Linear Algebra I

Skills, Concepts and Applications

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Contents

4 Vector Spaces and Subspaces

Outcome:

4. Understand subspaces of $\mathbb{R}^n$. Understand bases of vector spaces and subspaces. Find a least squares solution to an inconsistent system of equations.

   Performance Criteria:

   (a) Describe the span of a set of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ as a line or plane containing a given set of points.

   (b) Determine whether a vector **w** is in the span of a set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k\}$ of vectors. If it is, write $\mathbf{w}$ as a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k$.

   (c) Determine whether a set is closed under an operation. If it is, prove that it is; if it is not, give a counterexample.

   (d) Determine whether a subset of $\mathbb{R}^n$ is a subspace. If so, prove it; if not, give an appropriate counterexample.

   (e) Determine whether a vector is in the column space or null space of a matrix, based only on the definitions of those spaces.

   (f) Find the least-squares approximation to the solution of an inconsistent system of equations. Solve a problem using least-squares approximation.

   (g) Give the least squares error and least squares error vector for a least squares approximation to a solution to a system of equations.

   (h) Determine whether a set $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k$ of vectors is a linearly independent or linearly dependent. If the vectors are linearly dependent, (1) give a non-trivial linear combination of them that equals the zero vector, (2) give any one as a linear combination of the others when possible.

   (i) Determine whether a given set of vectors is a basis for a given subspace. Give a basis and the dimension of a subspace.

   (j) Find the dimensions of, and bases for, the column space and null space of a given matrix.

   (k) Given the dimension of the column space and/or null space of the coefficient matrix for a system of equations, say as much as you can about how many solutions the system has.

   (l) Determine, from given information about the coefficient matrix A and vector $\vec{\mathbf{b}}$ of a system $A\vec{\mathbf{x}}=\vec{\mathbf{b}}$, whether the system has any solutions and, if it does, whether there is more than one solution.

A very important concept in linear algebra is that all vectors of interest in a given situation can be constructed out of a small set of vectors, using linear combinations. That is the key idea that we will explore in this chapter. This will seem to take us farther from some of the more concrete ideas that we have used in applications, but these ideas have huge value in a practical sense as well.

4.1 Span of a Set of Vectors

Performance Criteria:

4. (a) Describe the span of a set of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ as a line or plane containing a given set of points.

   (b) Determine whether a vector **w** is in the span of a set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k\}$ of vectors. If it is, write $\mathbf{w}$ as a linear combination of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k$.

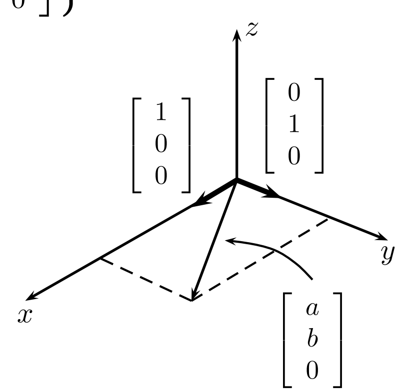
DEFINITION 4.1.1: The **span of a set** $\mathcal{S}$ **of vectors**, denoted $\text{span}(\mathcal{S})$ is the set of all linear combinations of those vectors.

◇ **Example 4.1(a):** Describe the span of the set $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

Solution: Note that *ANY* vector with a zero third component can be written as a linear combination of these two vectors:

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

All the vectors with $x_3 = 0$ (or $z = 0$) are the xy plane in $\mathbb{R}^3$, so the span of this set is the xy plane. Geometrically we can see the same thing in the picture to the right.



◇ **Example 4.1(b):** Describe $\text{span}\left(\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right)$.

Solution: By definition, the span of this set is all vectors $\vec{\mathbf{v}}$ of the form

$$\vec{\mathbf{v}} = c_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix},$$

which, because the two vectors are not scalar multiples of each other, we recognize as being a plane through the origin. It should be clear that all vectors created by such a linear combination will have a third component of zero, so the particular plane that is the span of the two vectors is the xy-plane. Algebraically we see that any vector $[a, b, 0]$ in the xy-plane can be created by

$$\left(\frac{a-3b}{7}\right) \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + \left(\frac{2a+b}{7}\right) \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{a-3b}{7} \\ \frac{-2a+6b}{7} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{6a+3b}{7} \\ \frac{2a+b}{7} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{7a}{7} \\ \frac{7b}{7} \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

You might wonder how one would determine the scalars $\dfrac{a-3b}{7}$ and $\dfrac{2a+b}{7}$. You will see how this is done in the exercises!

At this point we should make a comment and a couple observations:

- First, some language: we can say that the span of the two vectors in Example 4.1(b) is the xy-plane, but we also say that the two vectors span the xy-plane. That is, the word span can be either a noun or a verb, depending on how it is used.
- Note that in the two examples above we considered two different sets of two vectors, but in each case the span was the same. This illustrates that *different sets of vectors can have the same span.*
- Consider also the fact that if we were to include in either of the two sets additional vectors that are also in the xy-plane, it would not change the span. However, if we were to add another vector not in the xy-plane, the span would increase to all of $\mathbb{R}^3$.
- In either of the preceding examples, removing either of the two given vectors would reduce the span to a linear combination of a single vector, which is a line rather than a plane. But in some cases, removing a vector from a set does not change its span.
- The last two bullet items tell us that *adding or removing vectors from a set of vectors may or may not change its span.* This is a somewhat undesirable situation that we will remedy in Section 4.7.
- It may be obvious, but it is worth emphasizing that (in this course) we will consider spans of finite (and usually rather small) sets of vectors, but the span of a finite set $\mathcal{S}$ always contains infinitely many vectors (unless $\mathcal{S}$ consists of only the zero vector).

