix that rotates every vector in $\mathbb{R}^2$ through a fixed angle θ is called a **rotation matrix**. In this case the direction changes, but not the magnitude. (Of course the direction doesn't change if $\theta = 0°$ and, in some sense, if $\theta = 180°$. In the second case, even though the direction is opposite, the resulting vector is still just a scalar multiple of the original.)
- For most matrices there are certain vectors, called **eigenvectors** whose directions don't change (other than perhaps reversing) when acted on by by the matrix under consideration. In those cases, the effect of multiplying such a vector by the matrix is the same as multiplying the vector by a scalar. This has very useful applications.

In the following exercises you will see rotation matrices and eigenvectors, along with some other matrices that do interesting things to vectors geometrically.

Section 3.2 Exercises

For the following exercises you will be multiplying each of several vectors by a given matrix and trying to see what the matrix does to the vectors. This can be pretty tedious by hand, so I would suggest that you use the UCSMP Polygon Plotter that you can link to from the class web page (or find with a web search for "UCSMP polygon plotter"). You will need to enter each vector as a position vector from the origin, and then transform it by the transformation matrix you are working with. Here's how you do all that:

- Under "Enter New" you are asked "How Many Points". Enter 2, meaning you are creating a polygon with only two vertices (a line segment!).
- Below that you are to describe your polygon as a "matrix." The first column should be zeros, and the second column should be the components of your vector. (The first column is the coordinates of the initial point, and the second column is the coordinates of the terminal point.)
- Once you put the values in for the vector, click "Enter" and you should see your vector, in red.

- Enter your transformation matrix. For the ones with with entries containing roots, use their decimal representations rounded to the thousandth's place. **Fractions may be entered as they are**.

- Once you have entered the transformation matrix, click "Transform!" You will see the result of the transformation as a vector in black.

- If you now look below "Select Polygon:" you will see your vector under "Preimage: AB," and the result of the transformation as "Image: A'B'."

- To multiply another vector by the same transformation, enter the new vector and click "Enter" again, followed by "Transform!"

- To repeat all this with a different transformation matrix, click "Clear Grid" at the bottom and start over.

1. Let $A = \begin{bmatrix} 3 & -1 \\ -4 & 0 \end{bmatrix}$, $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\vec{\mathbf{w}} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

   (a) Find $A\,\vec{\mathbf{u}}$, $A\,\vec{\mathbf{v}}$ and $A\,\vec{\mathbf{w}}$ **by hand**. Check your answers with the polygon plotter.

   (b) Plot and label $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$, $A\,\vec{\mathbf{u}}$, $A\,\vec{\mathbf{v}}$ and $A\,\vec{\mathbf{w}}$ on one $\mathbb{R}^2$ coordinate grid. Label each vector by putting its name near its tip.

   You should not see any special relationship between the vectors $\vec{\mathbf{x}}$ and $A\,\vec{\mathbf{x}}$ (where $\vec{\mathbf{x}}$ is to represent any vector) here.

2. Let $B = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $\vec{\mathbf{u}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $\vec{\mathbf{w}} = \begin{bmatrix} 3 \\ 4.5 \end{bmatrix}$.

   (a) Plot and label $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$, $B\,\vec{\mathbf{u}}$, $B\,\vec{\mathbf{v}}$ and $B\,\vec{\mathbf{w}}$ on one $\mathbb{R}^2$ coordinate grid. Label each vector by putting its name near its tip.

   (b) You should be able to see that B does not seem to change the length of a vector. To verify this, find $\|\,\vec{\mathbf{w}}\,\|$ and $\|\,B\,\vec{\mathbf{w}}\,\|$ to the nearest hundredth.

   (c) What does the matrix B seem to do to every vector? **Think about the two attributes of any vector, direction and magnitude.**

   (d) The entries of B should look familiar to you. What is special about $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$?

3. Let $C = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}$, $\vec{\mathbf{u}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $\vec{\mathbf{w}} = \begin{bmatrix} 4.5 \\ 6 \end{bmatrix}$.

   (a) Plot and label $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$, $C\,\vec{\mathbf{u}}$, $C\,\vec{\mathbf{v}}$ and $C\,\vec{\mathbf{w}}$ on one $\mathbb{R}^2$ coordinate grid.

   (b) What does the matrix C seem to do to every vector? (Does the magnitude change? Does the direction change?)

(c) Try C times $\begin{bmatrix} -4 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$. Hmmm...

(d) Can you see the role of the entries of the matrix here?

4. Let $D = \begin{bmatrix} \frac{7}{25} & \frac{24}{25} \\ \frac{24}{25} & -\frac{7}{25} \end{bmatrix}$, $\vec{\mathbf{u}} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $\vec{\mathbf{w}} = \begin{bmatrix} 4.5 \\ 6 \end{bmatrix}$.

   (a) Plot and label $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$, $D\,\vec{\mathbf{u}}$, $D\,\vec{\mathbf{v}}$ and $D\,\vec{\mathbf{w}}$ on one $\mathbb{R}^2$ coordinate grid.

   (b) What does the matrix D seem to do to every vector? (Does the magnitude change? Does the direction change?)

   (c) Try D times $\begin{bmatrix} -4 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$. Hmmm...

   (d) Can you see the role of the entries of the matrix here?

5. Again let $A = \begin{bmatrix} 3 & -1 \\ -4 & 0 \end{bmatrix}$, (see Exercise 1) but let

   $\vec{\mathbf{u}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{\mathbf{w}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

   (a) Plot and label $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$, $A\,\vec{\mathbf{u}}$, $A\,\vec{\mathbf{v}}$ and $A\,\vec{\mathbf{w}}$ on one $\mathbb{R}^2$ coordinate grid.

   (b) For one of the vectors, there should be no apparent relationship between the vector and the result when it is multiplied by the matrix. Discuss what happened to the direction and magnitude of each of the other two vectors when the matrix acted on it.

   (c) Pick one of your two vectors for which something special happened and multiply it by three, and multiply the result by A; what is the effect of multiplying by A in this case?

   (d) Pick the other special vector, multiply it by five, then by A. What effect does multiplying by A have on the vector?

3.4 Multiplying Matrices

Performance Criteria:

3. (j) Know when two matrices can be multiplied, and know that matrix multiplication is not necessarily commutative. Multiply two matrices "by hand."

When two matrices have appropriate sizes they can be multiplied by a process you are about to see. Although the most reliable way to multiply two matrices and get the correct result is with a calculator or computer software, *it is very important that you get quite comfortable with the way that matrices are multiplied*. That will allow you to better understand certain conceptual things you will encounter later.

The process of multiplying two matrices is a bit clumsy to describe, but I'll do my best here. First I will try to describe it informally, then I'll formalize it with a definition based on some special notation. To multiply two matrices we just multiply each row of the first with each column of the second as we did when multiplying a matrix times a vector, with the results becoming the elements of the second matrix. Here is an informal description of the process:

(1) Multiply the first row of the first matrix times the first column of the second. The result is the (1,1) entry (first row, first column) of the product matrix.

(2) Multiply the first row of the first matrix times the second column of the second. The result is the (1,2) entry (first row, second column) of the product matrix.

(3) Continue multiplying the first row of the first matrix times each column of the second to fill out the first row of the product matrix, stopping after multiplying the first row of the first matrix times the last column of the second matrix.

(4) Begin filling out the second row of the product matrix by multiplying the *second* row of the first matrix times the first column of the second matrix to get the (2,1) entry (second row, first column) of the product matrix.

(5) Continue multiplying the second row of the first matrix times each column of the second until the second row of the product matrix is filled out.

(6) Continue multiplying each row of the first matrix times each column of the second until the last row of the first has been multiplied times the last column of the second, at which point the product matrix will be complete.

Note that for this to work the number of columns of the first matrix must be equal the number of rows of the second matrix. Let's look at an example.

◇ **Example 3.4(a):** For $A = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 7 & 3 \end{bmatrix}$, find the product AB.

Video Example

Solution:

$$AB = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 7 & 3 \end{bmatrix} = \begin{bmatrix} -5(1)+1(7) & -5(2)+1(3) \\ 0(1)+4(7) & 0(2)+4(3) \\ 2(1)+(-3)(7) & 2(2)+(-3)(3) \end{bmatrix} = \begin{bmatrix} 2 & -7 \\ 28 & 12 \\ -19 & -5 \end{bmatrix}$$

In order to make a formal definition of matrix multiplication, we need to remember the special notation from Section 3.2 for rows and columns of a matrix. Given a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

we refer to, for example the third row as $\vec{\mathbf{a}}_{3*}$. Here the first subscript 3 indicates that we are considering the third row, and the $*$ indicates that we are taking the elements from the third row *in all columns*. Therefore $\vec{\mathbf{a}}_{3*}$ refers to a $1 \times n$ matrix. Similarly, $\vec{\mathbf{a}}_{*2}$ is the vector that is the second column of A. So we have

$$\vec{\mathbf{a}}_{3*} = \begin{bmatrix} a_{31} & a_{32} & \cdots & a_{3n} \end{bmatrix} \qquad\qquad \vec{\mathbf{a}}_{*2} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{2m} \end{bmatrix}$$

A $1 \times n$ matrix like $\vec{\mathbf{a}}_{3*}$ can be thought of like a vector; in fact, we sometimes call such a matrix a **row vector**. Note that the transpose of such a vector is a column vector. We then define a product like product $\vec{\mathbf{a}}_{i*}\vec{\mathbf{b}}_{*j}$ by

$$\vec{\mathbf{a}}_{i*}\vec{\mathbf{b}}_{*j} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

This is the basis for the following formal definition of the product of two matrices.