It is often of interest to know whether a particular vector is in the span of a certain set of vectors. The next examples show how we do this.

◇ **Example 4.1(c):** Is $\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ -2 \\ -4 \\ 1 \end{bmatrix}$ in the span of $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$?

Solution: The question is, "can we find scalars c_1, c_2 and c_3 such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -4 \\ 1 \end{bmatrix} \text{?"} \qquad (1)$$

We should recognize this as the linear combination form of the system of equations below and to the left. The augmented matrix for the system row reduces to the matrix below and to the right.

$$\begin{aligned} c_1 + c_2 + 2c_3 &= 3 \\ 2c_1 - c_2 &= -2 \\ 3c_1 + c_2 - 3c_3 &= -4 \\ 4c_1 - c_2 + c_3 &= 1 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This tells us that the system above and to the left has no solution, so there are no scalars c_1, c_2 and c_3 for which equation (1) holds. Thus $\vec{\mathbf{v}}$ is not in the span of $\mathcal{S}$.

◇ **Example 4.1(d):** Is $\vec{\mathbf{v}} = \begin{bmatrix} 19 \\ 10 \\ -1 \end{bmatrix}$ in $\text{span}(\mathcal{S})$, where $\mathcal{S} = \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix} \right\}$?

Solution: Here we are trying to find scalars c_1, c_2 and c_3 such that

$$c_1 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 19 \\ 10 \\ -1 \end{bmatrix} \tag{2}$$

This is the linear combination form of the system of equations below and to the left, whose augmented matrix row reduces to the matrix below and to the right.

$$\begin{aligned} 3c_1 - 5c_2 + c_3 &= 19 \\ -c_1 + 7c_3 &= 10 \\ 2c_1 + c_2 - 4c_3 &= -1 \end{aligned} \qquad\qquad \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

This tells us that (2) holds for $c_1 = 4$, $c_2 = -1$ and $c_3 = 2$, so $\vec{\mathbf{v}}$ is in $\text{span}(\mathcal{S})$.

Sometimes, with a little thought, no computations are necessary to answer such questions, as the next examples show.

◇ **Example 4.1(e):** Is $\vec{\mathbf{v}} = \begin{bmatrix} -4 \\ 2 \\ 5 \end{bmatrix}$ in the span of $\mathcal{S} = \left\{ \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}$?

Solution: One can see that any linear combination of the two vectors in S will have zero as its second component:

$$c_1 \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3c_1 \\ 0 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} -5c_2 \\ 0 \\ 1c_2 \end{bmatrix} = \begin{bmatrix} -3c_1 - 5c_2 \\ 0 \\ 2c_1 + c_2 \end{bmatrix}$$

Since the second component of $\vec{\mathbf{v}}$ is not zero, $\vec{\mathbf{v}}$ is not in the span of the set $\mathcal{S}$.

◇ **Example 4.1(f):** Is $\vec{\mathbf{v}} = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$ in $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$?

Solution: Here we can see that if we multiply the three vectors in $\mathcal{S}$ by 4, 7 and -1, respectively, and add them, the result will be $\vec{\mathbf{v}}$:

$$4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ -1 \end{bmatrix}$$

Therefore $\vec{\mathbf{v}}$ is in $\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$.

Sometimes we will be given an infinite set of vectors, and we'll ask whether a particular finite set of vectors *spans the infinite set*. By this we are asking whether the span of the finite set is the infinite set. For example, we might ask whether the vector $\vec{\mathbf{v}}=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ spans $\mathbb{R}^2$. Because the span of the single vector $\vec{\mathbf{v}}$ is just a line, $\vec{\mathbf{v}}$ does not span $\mathbb{R}^2$. With the knowledge we have at this point, it can sometimes be difficult to tell whether a finite set of vectors spans a particular infinite set. In Sections 4.6 and 4.7 we will encounter some concepts that will give us a means for making such a judgement a bit easier.

We conclude with a few more observations. With a little thought, the following can be seen to be true. (Assume all vectors are non-zero.)

- The span of a single vector is all scalar multiples of that vector. In $\mathbb{R}^2$ or $\mathbb{R}^3$ the span of a single vector is a line through the origin.
- The span of a set of two non-parallel vectors in $\mathbb{R}^2$ is all of $\mathbb{R}^2$. In $\mathbb{R}^3$ it is a plane through the origin.
- The span of three vectors in $\mathbb{R}^3$ that do not lie in the same plane is all of $\mathbb{R}^3$.

Section 4.1 Exercises To Solutions

1. Describe the span of each set of vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ by telling what it is geometrically and, if it is a standard set like one of the coordinate axes or planes, specifically what it is. If it is a line that is not one of the axes, give two points on the line. If it is a plane that is not one of the coordinate planes, give three points on the plane.

   (a) The vector $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ in $\mathbb{R}^2$.

   (b) The set of vectors $\left\{ \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

   (c) The vectors $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ in $\mathbb{R}^2$.

   (d) The set $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

   (e) The vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ in $\mathbb{R}^3$.

2. For each of the following, determine whether the vector $\vec{\mathbf{w}}$ is in the span of the set S. If it is, write it as a linear combination of the vectors in S.

   (a) $\vec{\mathbf{w}} = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}, \qquad S = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -11 \\ -1 \end{bmatrix} \right\}$

(b) $\vec{\mathbf{w}} = \begin{bmatrix} -5 \\ -23 \\ 12 \\ 8 \end{bmatrix}$, $\quad S = \left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -4 \\ 5 \end{bmatrix} \right\}$

(c) $\vec{\mathbf{w}} = \begin{bmatrix} 8 \\ 38 \\ -14 \\ 11 \end{bmatrix}$, $\quad S = \left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ -4 \\ 5 \end{bmatrix} \right\}$

(d) $\vec{\mathbf{w}} = \begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix}$, $\quad S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Say we have a set $\mathcal{S}$ of vectors in $\mathbb{R}^n$, and we consider $\text{span}(\mathcal{S})$. Then suppose that we include an additional new vector $\vec{\mathbf{v}}$ in $\mathcal{S}$, to obtain a new set we'll call $\mathcal{S}'$. If $\vec{\mathbf{v}}$ is in $\text{span}(\mathcal{S})$, then $\text{span}(\mathcal{S}') = \text{span}(\mathcal{S})$. If $\vec{\mathbf{v}}$ is not in $\text{span}(\mathcal{S})$, then $\text{span}(\mathcal{S}')$ contains more vectors than $\text{span }(\mathcal{S})$ and the two spans are not the same. This concept will be useful for parts of the following exercise.

3. In each of the following, two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ are given. In each case, determine whether or not $\text{span}(\mathcal{S}_1)$ and $\text{span}(\mathcal{S}_2)$ are equal. You should be able to do (a) - (d) without any computations other than some simple mental arithmetic.

   (a) $S_1 = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \end{bmatrix} \right\}$, $\quad S_2 = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \end{bmatrix} \right\}$

   (b) $S_1 = \left\{ \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} \right\}$, $\quad S_2 = \left\{ \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix} \right\}$

   (c) $S_1 = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \right\}$, $\quad S_2 = \left\{ \begin{bmatrix} -9 \\ 3 \\ -12 \end{bmatrix} \right\}$

   (d) $S_1 = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \end{bmatrix} \right\}$, $\quad S_2 = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix} \right\}$

   (e) $S_1 = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} \right\}$, $\quad S_2 = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \right\}$

   (f) $S_1 = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} \right\}$, $\quad S_2 = \left\{ \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

4. (a) For the following sets $\mathcal{S}_1$ and $\mathcal{S}_2$, check to see if each vector in $\mathcal{S}_2$ is in $\text{span}(\mathcal{S}_1)$.

$$\mathcal{S}_1 = \left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ -1 \end{bmatrix} \right\}, \qquad \mathcal{S}_2 = \left\{ \begin{bmatrix} 15 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -6 \\ 0 \end{bmatrix} \right\}$$

Are $\text{span}(\mathcal{S}_1)$ and $\text{span}(\mathcal{S}_2)$ equal?

(b) Give an example of two sets $\mathcal{S}_1$ and $\mathcal{S}_2$, with the same number of vectors in each, for which every vector in $\mathcal{S}_2$ is in $\text{span}(\mathcal{S}_1)$, but $\text{span}(\mathcal{S}_1) \neq \text{span}(\mathcal{S}_2)$.

(c) How could we test two sets of vectors to see if their spans are the same?

5. This exercise will be important in the next section and those that follow. Read both parts of the exercise before attempting part (a) - the two parts are "inverses" of each other, in a sense.

(a) Letting $\mathcal{S} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$, what does any vector in $\text{span}(\mathcal{S})$ look like, *as a single vector*?

(b) Give a set $\mathcal{S}$ consisting of two *specific* vectors in $\mathbb{R}^3$ whose span consists of all vectors of the form

$\begin{bmatrix} a+b \\ 2a \\ 3b \end{bmatrix}$, where a and b are any real numbers.

6. Consider the sets

$$\mathcal{S}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \qquad \mathcal{S}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathcal{S}_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad \mathcal{S}_4 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\mathcal{S}_5 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(a) For each of the given sets, try to find a linear combination of the vectors in the set that equals the vector $\vec{\mathbf{v}} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}$. When you can find such a linear combination, write your results down, assuming the names of the vectors in each set are $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3, ...$

(b) In the cases where you *can* find a linear combination equaling $\vec{\mathbf{v}}$, is the linear combination unique? That is, is there only one linear combination equalling the vector $\vec{\mathbf{v}}$?

4.2 Closure of a Set Under an Operation

Performance Criteria:

4. (c) Determine whether a set is closed under an operation. If it is, prove that it is; if it is not, give a counterexample.