DEFINITION 3.4.1: **Matrix Multiplication**

Let A be an $m \times n$ matrix whose *rows* are the vectors $\vec{\mathbf{a}}_{1*}, \vec{\mathbf{a}}_{2*}, ..., \vec{\mathbf{a}}_{m*}$ and let B be an $n \times p$ matrix whose *columns* are the vectors $\vec{\mathbf{b}}_{*1}, \vec{\mathbf{b}}_{*2}, ..., \vec{\mathbf{b}}_{*p}$. Then AB is the $m \times p$ matrix

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} \vec{\mathbf{a}}_{1*} \\ \vec{\mathbf{a}}_{2*} \\ \vdots \\ \vec{\mathbf{a}}_{m*} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{b}}_{*1} & \vec{\mathbf{b}}_{*2} & \vec{\mathbf{b}}_{*3} & \cdots & \vec{\mathbf{b}}_{*p} \end{bmatrix}$$

$$= \begin{bmatrix} \vec{\mathbf{a}}_{1*}\vec{\mathbf{b}}_{*1} & \vec{\mathbf{a}}_{1*}\vec{\mathbf{b}}_{*2} & \cdots & \vec{\mathbf{a}}_{1*}\vec{\mathbf{b}}_{*p} \\ \vec{\mathbf{a}}_{2*}\vec{\mathbf{b}}_{*1} & \vec{\mathbf{a}}_{2*}\vec{\mathbf{b}}_{*2} & \cdots & \vec{\mathbf{a}}_{2*}\vec{\mathbf{b}}_{*p} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{\mathbf{a}}_{m*}\vec{\mathbf{b}}_{*1} & \vec{\mathbf{a}}_{m*}\vec{\mathbf{b}}_{*2} & \cdots & \vec{\mathbf{a}}_{m*}\vec{\mathbf{b}}_{*p} \end{bmatrix}$$

For the above computation to be possible, products in the last matrix.This implies that the number of columns of A must equal the number of rows of B.

◇ **Example 3.4(b):** For $C = \begin{bmatrix} -5 & 1 & -2 \\ 7 & 0 & 4 \\ 2 & -3 & 6 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -7 & 0 \\ 5 & 2 & 3 \end{bmatrix}$, find CD and DC.

Solution:

$$CD = \begin{bmatrix} -5 & 1 & -2 \\ 7 & 0 & 4 \\ 2 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -3 & -7 & 0 \\ 5 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -5-3-10 & -10-7-4 & 5+0-6 \\ 7+0+20 & 14+0+8 & -7+0+12 \\ 2+9+30 & 4+21+12 & -2+0+18 \end{bmatrix}$$

$$= \begin{bmatrix} -18 & -21 & -1 \\ 27 & 22 & 5 \\ 41 & 37 & 16 \end{bmatrix}$$

$$DC = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -7 & 0 \\ 5 & 2 & 3 \end{bmatrix} \begin{bmatrix} -5 & 1 & -2 \\ 7 & 0 & 4 \\ 2 & -3 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 4 & 0 \\ -34 & -3 & -22 \\ -5 & -4 & 16 \end{bmatrix}$$

We want to notice in the last example that $CD \neq DC$! This illustrates something very important:

Matrix multiplication is not necessarily commutative! That is, given two matrices A and B, it is not necessarily true that $AB = BA$. It is possible, but is not "usually" the case. In fact, one of AB and BA might exist and the other not.

This is not just a curiosity; the above fact will have important implications in how certain computations are done. The next example, along with Example 3.4(a), shows that one of the two products might exist and the other not.

◇ **Example 3.4(c):** For the same matrices $A = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 7 & 3 \end{bmatrix}$ from Example 3.4(a), find the product BA.

Solution: When we try to multiply $BA = \begin{bmatrix} 1 & 2 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}$ it is not even possible. We can't find the dot product of a row of B with a column of A because, as vectors, they don't have the same number of components. Therefore the product BA does not exist.

◇ **Example 3.4(d):** For $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 7 & 3 \end{bmatrix}$, find the products I_2B, BI_2, CB and BC.

Solution:

$$I_2B = BI_2 = \begin{bmatrix} 1 & 2 \\ 7 & 3 \end{bmatrix}, \qquad CB = BC = \begin{bmatrix} 3 & 6 \\ 21 & 9 \end{bmatrix}$$

The notation I_2 here means the 2×2 **identity matrix**. Note that when it is multiplied by another matrix A *on either side* the result is just the matrix A.

Let's take a minute to think a bit more about the idea of an "identity." In the real numbers we say zero is the **additive identity** because adding it to any real number a does not change the value of the number:

$$a + 0 = 0 + a = a$$

Similarly, the number one is the multiplicative identity:

$$a \times 1 = 1 \times a = a$$

Here the symbol $\times$ is just multiplication of real numbers. When we talk about an identity matrix, we are talking about a *multiplicative* identity, like the number one. There is no confusion because even though there are matrices that could be considered to be additive identities, they are not useful, so we don't consider them. When the size of the identity matrix is clear from the context, which is almost always the case, we omit the subscript and just write I.

There are many other special and/or interesting things that can happen when multiplying two matrices. Here's an example that shows that we can take powers of a matrix if it is a square matrix.

◇ **Example 3.4(e):** For the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 8 & 5 \\ 6 & -4 & -3 \end{bmatrix}$, find A^2 and A^3.

Solution:

$$A^2 = AA = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 8 & 5 \\ 6 & -4 & -3 \end{bmatrix}\begin{bmatrix} 3 & 1 & -1 \\ -2 & 8 & 5 \\ 6 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 15 & 5 \\ 8 & 42 & 27 \\ 8 & -14 & -17 \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 8 & 5 \\ 6 & -4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 15 & 5 \\ 8 & 42 & 27 \\ 8 & -14 & -17 \end{bmatrix} = \begin{bmatrix} 3 & 101 & 59 \\ 102 & 236 & 121 \\ -50 & -36 & - \end{bmatrix}$$

In Section 3.1 we saw that a scalar times a matrix is defined by multiplying each entry of the matrix by the scalar. With a little thought the following should be clear:

> THEOREM 3.4.2
>
> Let A and B be matrices for which the product AB is defined, and let c be any scalar. Then
> $$c(AB) = (cA)B = A(cB)$$

Note this carefully - when multiplying a product of two matrices by a scalar, we can instead multiply *one or the other, but NOT BOTH* of the two matrices by the scaler, then multiply the result with the remaining matrix.

Although one can do a great deal of study of matrices themselves, linear algebra is primarily concerned with the action of matrices on vectors. The following simple result is extremely important conceptually:

> THEOREM 3.4.3
>
> Let A and B be matrices and $\vec{\mathbf{x}}$ a vector. Assuming that all the indicated operations below are defined (possible), then
> $$(AB)\vec{\mathbf{x}} = A(B\vec{\mathbf{x}})$$

The following illustrates the difference between $(AB)\vec{\mathbf{x}}$ and $A(B\vec{\mathbf{x}})$ from a computational standpoint.

◇ **Example 3.4(g):** For the matrices $A = \begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ 7 & 0 \end{bmatrix}$ and the vector $\vec{\mathbf{x}} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$, find $(AB)\vec{\mathbf{x}}$ and $A(B\vec{\mathbf{x}})$.

Solution: Be sure to note the difference between the how the two calculations are performed, along with the fact that the resluts are the same:

$$(AB)\vec{\mathbf{x}}=\left(\begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 7 & 0 \end{bmatrix}\right)\begin{bmatrix} 3 \\ -6 \end{bmatrix}=\begin{bmatrix} -3 & -3 \\ 27 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ -6 \end{bmatrix}=\begin{bmatrix} 9 \\ 45 \end{bmatrix}$$

$$A(B\vec{\mathbf{x}})=\begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}\left(\begin{bmatrix} 4 & -3 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -6 \end{bmatrix}\right)=\begin{bmatrix} 1 & -1 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 30 \\ 21 \end{bmatrix}=\begin{bmatrix} 9 \\ 45 \end{bmatrix}$$

Let's now continue the analogy between functions and multiplication of a vector by a matrix. Consider the functions

$$f(x)=x^2 \qquad \text{and} \qquad g(x)=2x-1.$$

Suppose that we wanted to apply g to the number three, and then apply f to the result. We show this symbolically by

$$f\big[g(3)\big]=f\big[2(3)-1\big]=f[5]=25.$$

We can form a new function called the **composition** of f and g, $f\circ g$. This function is *defined* for any value of x by

$$(f\circ g)(x)=f\big[g(x)\big].$$

In the case of our particular f and g the composition is

$$(f\circ g)(x)=f\big[g(x)\big]=f[2x-1]=(2x-1)^2=4x^2-4x+1,$$

and

$$(f\circ g)(3)=4(3)^2-4(3)+1=36-12+1=25,$$

showing that $f\circ g$ does indeed act on the number three just as f and g did in sequence. Let's reiterate - $f\circ g$ is a *single* new function that is equivalent to performing g followed by f, *in that order*. If we were to perform the two functions on the number three, but in the opposite order, we would get

$$g\big[f(3)\big]=g\big[3^2\big]=g[9]=2(9)-1=17.$$

We also see that

$$(g\circ f)(x)=g\big[f(x)\big]=g[x^2]=2x^2-1,$$

so *the functions $f\circ g$ and $g\circ f$ are not the same!*

Now Theorem 3.4.3 tells us that, for two matrices A and B and any vector $\vec{\mathbf{x}}$,

$$(AB)\vec{\mathbf{x}}=A(B\vec{\mathbf{x}})$$

when all of the operations are defined. Here A and B can be thought of like the functions f and g above, except they act on vectors rather than numbers. The product AB is like the composition $f\circ g$ - it is a single matrix whose action $\vec{\mathbf{x}}$ is equivalent to B acting on $\vec{\mathbf{x}}$, and then A acting on the result $B\vec{\mathbf{x}}$. The fact that CD is not usually equal to DC for two matrices C and D for which both CD and DC exist (see Example 3.4(b)) is analogous to the fact that the two compositions $f\circ g$ and $g\circ f$ of two functions f and g are not generally the same.