Consider the set $\{0, 1, 2, 3, ...\}$, which are called the whole numbers. Notice that if we add or multiply any two whole numbers the result is also a whole number, but if we try subtracting two such numbers it is possible to get a number that is not in the set. We say that the whole numbers are **closed under addition and multiplication**, but the set of whole numbers is not closed under subtraction. If we enlarge our set to be the integers $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ we get a set that is closed under addition, subtraction and multiplication. These operations we are considering are called **binary operations** because they take two elements of the set and create a single new element. An operation that takes just one element of the set and gives another (possibly the same) element of the set is called a **unary operation**. An example would be absolute value; note that the set of integers is closed under absolute value.

<u>Definition 4.2.1</u>: **Closed Under an Operation**

A set $\mathcal{S}$ is said to be closed under a binary operation $*$ if for every s and t in $\mathcal{S}$, $s * t$ is in $\mathcal{S}$. $\mathcal{S}$ is closed under a unary operation $\langle\ \rangle$ if for every s in $\mathcal{S}$, $\langle s \rangle$ is in $\mathcal{S}$.

Notice that the term "closed," as defined here, only makes sense in the context of a set with an operation. Notice also that *it is the set that is closed*, not the operation. The operation is important as well; as we have seen, a given set can be closed under one operation but not another.

When considering closure of a set $\mathcal{S}$ under a binary operation $*$, our considerations are as follows:

- We first wish to determine whether we think $\mathcal{S}$ *IS* closed under $*$.
- If we do think that $\mathcal{S}$ is closed under $*$, we then need to prove that it is. To do this, we need to take two general, or *arbitrary* elements x and y of $\mathcal{S}$ and show that $x * y$ is in $\mathcal{S}$.
- If we think that $\mathcal{S}$ is not closed under $*$, we need to take two *specific* elements x and y of $\mathcal{S}$ and show that $x * y$ is not in $\mathcal{S}$.

◇ **Example 4.2(a):** The odd integers are the numbers $..., -5, -3, -1, 1, 3, 5, ...$ Are the odd integers closed under addition? Multiplication?

Solution: We see that $3 + 5 = 8$. Because 3 and 5 are both odd but their sum isn't, the odd integers are not closed under addition. Let's try multiplying some odds:

$$3 \times 5 = 15 \qquad -7 \times 9 = -63 \qquad -1 \times -7 = 7$$

Based on these three examples, it appears that the odd integers are perhaps closed under multiplication. Let's attempt to prove it. First we observe that any number of the form $2n + 1$, where n is any integer, is odd. (This is in fact the definition of an odd integer.) So if we have two *possibly different* odd integers, we can write them as $2m+1$ and $2n+1$, where m and n are not necessarily the same integers. Their product is

$$(2m+1)(2n+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1.$$

Because the integers are closed under multiplication and addition, $2mn + m + n$ is an integer and the product of $2m+1$ and $2n+1$ is of the form two times an integer, plus one, so it is odd as well. Therefore the odd integers are closed under multiplication.

Closure of a set under an operation is a fairly general concept; let's narrow our focus to what is important to us in linear algebra.

◇ **Example 4.2(b):** Prove that the span of a set $\mathcal{S} = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3, ..., \vec{\mathbf{v}}_k\}$ in $\mathbb{R}^n$ is closed under addition and scalar multiplication.

Solution: Suppose that $\vec{\mathbf{u}}$ and $\vec{\mathbf{w}}$ are in span($\mathcal{S}$). Then there are scalars $c_1, c_2, c_3, ..., c_k$ and $d_1, d_2, d_3, ..., d_k$ such that

$$\vec{\mathbf{u}} = c_1 \vec{\mathbf{v}}_1 + c_2 \vec{\mathbf{v}}_2 + c_3 \vec{\mathbf{v}}_3 + \cdots + c_k \vec{\mathbf{v}}_k \qquad \text{and} \qquad \vec{\mathbf{w}} = d_1 \vec{\mathbf{v}}_1 + d_2 \vec{\mathbf{v}}_2 + d_3 \vec{\mathbf{v}}_3 + \cdots + d_k \vec{\mathbf{v}}_k .$$

Therefore

$$\begin{aligned} \vec{\mathbf{u}} + \vec{\mathbf{w}} &= (c_1 \vec{\mathbf{v}}_1 + c_2 \vec{\mathbf{v}}_2 + c_3 \vec{\mathbf{v}}_3 + \cdots + c_k \vec{\mathbf{v}}_k) + (d_1 \vec{\mathbf{v}}_1 + d_2 \vec{\mathbf{v}}_2 + d_3 \vec{\mathbf{v}}_3 + \cdots + d_k \vec{\mathbf{v}}_k) \\ &= (c_1 + d_1) \vec{\mathbf{v}}_1 + (c_2 + d_2) \vec{\mathbf{v}}_2 + (c_3 + d_3) \vec{\mathbf{v}}_3 + \cdots + (c_k + d_k) \vec{\mathbf{v}}_k \end{aligned}$$

This last expression is a linear combination of the vectors in $\mathcal{S}$, so it is in span($\mathcal{S}$). Therefore span($\mathcal{S}$) is closed under addition. Now suppose that $\vec{\mathbf{u}}$ is as above and a is any scalar. Then

$$\begin{aligned} a \vec{\mathbf{u}} &= a(c_1 \vec{\mathbf{v}}_1 + c_2 \vec{\mathbf{v}}_2 + c_3 \vec{\mathbf{v}}_3 + \cdots + c_k \vec{\mathbf{v}}_k) \\ &= (ac_1) \vec{\mathbf{v}}_1 + (ac_2) \vec{\mathbf{v}}_2 + (ac_3) \vec{\mathbf{v}}_3 + \cdots + (ac_k) \vec{\mathbf{v}}_k \end{aligned}$$

which is also a linear combination of the vectors in $\mathcal{S}$, so it is also in span($\mathcal{S}$). Thus span($\mathcal{S}$) is closed under multiplication by scalars.