Section 3.4 Exercises To Solutions

1. Multiply (a) $\begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 5 & -1 \end{bmatrix}$ and (b) $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} -2 & 4 & 1 \\ 3 & -5 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ **by hand**.

2. For the following matrices, there are *THIRTEEN* multiplications possible, *including squaring* some of the matrices. Find and do as many of them do as many of them as you can. When writing your answers, tell which matrices you multiplied to get any particular answer. For example, it *IS* possible to multiply A times B (how about B times A?), and you would then write

$$AB = \begin{bmatrix} -10 & 0 & 25 \\ -14 & 21 & -4 \end{bmatrix}$$

to give your answer. Now you have twelve left to find and do.

$$A = \begin{bmatrix} 0 & 5 \\ -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -7 & 3 \\ -2 & 0 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} -5 \\ 4 \\ -7 \end{bmatrix}$$

$$D = \begin{bmatrix} 6 & 0 & 3 \\ -5 & 4 & 2 \\ 1 & 1 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 5 & -1 & 2 \end{bmatrix} \qquad F = \begin{bmatrix} 2 & -1 \\ 6 & 9 \end{bmatrix}$$

3. Fill in the blanks: $\begin{bmatrix} -5 & 1 & 3 \\ 2 & 4 & 0 \\ 1 & -1 & -6 \end{bmatrix}\begin{bmatrix} 6 & 0 & -1 \\ -5 & 7 & 2 \\ -4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} * & * & * \\ \underline{\qquad} & * & * \\ * & \underline{\qquad} & * \end{bmatrix}$

4. Suppose that $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & & \\ a_{31} & & \ddots & \\ \vdots & & & \end{bmatrix}$ is a 5×5 matrix. Write an expression for the third row, second column entry of A^2.

5. In the previous section you found that the matrix $P = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}$ projects every vector in $\mathbb{R}^2$ onto the line through the origin and the point $(4,3)$. If we wish to calculate P^2 we can apply Theorem 3.4.2 to factor $\frac{1}{25}$ out of each copy of the matrix, multiply the resulting matrices with integer entries, then multiply the product of the two $\frac{1}{25}$ scalars back in at the end. Do this using your calculator for multiplying numbers, but not the actual matrix multiplication, and reduce the entries when you are done. The result may surprise you a bit!

6. Let $A = \begin{bmatrix} -5 & 1 \\ 0 & 4 \\ 2 & -3 \end{bmatrix}$.

(a) Give A^T, the transpose of A.

(b) Find $A^T A$ and AA^T. Are they the same (equal)?

(c) Your answers to (b) are special in two ways. What are they? (What I'm looking for here is two of the special types of matrices described in Section 3.1.)

3.5 Inverse Matrices

Performance Criteria:

3. (k) Determine whether two matrices are inverses without finding the inverse of either.

   (l) Find the inverse of a 2×2 matrix using the formula. Find the inverse of a matrix using the Gauss-Jordan method. Describe the Gauss-Jordan method for finding the inverse of a matrix.

   (m) Solve a system of equations using an inverse matrix. Describe how to use an inverse matrix to solve a system of equations.

Inverse Matrices

Let's begin with an example!

◇ **Example 3.5(a):** Find AC and CA for $A = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$.

Solution:

$$AC = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad CA = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We see that $AC = CA = I_2$!

Now let's remember that the identity matrix is like the number one for multiplication of numbers. Note that, for example, $\frac{1}{5} \cdot 5 = 5 \cdot \frac{1}{5} = 1$. This is exactly what we are seeing in the above example. We say the numbers 5 and $\frac{1}{5}$ are **multiplicative inverses**, and we say that the matrices A and C above are inverses of each other.

Definition 3.5.1 Inverse Matrices

Suppose that for matrices A and B we have $AB = BA = I$, with the size of the identity being the same in both cases. Then we say that A and B are **inverse matrices**.

Notationally we write $B = A^{-1}$ or $A = B^{-1}$, and we will say that A and B are **invertible**. Note that in order for us to be able to do both multiplications AB and BA, both matrices must be square and of the same dimensions. It also turns out that that to test two square matrices to see if they are inverses we only need to multiply them in one order:

Theorem 3.5.2 Test for Inverse Matrices

To test two *square* matrices A and B to see if they are inverses, compute AB. If it is the identity, then the matrices are inverses.

Here are a few notes about inverse matrices:

- Not every square matrix has an inverse, but "many" do. If a matrix does have an inverse, it is said to be **invertible**.
- The inverse of a matrix is unique, meaning there is only one.
- Matrix multiplication *IS* commutative for inverse matrices.

Two questions that should be occurring to you now are

1) How do we know whether a particular matrix has an inverse?
2) If a matrix does have an inverse, how do we find it?

There are a number of ways to answer the first question; here is one:

<u>THEOREM 3.5.3</u> **Test for Invertibility of a Matrix**

A square matrix A is invertible if, and only if, *rref* $(A) = I$.

Finding Inverse Matrices

Here is the answer to the second question above in the case of a 2×2 matrix:

<u>THEOREM 3.5.4</u> **Inverse of a 2×2 Matrix**

The inverse of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

◇ **Example 3.5(b):** Find the inverse of $A = \begin{bmatrix} -2 & 7 \\ 1 & -5 \end{bmatrix}$.

Solution:

$$A^{-1} = \frac{1}{(-2)(-5) - (1)(7)} \begin{bmatrix} -5 & -7 \\ -1 & -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -5 & -7 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} & -\frac{7}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Before showing how to find the inverse of a larger matrix we need to go over the idea of **augmenting** a matrix with a vector or another matrix. To augment a matrix A with a matrix B, both matrices must have the same number of rows. A new matrix, denoted $[\,A\,|\,B\,]$ is formed as follows: the first row of $[A|B]$ is the first row of A followed by the first row of B, and every other row in $[A|B]$ is formed the same way.

◇ **Example 3.5(c):** Let $A = \begin{bmatrix} -5 & 1 & -2 \\ 7 & 0 & 4 \\ 2 & -3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 9 & 1 \\ -1 & 8 \\ -6 & -3 \end{bmatrix}$ and $\vec{\mathbf{x}} = \begin{bmatrix} -7 \\ 10 \\ 4 \end{bmatrix}$.

Give the augmented matrices $[\, A \mid \vec{\mathbf{x}} \,]$ and $[A \,|\, B]$.

Solution: $[\, A \mid \vec{\mathbf{x}} \,] = \left[\begin{array}{rrr|r} -5 & 1 & -2 & -7 \\ 7 & 0 & 4 & 10 \\ 2 & -3 & 6 & 4 \end{array}\right]$, $\quad [A \,|\, B] = \left[\begin{array}{rrr|rr} -5 & 1 & -2 & 9 & 1 \\ 7 & 0 & 4 & -1 & 8 \\ 2 & -3 & 6 & 6 & -3 \end{array}\right]$

Gauss-Jordan Method for Finding Inverse Matrices

Let A be an $n \times n$ invertible matrix and I_n be the $n \times n$ identity matrix. Form the augmented matrix $[\, A \mid I_n \,]$ and find *rref* $([\, A \mid I_n \,]) = [\, I_n \mid B \,]$. (The result of row-reduction will have this form.) Then $B = A^{-1}$.

◇ **Example 3.5(d):** Find the inverse of $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$, if it exists.

Solution: We begin by augmenting with the 3×3 identity: $[A \,|\, I_3] = \left[\begin{array}{rrr|rrr} 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$.

Row reducing then gives $\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 2 & 3 & -3 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & -4 & -6 & 7 \end{array}\right]$, so $A^{-1} = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ -4 & -6 & 7 \end{bmatrix}$.

The above example is a bit unusual; the inverse of a randomly generated matrix will usually contain fractions.

◇ **Example 3.5(e):** Find the inverse of $B = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 0 & 2 & -2 \end{bmatrix}$, if it exists.

Solution: We compute

$$[\, B \mid I_n \,] = \left[\begin{array}{rrr|rrr} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \overset{rref}{\Longrightarrow} \left[\begin{array}{rrr|rrr} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & -1 & \frac{3}{2} \end{array}\right].$$

Because the left side of the reduced matrix is not the identity, the matrix B is not invertible.

$\diamond$ **Example 3.5(f):** Find a matrix B such that $AB = C$, where $A = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -3 \\ -2 & 3 \end{bmatrix}$.