The result of the above example is that

> THEOREM 4.2.2: The span of a set $\mathcal{S}$ of vectors is closed under vector addition and scalar multiplication.

This seemingly simple observation is the beginning of one of the most important stories in the subject of linear algebra. The remainder of this chapter will fill out the rest of that story.

Section 4.2 Exercises To Solutions

1. Determine whether each of sets described is closed under addition and scalar multiplication. For those that are not closed, give a *specific* example demonstrating that.

   (a) $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \right\}$

   (b) $\mathcal{S} = \left\{ t \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \right\}$, where t ranges over all real numbers

   (c) $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \right\}$, where t ranges over all real numbers

   (d) $\mathcal{S} = \left\{ s \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \right\}$, where s and t range over all real numbers

   (e) $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \right\}$

2. Determine whether each of the following sets (considered as sets of position vectors) is closed under addition and scalar multiplication. then

   - for those that are not, give a specific counterexample demonstrating that,
   - for those that are, give a set of *specific* vectors that span the given set.

   (a) The line with equation $y = 2x$ in $\mathbb{R}^2$.

   (b) The yz-plane in $\mathbb{R}^3$.

   (c) The line with equation $y = 2x + 1$ in $\mathbb{R}^2$.

   (d) $\mathcal{S} = \left\{ s \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\}$, where s and t range over all real numbers

   (e) The set of all vectors of the form $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$, where a and b can be any real number.

   (f) The set of all vectors of the form $\begin{bmatrix} a \\ b \\ ab \end{bmatrix}$, where a and b can be any real number.

4.3 Vector Spaces and Subspaces

Performance Criterion:

4. (d) Determine whether a subset of $\mathbb{R}^n$ is a subspace. If so, prove it; if not, give an appropriate counterexample.

Vector Spaces

The term "space" in math simply means a set of objects with some additional special properties. There are metric spaces, function spaces, topological spaces, Banach spaces, and more. The vectors that we have been dealing with make up the **vector spaces** called $\mathbb{R}^2$, $\mathbb{R}^3$ and, for larger values, $\mathbb{R}^n$. In general, a vector space is simply a collection of objects called vectors, a set of scalars, and two operations that satisfy certain properties.

Definition 4.3.1: **Vector Space**

A **vector space** is a set V of objects called **vectors** and a set of scalars (usually the real numbers $\mathbb{R}$), with the operations of vector addition and scalar multiplication, for which the following properties hold for all $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, $\vec{\mathbf{w}}$ in V and scalars c and d.

1. $\vec{\mathbf{u}}+\vec{\mathbf{v}}$ is in V
2. $\vec{\mathbf{u}}+\vec{\mathbf{v}}=\vec{\mathbf{v}}+\vec{\mathbf{u}}$
3. $(\vec{\mathbf{u}}+\vec{\mathbf{v}})+\vec{\mathbf{w}}=\vec{\mathbf{u}}+(\vec{\mathbf{v}}+\vec{\mathbf{w}})$
4. There exists a vector $\vec{\mathbf{0}}$ in V such that $\vec{\mathbf{u}}+\vec{\mathbf{0}}=\vec{\mathbf{u}}$. This vector is called the **zero vector**.
5. For every $\vec{\mathbf{u}}$ in V there exists a vector $-\vec{\mathbf{u}}$ in V such that $\vec{\mathbf{u}}+(-\vec{\mathbf{u}})=\vec{\mathbf{0}}$.
6. $c\,\vec{\mathbf{u}}$ is in V.
7. $c\,(\vec{\mathbf{u}}+\vec{\mathbf{v}})=c\vec{\mathbf{u}}+c\vec{\mathbf{v}}$
8. $(c+d)\vec{\mathbf{u}}=c\vec{\mathbf{u}}+d\vec{\mathbf{u}}$
9. $c\,(d\vec{\mathbf{u}})=(cd)\,\vec{\mathbf{u}}$
10. $1\vec{\mathbf{u}}=\vec{\mathbf{u}}$

Note that items 1 and 6 of the above definition say that a vector space is closed under addition and scalar multiplication.

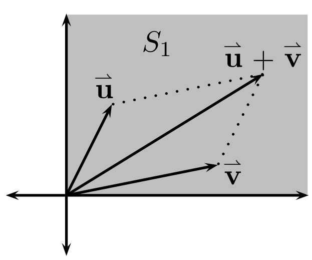
When working with vector spaces, we will be very interested in certain subsets of those vector spaces that are the span of a set of vectors. As you proceed, recall Example 4.2(b), where we showed that *the span of a set of vectors is closed under addition and scalar multiplication.*

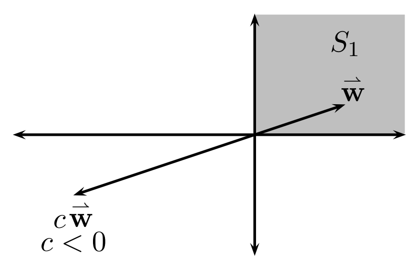
Subspaces of Vector Spaces

As you should know by now, the two main operations with vectors are multiplication by scalars and addition of vectors. (Note that these two combined give us linear combinations, the foundation of almost everything we've done.) A given vector space can have all sorts of subsets; consider the following subsets of $\mathbb{R}^2$.