Solution: Note that if we multiply both sides of $AB = C$ on the left by A^{-1} we get $A^{-1}AB = A^{-1}C$. But $A^{-1}AB = IB = B$, so we have

$$B = A^{-1}C = \begin{bmatrix} -1 & -1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & -3 \end{bmatrix}$$

Inverse Matrices and Systems of Equations

Let's consider a simple algebraic equation of the form $ax = b$, where a and b are just constants. If we multiply both sides on the left by $\frac{1}{a}$, the multiplicative inverse of a, we get $x = \frac{1}{a} \cdot b$. for example,

$$\begin{aligned} 3x &= 5 \\ \tfrac{1}{3}(3x) &= \tfrac{1}{3} \cdot 5 \\ \left(\tfrac{1}{3} \cdot 3\right) x &= \tfrac{5}{3} \\ 1x &= \tfrac{5}{3} \\ x &= \tfrac{5}{3} \end{aligned}$$

The following shows how an inverse matrix can be used to solve a system of equations by exactly the same idea:

$$\begin{aligned} A\vec{\mathbf{x}} &= \vec{\mathbf{b}} \\ A^{-1}(A\vec{\mathbf{x}}) &= A^{-1}\vec{\mathbf{b}} \\ (A^{-1}A)\vec{\mathbf{x}} &= A^{-1}\vec{\mathbf{b}} \\ I\vec{\mathbf{x}} &= A^{-1}\vec{\mathbf{b}} \\ \vec{\mathbf{x}} &= A^{-1}\vec{\mathbf{b}} \end{aligned}$$

Note that this only "works" if A is invertible! The upshot of all this is that when A is invertible the solution to the system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is given by $\vec{\mathbf{x}} = A^{-1}\vec{\mathbf{b}}$. The above sequence of steps shows the details of why this is. Although this may seem more straightforward than row reduction, it is more costly in terms of computer time than row reduction or LU-factorization and can lead to poor results. Therefore it is not used in practice.

$\diamond$ **Example 3.5(g):** Solve the system of equations $\begin{aligned} 5x_1 + 4x_2 &= 25 \\ -2x_1 - 2x_2 &= -12 \end{aligned}$ using an inverse matrix, showing all steps given above.

Solution: The matrix form of the system is $\begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 25 \\ -12 \end{bmatrix}$, and $A^{-1} = -\frac{1}{2}\begin{bmatrix} -2 & -4 \\ 2 & 5 \end{bmatrix}$. A^{-1} can now be used to solve the system:

$$\begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 25 \\ -12 \end{bmatrix}$$

$$-\frac{1}{2}\begin{bmatrix} -2 & -4 \\ 2 & 5 \end{bmatrix}\left(\begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = -\frac{1}{2}\begin{bmatrix} -2 & -4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 25 \\ -12 \end{bmatrix}$$

$$\left(-\frac{1}{2}\begin{bmatrix} -2 & -4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

The solution to the system is $(1, 5)$.

Section 3.5 Exercises

To Solutions

1. Determine whether $A = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$ and $C = \begin{bmatrix} 8 & -4 \\ -3 & 2 \end{bmatrix}$ are inverses, *without actually finding the inverse of either*. Show clearly how you do this.

2. Consider the matrices $A = \begin{bmatrix} 3 & 0 & -1 \\ -1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 5 & 3 \end{bmatrix}$. Find AB, then give two reasons why A and B are not inverses.

3. Consider the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$.

   (a) Apply row reduction (*"by hand"*) to $[\,A \mid I_2\,]$ until you obtain $[\,I_2 \mid B\,]$. That is, find the *reduced* row-echelon form of $[\,A \mid I_2\,]$.

   (b) Find AB and BA.

   (c) What does this illustrate?

4. Assume that you have a system of equations $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ for some invertible matrix A. Show how the inverse matrix is used to solve the system, showing all steps in the process clearly. Check your answer against what is shown at the bottom of page 111.

5. Consider the system of equations $\begin{aligned} 2x_1 - 3x_2 &= 4 \\ 4x_1 + 5x_2 &= 3 \end{aligned}$.

(a) Write the system in matrix times a vector form $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(b) Apply the formula in Theorem 3.5.4 to obtain the inverse matrix A^{-1}. **Show a step or two in how you do this.**

(c) **Demonstrate** that your answer to (b) really is the inverse of A.

(d) Use the inverse matrix to solve the system. **Show ALL steps outlined in Example 3.5(g), and give your answer in exact form.**

(e) Apply row reduction (**"by hand"**) to $[\,A \mid I_2\,]$ until you obtain $[\,I_2 \mid B\,]$. That is, find the *reduced* row-echelon form of $[\,A \mid I_2\,]$. What do you notice about B?

6. Consider the system of equations $\begin{aligned} 5x + 7y &= -1 \\ 2x + 3y &= 4 \end{aligned}$

(a) Write the system in $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ form.

(b) Use Theorem 3.5.4 to find A^{-1}.

(c) Give the matrix that is to be row reduced to find A^{-1} by the Gauss-Jordan method. Then give the reduced row-echelon form obtained using your calculator.

(d) Repeat *EVERY* step of the process for solving $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ using the inverse matrix.

3.6 Determinants and Systems of Equations

Performance Criterion:

3. (n) Find the determinant of a 2×2 or 3×3 matrix by hand. Use a calculator to find the determinant of an $n\times n$ matrix.
   (o) Use the determinant to determine whether a system of equations has a unique solution.
   (p) Determine whether a homogeneous system has more than one solution.
   (q) Use Cramer's rule to solve a system of equations.

Associated with every *square* matrix is a scalar that is called the **determinant** of the matrix, and determinants have numerous conceptual and practical uses. For a square matrix A, the determinant is denoted by $\det(A)$. This notation implies that the determinant is a function that takes a matrix returns a scalar. The determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ written as $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

There is a simple formula for finding the determinant of a 2×2 matrix:

DEFINITION 3.6.1: **Determinant of a 2×2 Matrix**

The determinant of the matrix $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A)=ad-bc$.

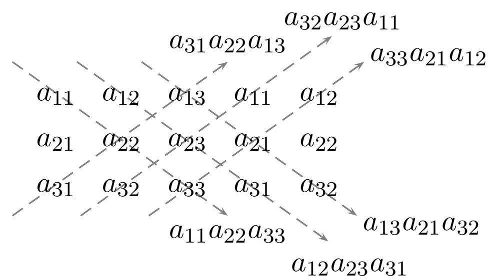
◇ **Example 3.6(a):** Find the determinant of $A=\begin{bmatrix} 5 & 4 \\ -2 & -2 \end{bmatrix}$

$$\det(A)=(5)(-2)-(-2)(4)=-10+8=-2$$

There is a fairly involved method of breaking the determinant of a larger matrix down to where it is a linear combination of determinants of 2×2 matrices, but we will not go into that here. It is called the **cofactor expansion** of the determinant, and can be found in any other linear algebra book, or online. Of course your calculator will find determinants of matrices whose entries are numbers, as will online matrix calculators and various software like *MATLAB*.

Later we will need to be able to find determinants of matrices containing an unknown parameter, and it will be necessary to find determinants of 3×3 matrices. For that reason, we now show a relatively simple method for finding the determinant of a 3×3 matrix. (This will not look simple here, but it is once you are familiar with it.) *This method only works for 3×3 matrices.* To perform this method we begin by augmenting the matrix with its own first two columns, as shown below.

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \Longrightarrow \qquad \begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

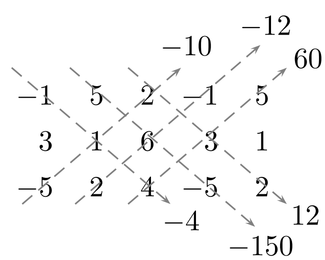


We get the determinant by adding up each of the results of the downward multiplications and then subtracting each of the results of the upward multiplications. This is shown below.

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

◇ **Example 3.6(b):** Find the determinant of $A = \begin{bmatrix} -1 & 5 & 2 \\ 3 & 1 & 6 \\ -5 & 2 & 4 \end{bmatrix}$. Video Example

$$\det \begin{bmatrix} -1 & 5 & 2 \\ 3 & 1 & 6 \\ -5 & 2 & 4 \end{bmatrix} \implies$$



$$\det(A) = (-4) + (-150) + 12 - (-10) - (-12) - 60 = -4 - 150 + 12 + 10 + 12 - 60 = -180$$

In the future we will need to compute determinants like the following.

◇ **Example 3.6(c):** Find the determinant of $B = \begin{bmatrix} 1-\lambda & 0 & 3 \\ 1 & -1-\lambda & 2 \\ -1 & 1 & -2-\lambda \end{bmatrix}$.

$$\begin{aligned} \det(B) &= (1-\lambda)(-1-\lambda)(-2-\lambda) + (0)(2)(-1) + (3)(1)(1) \\ &\qquad - (-1)(-1-\lambda)(3) - (1)(2)(1-\lambda) - (-2-\lambda)(1)(0) \\ &= (-1+\lambda^2)(-2-\lambda) + 3 - 3 - 3\lambda - 2 + 2\lambda \\ &= 2 + \lambda - 2\lambda^2 - \lambda^3 - \lambda - 2 \\ &= -\lambda^3 - 2\lambda^2 \end{aligned}$$

Here is why we care about determinants right now:

Theorem 3.6.2: **Determinants and Invertibility, Systems**

Let A be a square matrix.

(a) A is invertible if, and only if, $\det(A) \neq 0$.

(b) The system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution if, and only if, A is invertible.

(c) If A is not invertible, the system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ will have either no solution or infinitely many solutions.