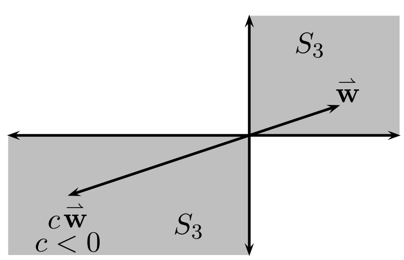
- The set S_1 consisting of the first quadrant and the nonnegative parts of the two axes, or all vectors of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $x_1 \geq 0$ and $x_2 \geq 0$.
- The set S_2 consisting of the line containing the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Algebraically this is all vectors of the form $t\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ where t ranges over all real numbers.
- The set S_3 consisting of the first and third quadrants and both axes. This can be described as the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $x_1x_2 \geq 0$.

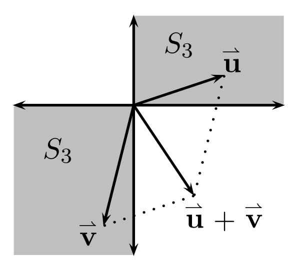
Our current concern is whether these subsets of $\mathbb{R}^2$ are closed under addition and scalar multiplication. With a bit of thought you should see that S_1 is closed under addition, but not scalar multiplication when the scalar is negative:





In some sense we can solve the problem of not being closed under scalar multiplication by including the third quadrant as well to get S_3, but then the set isn't closed under addition:





Finally, the set S_2 is closed under both addition and scalar multiplication. That is a bit messy to show with a diagram, but consider the following. S_2 is the span of the single vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and we showed in the last section that the span of any set of vectors is closed under addition and scalar multiplication.

It turns out that when working with vector spaces the only subsets of any real interest are the ones that are closed under both addition and scalar multiplication. We give such subsets a name:

> **DEFINITION 4.3.2: Subspace of $\mathbb{R}^n$**
>
> A subset S of $\mathbb{R}^n$ is called a **subspace** of $\mathbb{R}^n$ if for every scalar c and any vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in S, $c\vec{\mathbf{u}}$ and $\vec{\mathbf{u}}+\vec{\mathbf{v}}$ are also in S. That is, S is closed under scalar multiplication and addition.

You will be asked whether certain subsets of $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathbb{R}^n$ are subspaces, and it is your job to back your answers up with some reasoning. This is done as follows:

- When a subset *IS* a subspace a general proof is required. That is, we must show that the set is closed under scalar multiplication and addition, for *ALL* scalars and *ALL* vectors. We may have to do this outright, but if it is clear that the set of vectors is the span of some set of vectors, then we know from the argument presented in Example 4.2(b) that the set is closed under addition and scalar multiplication, so it is a subspace.
- When a subset *IS NOT* a subspace, we demonstrate that fact with a *SPECIFIC* example. Such an example is called a **counterexample**. Notice that all we need to do to show that a subset is not a subspace is to show that either it is not closed under scalar multiplication *OR* it is not closed under scalar multiplication vector addition. If either is the case, then the set in question is not a subspace. *Even if both are the case, we need only show one.*

The following examples illustrate these things.

◇ **Example 4.3(a):** Show that the set S_1 consisting of all vectors of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $x_1 \geq 0$ and $x_2 \geq 0$ is not a subspace of $\mathbb{R}^2$.

Solution: As mentioned before, this set is not closed under multiplication by negative scalars, so we just need to give a specific example of this. Let $\vec{\mathbf{u}}=\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $c=-2$. Clearly $\vec{\mathbf{u}}$ is in S_1 and $c\vec{\mathbf{u}}=(-2)\begin{bmatrix} 3 \\ 5 \end{bmatrix}=\begin{bmatrix} -6 \\ -10 \end{bmatrix}$, which is not in S_1. Therefore S_1 is not closed under scalar multiplication so it is not a subspace of $\mathbb{R}^2$.

◇ **Example 4.3(b):** Show that the set S_2 consisting of all vectors of the form $t\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, where t ranges over all real numbers, is a subspace of $\mathbb{R}^3$.

Solution: Let c be any scalar and let $\vec{\mathbf{u}}=s\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{\mathbf{v}}=t\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. Then $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are both in S_2 and

$$c\,\vec{\mathbf{u}}=c\left(s\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}\right)=(cs)\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{u}}+\vec{\mathbf{v}}=s\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}+t\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}=(s+t)\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

Because cs and $s+t$ are scalars, we can see that both $c\vec{\mathbf{u}}$ and $\vec{\mathbf{u}}+\vec{\mathbf{v}}$ are in S_2, so S_2 is a subspace of $\mathbb{R}^3$.