Recall that when things are "nice" the system $A\,\vec{\mathbf{x}}=\vec{\mathbf{b}}$ can be solved as follows:

$$\begin{aligned}
A\vec{\mathbf{x}} &= \vec{\mathbf{b}}\\
A^{-1}(A\vec{\mathbf{x}}) &= A^{-1}\vec{\mathbf{b}}\\
(A^{-1}A)\vec{\mathbf{x}} &= A^{-1}\vec{\mathbf{b}}\\
I\vec{\mathbf{x}} &= A^{-1}\vec{\mathbf{b}}\\
\vec{\mathbf{x}} &= A^{-1}\vec{\mathbf{b}}
\end{aligned}$$

In this case the system will have the unique solution $\vec{\mathbf{x}} = A^{-1}\vec{\mathbf{b}}$. (When we say unique, we mean only one.) *If A is not invertible, the above process cannot be carried out, and the system will not have a single unique solution*. In that case there will either be no solution or infinitely many solutions.

We previously discussed the fact that the above computation is analogous to the following one involving simple numbers and an unknown number x:

$$\begin{aligned}
3x &= 5\\
\tfrac{1}{3}(3x) &= \tfrac{1}{3}\cdot 5\\
\left(\tfrac{1}{3}\cdot 3\right)x &= \tfrac{5}{3}\\
1x &= \tfrac{5}{3}\\
x &= \tfrac{5}{3}
\end{aligned}$$

Now let's consider the following two equations, of the same form $ax = b$ but for which $a = 0$:

$$0x = 5 \qquad\qquad 0x = 0$$

We first recognize that we can't do as before and multiply both sides of each by $\frac{1}{0}$, since that is undefined. The first equation has no solution, since there is no number x that can be multiplied by zero and result in five! In the second case, every number is a solution, so the system has infinitely many solutions. *These equations are analogous to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ when $\det(A) = 0$. The one difference is that $A\,\vec{\mathbf{x}}=\vec{\mathbf{b}}$ can have infinitely many solutions even when $\vec{\mathbf{b}}$ is NOT the zero vector.*

Homogenous systems are important and will come up in a couple places in the future, but there is not a whole lot that can be said about them! A **homogeneous system** is one of the form $A\,\vec{\mathbf{x}}= \mathbf{0}$. With a tiny bit of thought this should be clear: *Every homogenous system has at least one solution - the zero vector!* Given the Theorem 3.6.2, if A is invertible (so $\det(A) \neq 0$), that is the only solution. If A is not invertible there will be infinitely many solutions, the zero vector being just one of them.

Cramer's Rule

Cramer's rule is a method for finding solutions to systems of equations. It is not generally used for solving large systems with numerical solutions, but it is used sometimes for solving smaller systems containing an unknown parameter.

Let's consider the system of equations $\begin{array}{rcl} 5x+3y &=& 1 \\ 4x+2y &=& 2 \end{array}$. To solve for x we can multiply the first equation by 2 and the second equation by -3 to obtain

$$\begin{array}{rcr} 10x+6y &=& 2 \\ -12x-6y &=& -6 \end{array}.$$

We then add the two equations to obtain the equation $-2x=-4$, so $x=2$. Note that we could instead multiply the first equation by 2, the second by 3, and then *subtract* the two equations to get

$$\begin{array}{rcl} 5x+3y=1 & \Longrightarrow & 10x+6y=2 \\ 4x+2y=2 & \Longrightarrow & \underline{12x+6y=6} \\ & & \quad -2x=-4, \end{array}$$

solving the resulting equation to get $x=2$. Using this process on a general system of two equations we get

$$\begin{array}{rcl} ax+by=e & \Longrightarrow & adx+bdy=ed \\ cx+dy=f & \Longrightarrow & \underline{bcx+bdy=bf} \\ & & adx-bcx=ed-bf \end{array}$$

$$\Longrightarrow \quad (ad-bc)x=ed-bf \quad \Longrightarrow \quad x=\frac{ed-bf}{ad-bc} \tag{1}$$

We can also multiply the top equation by c and the bottom equation by a and then subtract the top equation from the bottom one to get

$$\begin{array}{rcl} ax+by=e & \Longrightarrow & acx+bcy=ce \\ cx+dy=f & \Longrightarrow & \underline{acx+ady=af} \\ & & ady-bcy=af-ce \end{array}$$

$$\Longrightarrow \quad (ad-bc)y=af-ce \quad \Longrightarrow \quad y=\frac{af-ce}{ad-bc} \tag{2}$$

Note the matrix form of the system of equations: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} e \\ f \end{bmatrix}$. If we look carefully at the expressions for x and y that were obtained in (1) and (2) above, we see that they both have the same denominator $ad-bc$, the determinant of the coefficient matrix! How about the numerators of the expressions for x and y? We can see that they are the determinants

$$\begin{vmatrix} e & b \\ f & d \end{vmatrix} \qquad \text{and} \qquad \begin{vmatrix} a & e \\ c & f \end{vmatrix},$$

respectively. This leads us to

THEOREM 3.6.3: **Cramer's Rule**

The system of two equations in two unknowns with standard and matrix forms

$$\begin{array}{rcl} ax+by & = & e \\ cx+dy & = & f \end{array} \qquad \text{and} \qquad \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\left[\begin{array}{c} x \\ y \end{array}\right]=\left[\begin{array}{c} e \\ f \end{array}\right].$$

has solution given by

$$x=\frac{\left|\begin{array}{cc} e & b \\ f & d \end{array}\right|}{\left|\begin{array}{cc} a & b \\ c & d \end{array}\right|} \qquad \text{and} \qquad y=\frac{\left|\begin{array}{cc} a & e \\ c & f \end{array}\right|}{\left|\begin{array}{cc} a & b \\ c & d \end{array}\right|}.$$

when the determinant of the coefficient matrix is not zero.

We reiterate: the denominators of both of these fractions are the determinant of the coefficient matrix. The numerator for finding x is the determinant of the matrix obtained when the coefficient matrix has its first column (the coefficients of x) replaced with the numbers to the right of the equal signs. Let's see an example:

◇ **Example 3.6(d):** Use Cramer's rule to solve the system of equations $\begin{array}{rcl} 5x+3y & = & 1 \\ 4x+2y & = & 2 \end{array}$.

Solution: Cramer's Rule gives us

$$x=\frac{\left|\begin{array}{cc} 1 & 3 \\ 2 & 2 \end{array}\right|}{\left|\begin{array}{cc} 5 & 3 \\ 4 & 2 \end{array}\right|}=\frac{1\cdot 2-2\cdot 3}{5\cdot 2-4\cdot 3}=\frac{-4}{-2}=2 \qquad y=\frac{\left|\begin{array}{cc} 5 & 1 \\ 4 & 1 \end{array}\right|}{\left|\begin{array}{cc} 5 & 3 \\ 4 & 2 \end{array}\right|}=\frac{5\cdot 2-4\cdot 1}{5\cdot 2-4\cdot 3}=\frac{6}{-2}=-3$$

We conclude by noting that the use of Cramer's rule is not restricted to systems of two equations in two unknowns:

THEOREM 3.6.4: **Cramer's Rule for More Unknowns**

Any system of n equations in n unknowns whose coefficient matrix has nonzero determinant can be solved in the same manner as above. That is, the value of each unknown is obtained by replacing the column of the coefficient matrix corresponding to that unknown with the right hand side vector, then dividing the determinant of the resulting matrix by the determinant of the coefficient matrix.

Section 3.6 Exercises To Solutions

1. Find the determinant of each matrix *by hand*, giving your answer in fraction form.

(a) $A = \begin{bmatrix} 3 & 5 \\ 1 & 3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 2 \\ -1 & 4 \end{bmatrix}$ (c) $C = \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}$

(d) $A = \begin{bmatrix} -1 & -2 \\ 3 & 1 \end{bmatrix}$ (e) $B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ (f) $C = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ (f) $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

2. (a) $\begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$ (b) $\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}$

(c) $\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix}$ (d) $\begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & 0-\lambda & 2 \\ 4 & -2 & 3-\lambda \end{bmatrix}$

3. Explain/show how to use the determinant to determine whether
$$\begin{aligned} x + 3y - 3z &= -5 \\ 2x - \ y + \ z &= -3 \\ -6x + 3y - 3z &= 4 \end{aligned}$$
has a unique solution. **You may use your calculator for finding determinants - be sure to conclude by saying whether or not this particular system has a solution!**

4. Suppose that you hope to solve a system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ of n equations in n unknowns.

   (a) If the determinant of A is zero, what does it tell you about the nature of the solution? (By "the nature of the solution" I mean no solution, a unique solution or infinitely many solutions.)

   (b) If the determinant of A is *NOT* zero, what does it tell you about the nature of the solution?

5. Suppose that you hope to solve a system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ of n equations in n unknowns.

   (a) If the determinant of A is zero, what does it tell you about the nature of the solution? (By "the nature of the solution" I mean no solution, a unique solution or infinitely many solutions.)

   (b) If the determinant of A is *NOT* zero, what does it tell you about the nature of the solution?

6. Use Cramer's Rule to solve each of the following systems of equations. Check your answers by solving with *rref*.

(a) $$\begin{aligned} -2x + 5y &= 13 \\ 4x + 7y &= 25 \end{aligned}$$

(b) $$\begin{aligned} 1x - 3y &= -17 \\ -2x + 5y &= 29 \end{aligned}$$

(c) $$\begin{aligned} 8x - 3y &= 32 \\ 4x - 5y &= 16 \end{aligned}$$

(d) $$\begin{aligned} 1x + 7y &= 48 \\ 4y &= 28 \end{aligned}$$

3.7 Applications: Transformation Matrices, Graph Theory

Performance Criteria:

3. (r) Give the geometric or algebraic representations of the inverse or square of a rotation. Demonstrate that the geometric and algebraic versions are the same

   (s) Give the incidence matrix of a graph or digraph. Given the incidence matrix of a graph or digraph, identify the vertices and edges using correct notation, and draw the graph.

   (t) Determine the number of k-paths from one vertex of a graph to another. Solve problems using incidence matrices.

Rotation, Projection and Reflection Matrices

In the Section 3.3 Exercises you encountered matrices that rotated every vector in $\mathbb{R}^2$ thirty degrees counterclockwise, projected every vector in $\mathbb{R}^2$ onto the line $y = \frac{3}{4}x$, and reflected every vector in $\mathbb{R}^2$ across the line $y = \frac{3}{4}x$. Here are the general formulas for rotation, projection and reflection matrices in $\mathbb{R}^2$:

Rotation Matrix in $\mathbb{R}^2$

For the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and any position vector $\vec{\mathbf{x}}$ in $\mathbb{R}^2$, the product

$A\,\vec{\mathbf{x}}$ is the vector resulting when $\vec{\mathbf{x}}$ is rotated counterclockwise around the origin by the angle θ.

Projection Matrix in $\mathbb{R}^2$

For the matrix $B = \begin{bmatrix} \dfrac{a^2}{a^2+b^2} & \dfrac{ab}{a^2+b^2} \\ \dfrac{ab}{a^2+b^2} & \dfrac{b^2}{a^2+b^2} \end{bmatrix}$ and any position vector $\vec{\mathbf{x}}$ in $\mathbb{R}^2$, the

product $B\,\vec{\mathbf{x}}$ is the vector resulting when $\vec{\mathbf{x}}$ is projected onto the line containing the origin and the point (a, b).

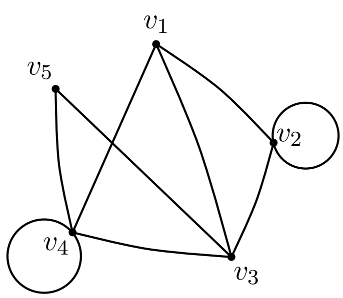
Reflection Matrix in $\mathbb{R}^2$

For the matrix $C = \begin{bmatrix} \dfrac{a^2-b^2}{a^2+b^2} & \dfrac{2ab}{a^2+b^2} \\ \dfrac{2ab}{a^2+b^2} & \dfrac{b^2-a^2}{a^2+b^2} \end{bmatrix}$ and any position vector $\vec{\mathbf{x}}$ in $\mathbb{R}^2$, the

product $C\,\vec{\mathbf{x}}$ is the vector resulting when $\vec{\mathbf{x}}$ is reflected across the line containing the origin and the point (a,b).

Graphs and Digraphs

A **graph** is a set of dots, called **vertices**, connected by segments of lines or curves, called **edges**. An example is shown to the left below. We will usually label each of the vertices with a subscripted v, as shown. Note that a vertex can be connected to itself, as shown by the circles at v_2 and v_4. We can then create a matrix, called an **incidence matrix** to show which pairs of vertices are connected (and which are not). The (i,j) and (j,i) entries of the matrix are one if v_i and v_j are connected by a single edge and a zero if they are not. If $i=j$ the entry is a one if that vertex is connected to itself by an edge, and zero if it is not. You should be able to see this by comparing the graph and and corresponding incidence matrix below. *Note that the incidence matrix is symmetric; that is the case for all incidence matrices of graphs.*



$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

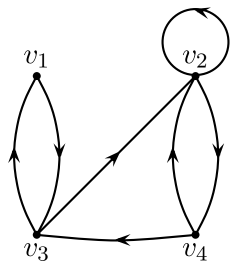
Even though there is no edge from vertex one to vertex five for the graph shown, we can get from vertex one to vertex five via vertex three or vertex four. We call such a "route" a **path**, and we denote the paths by the sequence of vertices, like $v_1v_3v_5$ or $v_1v_4v_5$. These paths, in particular, are called **2-paths**, since they consist of two edges. There are other paths from vertex one to vertex five, like the 3-path $v_1v_2v_3v_5$ and the 4-path $v_1v_2v_2v_3v_5$.

◇ **Example 3.7(a):** Give two more 3-paths from vertex v_1 to vertex v_5.

Solution: $v_1v_3v_4v_5$ is a fairly obvious 3-path from v_1 to vertex v_5. Less obvious is $v_1v_4v_4v_5$, which travels the edge from v_4 to itself.

We should note that when forming a path we are allowed to travel the same edge multiple times, including reversing the direction. Thus, for example, $v_1v_4v_5v_4v_5$ is a 4-path from v_1 to v_5. It is often desired to find all n-paths from one vertex to another, and it can be difficult to determine when all of them have been found. You will find in the exercises a clever way to determine how many n-paths there are from one vertex to another, so we know how many we are looking for. This doesn't necessarily make it any easier to find them, but we can know whether we have them all or not.

In some cases we want the edges of a graph to be "one-way." We indicate this by placing an arrow on each edge, indicating the direction it goes. *We will not put two arrowheads on one edge; if we can travel both ways between two vertices, we will show that by drawing TWO edges between them.* Such a graph is called a **directed graph**, or **digraph** for short. Below is a digraph and its incidence matrix. The (i, j) entry of the incidence matrix is one only if there is a directed edge from v_i to v_j. Of course *the incidence matrix for a digraph need not be symmetric*, since there may be an edge going one way between two vertices but not the other way. Digraphs have incidence matrices as well. Below is a digraph and its incidence matrix.



$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Both the graph and the digraph above are what we call **connected graphs**, meaning that every two vertices are connected by some path (but not necessarily an edge). A graph that is not connected will appear to be two or more separate graphs. All graphs that we will consider will be connected; we will leave further discussion/investigation of graphs and incidence matrices to the exercises.

Section 3.7 Exercises

To Solutions

1. Here we investigate projections.

   (a) Sketch a set of coordinate axes in $\mathbb{R}^2$, with an additional line passing through the origin. Let P be the matrix that projects all vectors onto the line.

   (b) Sketch a position vector $\vec{\mathbf{u}}$ that is on the line. What can you say about the vector $P\vec{\mathbf{u}}$ in this case?

   (c) Sketch a position vector $\vec{\mathbf{v}}$ that is perpendicular to the line. What is $P\vec{\mathbf{v}}$?

   (d) Sketch a third vector $\vec{\mathbf{w}}$ that is not on the line or perpendicular to it. Draw the vector $P\vec{\mathbf{w}}$. Now suppose that we applied P to $P\vec{\mathbf{w}}$ to get $P(P\vec{\mathbf{w}}) = P^2\vec{\mathbf{w}}$. How does that result compare with $P\vec{\mathbf{w}}$? What does this tell us about P^2?

   (e) Discuss P^3, P^4, ..., P^n.

   (f) Suppose that we know $P\vec{\mathbf{x}}$ for some unknown vector $\vec{\mathbf{x}}$. Can we determine $\vec{\mathbf{x}}$? What does this tell us about P^{-1}?

2. And now we investigate reflections.

   (a) Sketch another set of coordinate axes in $\mathbb{R}^2$, with an additional line passing through the origin. Let C be the matrix that reflects all vectors across the line.

   (b) Sketch a position vector $\vec{\mathbf{u}}$ that is on the line. What can you say about the vector $C\vec{\mathbf{u}}$ in this case?

(c) Sketch a position vector $\vec{\mathbf{v}}$ that is perpendicular to the line. What is $C\vec{\mathbf{v}}$?

(d) Sketch a third vector $\vec{\mathbf{w}}$ that is not on the line or perpendicular to it. Draw the vector $C\vec{\mathbf{w}}$. Now suppose that we applied C to $C\vec{\mathbf{w}}$ to get $C(C\vec{\mathbf{w}}) = C^2\vec{\mathbf{w}}$. What does this tell us about C^2?

(e) Discuss C^3, C^4, ..., C^n.

(f) Suppose that we know $C\vec{\mathbf{x}}$ for some unknown vector $\vec{\mathbf{x}}$. Can we determine $\vec{\mathbf{x}}$? What does this tell us about C^{-1}?

Some Trigonometric Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \cos(-\theta) = \cos\theta \qquad \sin(-\theta) = -\sin\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

3. Consider the general rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

   (a) Suppose that we were to apply A to a vector $\vec{\mathbf{x}}$, then apply A again, to the result. Thinking only geometrically (don't do any calculations), give a single matrix B that should have the same effect.

   (b) Find the matrix A^2 algebraically, by multiplying A by itself.

   (c) Use some of the trigonometric facts above to continue your calculations from part (b) until you arrive at matrix B. This of course shows that that $A^2 = B$.

4. Consider again the general rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

   (a) Give a matrix C that should "undo" what A does. Do this thinking only geometrically.

   (b) Find the matrix A^{-1} algebraically, using the formula for the inverse of a 2×2 matrix..

   (c) Use some of the trigonometric facts above to show that $C = A^{-1}$. Do this by starting with C, then modifying it a step at a time to get to A^{-1}.

   (d) Give the transpose matrix A^T. It should look familiar - tell how.

5. Let R_θ be the matrix that rotates all vectors counter-clockwise by the angle θ.

   (a) R_θ^2 is equal to R_ϕ for what angle ϕ, in terms of θ?

   (b) R_θ^{-1} is equal to R_ϕ for what angle ϕ, in terms of θ?

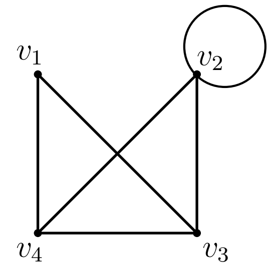
   (c) Give an angle θ for which $R_\theta^3 = I$.