This last example demonstrates the general method for showing that a set of vectors is closed under addition and scalar multiplication. That said, the given subspace could have been shown to be a subspace by simply observing that it is the span of the set consisting of the single vector $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$. From Theorem 4.2.2 we know that the span of any set of vectors is closed under addition and scalar multiplication, so a span of a set is a subspace. Therefore the set described in the above example is a subspace of $\mathbb{R}^2$. Let's formalize all that:

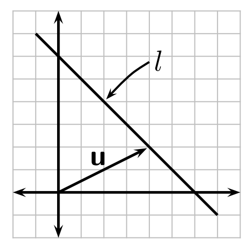
<u>Corollary 4.3.3</u>: **The Span of a Set is a Subspace**

The span of a set $\mathcal{S}$ in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

In Example 4.3(d) you will see how we can use the above to prove that a set is a subspace. First we look at another example.

◇ **Example 4.3(c):** Determine whether the subset S of $\mathbb{R}^3$ consisting of all vectors of the form $\vec{\mathbf{x}} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + t \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$ is a subspace. If it is, prove it. If it is not, provide a counterexample.

Solution: We recognize this as a line in $\mathbb{R}^3$ passing through the point $(2, 5, -1)$, and it is not hard to show that the line does not pass through the origin. Remember that what we mean by the line is really all position vectors (so with tails at the origin) whose tips are on the line. Considering a similar situation in $\mathbb{R}^2$, we see that $\vec{\mathbf{u}}$ is such a vector for the line l shown. It should be clear that if we multiply $\vec{\mathbf{u}}$ by any scalar other than one, the resulting vector's tip will not lie on the line. Thus we would guess that the set S, even though it is in $\mathbb{R}^3$, is probably not closed under scalar multiplication.



Now let's prove that it isn't. To do this we first let $\vec{\mathbf{u}} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$, which is in S. Let $c = 2$, so $2\vec{\mathbf{u}} = \begin{bmatrix} 12 \\ 8 \\ 4 \end{bmatrix}$. We need to show that this vector is not in S. If it were, there would have to be a scalar t such that $\begin{bmatrix} 12 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + t \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$. Subtracting $\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ from both sides we get $\begin{bmatrix} 10 \\ 3 \\ 5 \end{bmatrix} = t \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$. We can see that the value of t that would be needed to give the correct second component would be -3, but this would result in a third component of -9, which is not correct. Thus there is no such t and the vector $\begin{bmatrix} 12 \\ 8 \\ 4 \end{bmatrix}$ is not in S. Thus S is not closed under scalar multiplication, so it is not a subspace of $\mathbb{R}^3$.

This example may seem to contradict Corollary 4.3.3. The reason it does not is because the vectors of the form $\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + t\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$ do not include *ALL* linear combinations of the vectors $\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$.

We should compare the results of Examples 4.3(b) and 4.3(c). Note that both are lines in their respective $\mathbb{R}^n$'s, but the line in 4.3(b) passes through the origin, and the one in 4.3(c) does not. *It is no coincidence that the set in 4.3(b) is a subspace and the set in 4.3(c) is not.* If a set S is a subspace, being closed under scalar multiplication means that zero times any vector in the subspace must also be in the subspace. But zero times a vector is the zero vector $\mathbf{0}$. Therefore

> Theorem 4.3.4: **Subspaces Contain the Zero Vector**
>
> If a subset S of $\mathbb{R}^n$ is a subspace, then the zero vector of $\mathbb{R}^n$ is in S.

This type of a statement is called a **conditional statement**. Related to any conditional statement are two other statements called the **converse** and **contrapositive** of the conditional statement. In this case we have

- **Converse:** If the zero vector is in a subset S of $\mathbb{R}^n$, then S is a subspace.
- **Contrapositive:** If the zero vector is *not* in a subset S of $\mathbb{R}^n$, then S is *not* a subspace.

When a conditional statement is true, its converse may or may not be true. In this case the converse is not true. This is easily seen in Example 4.3(a), where the set contains the zero vector but is not a subspace. However, when a conditional statement is true, its contrapositive is true as well. Therefore the second statement above is the most useful of the three statements, since it gives us a quick way to rule out a set as a subspace. Let's recognize it formally.

> Corollary 4.3.5: **Test For Not a Subspace**
>
> If a subset S of $\mathbb{R}^n$ does not the zero vector of $\mathbb{R}^n$, then it is not a subspace of $\mathbb{R}^n$.

In Example 4.3(c) this would have saved us the trouble of providing a counterexample, although we'd still need to convincingly show that the zero vector is not in the set.

◇ **Example 4.3(d):** Determine whether the set of all vectors of the form $\vec{\mathbf{x}} = \begin{bmatrix} a \\ a+b \\ b \\ a-b \end{bmatrix}$, for some real numbers a and b, is a subspace of $\mathbb{R}^4$.

Solution: We first note that a vector $\vec{\mathbf{x}}$ of the given form will be the zero vector if $a = b = 0$. We cannot then use the above result to rule out the possibility that the given set is a subspace, but neither do we yet know it *IS* a subspace. But we observe that

$$\vec{\mathbf{x}} = \begin{bmatrix} a \\ a+b \\ b \\ a-b \end{bmatrix} = \begin{bmatrix} a \\ a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ b \\ -b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

Thus, by Corollary 4.3.3, the set of vectors under consideration is the span of $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$, so it is a subspace of $\mathbb{R}^4$.

We conclude this section with an example that gives us the "largest" and "smallest" subspaces of $\mathbb{R}^n$.