   (d) Give an angle θ for which $R_\theta^{-1} = R_\theta$. That is, what rotation is its own inverse?

6. Use the graph to the right for the following.

   (a) Give all 2-paths from v_3 to v_4.

   (b) Give all three paths from v_1 to v_4. Don't forget that you can follow the same edge more than once, including in opposite directions.



   (c) Give the incidence matrix A for the graph, and give the additional matrices A^2 and A^3.

   (d) Look at the $(3,4)$ and $(4,3)$ entries of A^2. How do they relate to the number of 2-paths from v_3 to v_4?

   (e) Look at the $(1,4)$ and $(4,1)$ entries of A^3. How do they relate to the number of 3-paths from v_1 to v_4? (You won't see the connection here if you didn't find all of the 3-paths from v_1 to v_4. You should have found five.)

   (f) Based on what you observed in parts (d) and (e), how many 3-paths from v_2 to v_3 are there? Give all 3-paths from v_2 to v_3. In this case, figure that you can follow the same edge more than once, including in opposite directions, and that you can travel the edge from v_2 to itself.

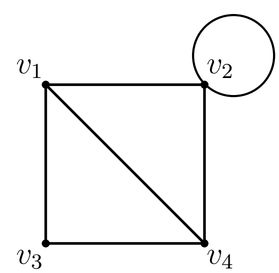
7. (a) Draw the graph with the incidence matrix A shown below, with vertices labeled v_1, v_2, ...

   (b) Draw the directed graph with the incidence matrix B shown below.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

8. Use the graph to the right for the following.

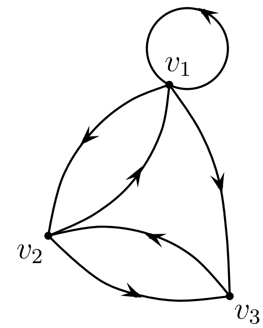
   (a) Give the incidence matrix for the graph; call it A. Find and give A^3 also.



   (b) How many 3-paths from v_1 to v_4 do you expect? Give all of them by listing the vertices the path goes through, *in order and including* v_2 *and* v_4, as done in class. (For example, then, a path from v_4 to v_3, through v_2 would be denoted $v_4v_2v_3$.)

   (c) How many 3-paths are there from v_4 to v_1? What characteristic of the matrix A^3 relates your answer to the number of 3-paths from v_1 to v_4?

9. Use the directed graph to the right for the following.



   (a) Give the incidence matrix; again, call it A.

   (b) The number of n-paths from vertex i to vertex j is given by the (i, j) entry of A^n. How many 4-paths are there from v_1 to v_3? Show how you get your answer.

   (c) Give all 4-paths from v_2 to v_3.

   (d) Give all 3-paths from v_3 to v_2. Is it the same as the number from v_2 to v_3?

B Solutions to Exercises

B.3 Chapter 3 Solutions

Section 3.1 Solutions Back to 3.1 Exercises

1. (a) A is 3×3, B is 3×2, C is 3×4

   (b) $b_{31} = 4$, $c_{23} = 2$

2. (a) all but B, F (b) C, D, G, J (c) C (d) C, E (e) A, C

3. The following are *possible* answers:

   (a) $\begin{bmatrix} 2 & 0 & 0 \\ -5 & 1 & 0 \\ 7 & 3 & -8 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -8 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -5 & 7 \\ -5 & 1 & 3 \\ 7 & 3 & -8 \end{bmatrix}$

   (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 2 & -1 & 5 \\ 0 & 4 & 3 \\ 0 & 0 & -8 \end{bmatrix}$ (f) see (c)

   (g) see (c) (h) see (b)

4. $A^T = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & 7 \\ 5 & -2 & 0 \end{bmatrix}$ $B^T = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & 7 \end{bmatrix}$ $C^T = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & 7 \\ -1 & 2 & 0 \\ 3 & 0 & -2 \end{bmatrix}$

5. $B + D = \begin{bmatrix} 2 & -3 \\ -2 & 3 \\ 5 & 11 \end{bmatrix}$, $B - D = \begin{bmatrix} 0 & 3 \\ -4 & -1 \\ 3 & 3 \end{bmatrix}$, $D - B = \begin{bmatrix} 0 & -3 \\ 4 & 1 \\ -3 & -3 \end{bmatrix}$

6. (a) The matrix is square and symmetric. (b) The matrix is square and symmetric.

   (c) The matrix is square, symmetric, diagonal and both upper and lower triangular.

   (d) The matrices are square, symmetric, diagonal and both upper and lower triangular. They are also called **zero matrices**.

7. (a) $B = \begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$ (b) B is a symmetric matrix.

Section 3.2 Solutions Back to 3.2 Exercises

1. $\begin{bmatrix} 3 \\ -12 \\ 9 \end{bmatrix}$, $\begin{bmatrix} -23 \\ 28 \end{bmatrix}$

2. (a) $\begin{bmatrix} -13 \\ -18 \end{bmatrix}$ (b) $\begin{bmatrix} -5 \\ 18 \end{bmatrix}$ (c) not possible

(d) $\begin{bmatrix} -3 \\ 1 \\ -28 \end{bmatrix}$ (e) $\begin{bmatrix} -22 \\ 4 \\ 0 \\ 27 \end{bmatrix}$ (f) $\begin{bmatrix} 33 \\ 0 \\ -14 \end{bmatrix}$

3. (a) $-4\begin{bmatrix} 3 \\ 5 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (b) $1\begin{bmatrix} 1 \\ 6 \end{bmatrix} + 0\begin{bmatrix} -5 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 2 \\ -4 \end{bmatrix}$

(c) not possible (d) $3\begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} + 5\begin{bmatrix} 6 \\ 3 \\ -4 \end{bmatrix}$

(e) $-4\begin{bmatrix} 7 \\ 1 \\ 2 \\ -3 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 5 \\ 1 \\ 7 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 3 \\ -1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ -3 \\ 5 \\ 1 \end{bmatrix}$

(f) $2\begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix} + 1\begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$

4. (a) $\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}$ (b) $x_1\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$

5. (a) $A = \begin{bmatrix} 3 & -5 \\ 1 & 1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 5 & 4 \end{bmatrix}$

(c) $C = \begin{bmatrix} 2 & 4 & -1 \\ -5 & 1 & 2 \\ 1 & 3 & 0 \end{bmatrix}$ (d) $D = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

6. (a) $\begin{bmatrix} 1 & 1 & 3 \\ -3 & 2 & -1 \\ 2 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 3 \\ -3 & 2 & -1 \\ 2 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0.5 \\ 1 & 0.5 \\ 1 & 0.5 \\ 1 & 0.5 \\ 1 & 0.5 \\ 1 & 0.5 \\ 1 & 0.5 \end{bmatrix}\begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 8.1 \\ 6.9 \\ 6.2 \\ 5.3 \\ 4.5 \\ 3.8 \\ 3.0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -4 & 1 & 2 \\ 3 & 2 & -1 & -7 \\ -2 & 1 & -4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

7. (a) $\vec{\mathbf{x}}=\begin{bmatrix} -5 \\ 2 \\ 4 \end{bmatrix}$ (b) $\vec{\mathbf{x}}=\begin{bmatrix} 3 \\ -7 \end{bmatrix}$ (c) $\vec{\mathbf{x}}=\begin{bmatrix} 1.7 \\ 0.4 \end{bmatrix}$ (d) $\vec{\mathbf{x}}=\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

8. (a) $-\dfrac{\nu\sigma_{xx}}{E}+\dfrac{\sigma_{yy}}{E}-\dfrac{\nu\sigma_{zz}}{E}=\epsilon_{yy}$ (b) $-\dfrac{\nu\sigma_{xx}}{E}-\dfrac{\nu\sigma_{yy}}{E}+\dfrac{\sigma_{zz}}{E}=\epsilon_{zz}$ (c) $\dfrac{\tau_{zx}}{\gamma_{zx}}$

(d) $$\begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} \end{bmatrix}\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{bmatrix}=\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{G} & 0 & 0 \\ 0 & \frac{1}{G} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix}\begin{bmatrix} \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}=\begin{bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

Section 3.4 Solutions

Back to 3.4 Exercises

1. (a) $\begin{bmatrix} 3 & 3 \\ 8 & -7 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & -7 & -3 \\ 18 & -31 & -5 \\ -7 & 16 & 9 \end{bmatrix}$

2. $A^2=\begin{bmatrix} -15 & 5 \\ -3 & -14 \end{bmatrix}$ $AF=\begin{bmatrix} 30 & 45 \\ 0 & 12 \end{bmatrix}$ $BC=\begin{bmatrix} -69 \\ -25 \end{bmatrix}$

$BD=\begin{bmatrix} 62 & -25 & -2 \\ -121 & 80 & 36 \end{bmatrix}$ $CE=\begin{bmatrix} -25 & 5 & -10 \\ 20 & -4 & 8 \\ -35 & 7 & -14 \end{bmatrix}$ $DC=\begin{bmatrix} -51 \\ 27 \\ -1 \end{bmatrix}$

$D^2=\begin{bmatrix} 39 & 3 & 18 \\ -48 & 18 & -7 \\ 1 & 4 & 5 \end{bmatrix}$ $EC=\begin{bmatrix} -43 \end{bmatrix}$ $ED=\begin{bmatrix} 37 & -2 & 13 \end{bmatrix}$

$FA=\begin{bmatrix} 3 & 9 \\ -27 & 39 \end{bmatrix}$ $FB=\begin{bmatrix} 10 & -35 & 10 \\ 6 & 147 & -18 \end{bmatrix}$ $F^2=\begin{bmatrix} -2 & -11 \\ 66 & 75 \end{bmatrix}$

3. The $(2,1)$ entry is -8 and the $(3,2)$ entry is -13.

4. $a_{31}a_{12}+a_{32}a_{22}+a_{33}a_{32}+a_{34}a_{42}+a_{35}a_{52}$

5. $$P^2=\begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}\begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}=\left(\tfrac{1}{25}\right)\left(\tfrac{1}{25}\right)\begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix}\begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix}=$$

$$\left(\tfrac{1}{625}\right)\begin{bmatrix} 400 & 300 \\ 300 & 225 \end{bmatrix}=\begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}$$

6. (a) $A^T=\begin{bmatrix} -5 & 0 & 2 \\ 1 & 4 & -3 \end{bmatrix}$ (b) $A^TA=\begin{bmatrix} 29 & -11 \\ -11 & 26 \end{bmatrix}$, $AA^T=\begin{bmatrix} 26 & 4 & -13 \\ 4 & 16 & -12 \\ -13 & -12 & 13 \end{bmatrix}$

(c) A^TA and AA^T are both square, symmetric matrices

Section 3.5 Solutions Back to 3.5 Exercises

1. Because $\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \neq I_2$, the matrices are not inverses.

2. $AB = I_2$ but A and B are not inverses because (1) neither is square and (2) $BA \neq I$

3. (a) $[I_2 \mid B] = \begin{bmatrix} 1 & 0 & -\frac{5}{2} & \frac{3}{2} \\ 0 & 1 & 2 & -1 \end{bmatrix}$

   (b) $AB = BA = I_2$ (c) B is the inverse of A

5. (a) $\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (b) $A^{-1} = \dfrac{1}{10-(-12)}\begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}$

   (c) $\frac{1}{22}\begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix} = \frac{1}{22}\begin{bmatrix} 22 & 0 \\ 0 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (d)
$$\begin{aligned}
\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\
\frac{1}{22}\begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}\left(\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= \frac{1}{22}\begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} \\
\left(\frac{1}{22}\begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \frac{1}{22}\begin{bmatrix} 5 & 3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \frac{1}{22}\begin{bmatrix} 29 \\ -10 \end{bmatrix} \\
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{29}{22} \\ -\frac{10}{22} \end{bmatrix}
\end{aligned}$$

   (e)
$$\begin{bmatrix} 2 & -3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -3 & 1 & 0 \\ 0 & 11 & -2 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -3 & 1 & 0 \\ 0 & 1 & -\frac{2}{11} & \frac{1}{11} \end{bmatrix}$$
$$\begin{bmatrix} 2 & 0 & \frac{5}{11} & \frac{3}{11} \\ 0 & 1 & -\frac{2}{11} & \frac{1}{11} \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & \frac{5}{22} & \frac{3}{22} \\ 0 & 1 & -\frac{2}{11} & \frac{1}{11} \end{bmatrix}$$

6. (a) $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ (b) $A^{-1} = \dfrac{1}{(5)(3)-(2)(7)}\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

   (c) $\begin{bmatrix} 5 & 7 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \overset{rref}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 3 & -7 \\ 0 & 1 & -2 & 5 \end{bmatrix}$, so $A^{-1} = \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$

   (d)
$$\begin{aligned}
\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -1 \\ 4 \end{bmatrix} \\
\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}\left(\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} \\
\left(\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -31 \\ 22 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -31 \\ 22 \end{bmatrix} \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -31 \\ 22 \end{bmatrix}
\end{aligned}$$

Section 3.6 Solutions

Back to 3.6 Exercises

1. (a) $\det(A) = 4$ (b) $\det(B) = 10$ (c) $\det(C) = 8$

   (d) $\det(A) = 5$ (e) $\det(B) = 1$ (f) $\det(C) = 0$

   (e) $\det(A) = 2$ (f) $\det(B) = -1$

2. (a) $\lambda^2 - 5\lambda$ (b) $\lambda^2 - 4\lambda + 3$

   (c) $-\lambda^3 + 4\lambda^2 - 3\lambda$ (d) $-\lambda^3 + 6\lambda^2 + 15\lambda + 8$

3. The determinant of the coefficient matrix is zero, so the system *DOES NOT* have a unique solution.

4. (a) If the determinant of A is zero, then the system has no solution or infinitely many solutions.
   (b) If the determinant of A is not zero, then the system has a unique solution.

5. (a) If the determinant of A is zero, then the system has infinitely many solutions. (It can't have no solutions, because $\vec{\mathbf{x}} = \mathbf{0}$ is a solution.
   (b) If the determinant of A is not zero, then the system has the unique solution $\vec{\mathbf{x}} = \mathbf{0}$.

Section 3.7 Solutions

Back to 3.7 Exercises

1. (b) $P\vec{\mathbf{u}} = \vec{\mathbf{u}}$ (c) $P\vec{\mathbf{v}} = \vec{\mathbf{0}}$ (d) $P(P\vec{\mathbf{w}}) = P^2\vec{\mathbf{w}} = P\vec{\mathbf{w}}$, so $P^2 = P$
   (e) $P^n = P$ for $n = 1, 2, 3, 4, \ldots$ This says that once we have projected a vector, any additional projecting of the result is just what we got on the first projection.
   (f) (Infinitely) many different vectors project to the same vector, so if we know the result of a projection we cannot determine the original vector. Therefore P^{-1} does not exist for a projection matrix P.

2. (b) $C\vec{\mathbf{u}} = \vec{\mathbf{u}}$ (c) $C\vec{\mathbf{v}} = -\vec{\mathbf{v}}$ (d) $C(C\vec{\mathbf{w}}) = C^2\vec{\mathbf{w}} = \vec{\mathbf{w}}$, so $C^2 = I$
   (e) $C^n = I$ if n is even, and $C^n = C$ if n is odd.
   (f) Yes, we can determine $\vec{\mathbf{x}}$ by simply applying C to $C\vec{\mathbf{x}}$. C is invertible and, in fact, C is its own inverse!

5. (a) $R_\theta^2 = R_{2\theta}$ (b) $R_\theta^{-1} = R_{-\theta}$ (c) $R_{120^\circ}^3 = R_{360^\circ} = I$
   (d) $R_{180^\circ}^{-1} = R_{180^\circ}$ This holds for any multiple of 180°.

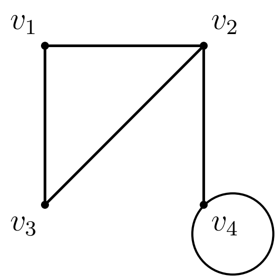
6. (a) $v_3v_2v_4,\ v_3v_1v_4$ (b) $v_1v_3v_1v_4,\ v_1v_3v_2v_4,\ v_1v_4v_1v_4,\ v_1v_4v_2v_4,\ v_1v_4v_3v_4$

   (c) $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $A^2 = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 2 & 3 & 2 & 2 \\ 1 & 2 & 3 & 2 \\ 1 & 2 & 2 & 3 \end{bmatrix}$ $A^3 = \begin{bmatrix} 2 & 4 & 5 & 5 \\ 4 & 7 & 7 & 7 \\ 5 & 7 & 5 & 6 \\ 5 & 7 & 6 & 5 \end{bmatrix}$

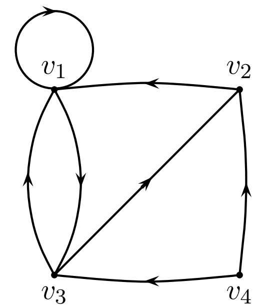
   (d) The $(3,4)$ and $(4,3)$ entries of A^2 are both two, the number of 2-paths from v_3 to v_4.
   (e) The $(1,4)$ and $(4,1)$ entries of A^3 are both five, the number of 3-paths from v_1 to v_4.

(e) There are seven 3-paths from v_2 to v_3. They are $v_1v_2v_2v_3$, $v_2v_2v_4v_3$, $v_2v_3v_1v_3$, $v_2v_3v_2v_3$, $v_2v_3v_4v_3$, $v_2v_4v_1v_3$, $v_2v_4v_2v_3$

7. (a)



(b)



8. (a)

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 5 & 7 & 5 & 6 \\ 7 & 7 & 4 & 7 \\ 5 & 4 & 2 & 5 \\ 6 & 7 & 5 & 5 \end{bmatrix}$$

(b) We expect six 3-paths from v_1 to v_4. they are

$v_1v_2v_1v_4$

$v_1v_2v_2v_4$

$v_1v_4v_1v_4$

$v_1v_4v_2v_4$

$v_1v_4v_3v_4$

$v_1v_3v_1v_4$

(c) There are six 3-paths from v_4 to v_1. This is indicated by the fact that the matrix A^3 is symmetric.

9. (a)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) There are eight 4-paths from v_1 to v_3, as indicated by the $(1,3)$ entry of

$$A^4 = \begin{bmatrix} 8 & 8 & 8 \\ 5 & 6 & 5 \\ 3 & 2 & 3 \end{bmatrix}$$

(c) $v_2v_3v_2v_1v_3$

$v_2v_1v_2v_1v_3$

$v_2v_1v_1v_1v_3$

$v_2v_1v_1v_2v_3$

$v_2v_1v_3v_2v_3$

(d) $v_3v_2v_3v_2$

$v_3v_2v_1v_2$

No, there are two 3-paths from v_3 to v_2, but three 3-paths from v_2 to v_3.