⋄ **Example 4.3(e):** Of course a scalar times any vector in $\mathbb{R}^n$ is also in $\mathbb{R}^n$, and the sum of any two vectors in $\mathbb{R}^n$ is in $\mathbb{R}^n$, so $\mathbb{R}^n$ is a subspace of itself. Also, the zero vector by itself is a subspace of $\mathbb{R}^n$ as well, often called the **trivial subspace**.

At this point we have seen a variety of subspaces, and some sets that are not subspaces as well. Now supose that we have two vectors in $\mathbb{R}^3$ that are not scalar multiples of each other. We know that the span of the two vectors is a plane through the origin in $\mathbb{R}^3$. Note that we could impose a coordinate system on any plane to make it essentially $\mathbb{R}^2$, so we can think of this particular variety of subspace as just being a copy of $\mathbb{R}^2$ "sitting inside" $\mathbb{R}^3$. This illustrates what is in fact a general principle: *any subspace of* $\mathbb{R}^n$ *is essentially a copy of* $\mathbb{R}^m$*, for some* $m \leq n$*, sitting inside* $\mathbb{R}^n$ *with its origin at the origin of* $\mathbb{R}^n$*.* More formally we have the following:

Subspaces of $\mathbb{R}^n$

- The only non-trivial subspaces of $\mathbb{R}^2$ are lines through the origin and all of $\mathbb{R}^2$.
- The only non-trivial subspaces of $\mathbb{R}^3$ are lines through the origin, planes through the origin, and all of $\mathbb{R}^3$.
- The only non-trivial subspaces of $\mathbb{R}^n$ are hyperplanes (including lines) through the origin and all of $\mathbb{R}^n$.

Section 4.3 Exercises To Solutions

1. For each of the following sub*sets* of $\mathbb{R}^3$, think of each point as a position vector; each set then becomes a set of vectors rather than points. For each,

   - determine whether the set is a sub*space* and
   - if it is *NOT* a subspace, give a reason why it isn't by doing one of the following:

     ◇ stating that the set does not contain the zero vector

     ◇ giving a vector that is in the set and a scalar multiple that isn't (show that it isn't)

     ◇ giving two vectors that in the set and showing that their sum is not in the set

   (a) All points on the horizontal plane at $z = 3$.

   (b) All points on the xz-plane.

   (c) All points on the line containing $\vec{\mathbf{u}} = [-3, 1, 4]$.

   (d) All points on the lines containing $\vec{\mathbf{u}} = [-3, 1, 4]$ and $\vec{\mathbf{v}} = [5, 0, 2]$.

   (e) All points for which $x \geq 0, y \geq 0$ and $z \geq 0$.

   (f) All points $\vec{\mathbf{x}}$ given by $\vec{\mathbf{x}} = \vec{\mathbf{w}} + s\,\vec{\mathbf{u}} + t\,\vec{\mathbf{v}}$, where $\vec{\mathbf{w}} = [1, 1, 1]$ and $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are as in (d).

   (g) All points $\vec{\mathbf{x}}$ given by $\vec{\mathbf{x}} = s\,\vec{\mathbf{u}} + t\,\vec{\mathbf{v}}$, where $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are as in (d).

   (h) The vector **0**.

   (i) All of $\mathbb{R}^3$.

2. Consider the vectors $\vec{\mathbf{u}} = \begin{bmatrix} 8 \\ -2 \\ 4 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} -16 \\ 4 \\ -8 \end{bmatrix}$.

   (a) Is the set of all vectors $\vec{\mathbf{x}} = \vec{\mathbf{u}} + t\,\vec{\mathbf{v}}$, where t ranges over all real numbers, a subspace of $\mathbb{R}^3$? If not, tell why not.

   (b) Is the set of all vectors $\vec{\mathbf{x}} = \vec{\mathbf{u}} + t\,\vec{\mathbf{w}}$, where t ranges over all real numbers, a subspace of $\mathbb{R}^3$? If not, tell why not.

4.4 Column Space and Null Space of a Matrix

Performance Criteria:

4. (e) Determine whether a vector is in the column space or null space of a matrix, based only on the definitions of those spaces.

In this section we will define two important subspaces associated with a matrix A, its **column space** and its **null space**.

DEFINITION 4.4.1: **Column Space of a Matrix**

The **column space** of an $m \times n$ matrix A is the span of the columns of A. It is a subspace of $\mathbb{R}^m$ and we denote it by col(A).

◇ **Example 4.4(a):** Determine whether $\vec{\mathbf{u}} = \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$ are in the column space of $A = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix}$.

Solution: We need to solve the two vector equations of the form

$$c_1 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ -7 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix} = \vec{\mathbf{b}}, \qquad (1)$$

with $\vec{\mathbf{b}}$ first being $\vec{\mathbf{u}}$, then $\vec{\mathbf{v}}$. The respective reduced row-echelon forms of the augmented matrices corresponding to the two systems are

$$\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore we can find scalars c_1, c_2 and c_3 for which (1) holds when $\vec{\mathbf{b}} = \vec{\mathbf{u}}$, but not when $\vec{\mathbf{b}} = \vec{\mathbf{v}}$. From this we deduce that $\vec{\mathbf{u}}$ is in col(A), but $\vec{\mathbf{v}}$ is not.

Recall that the system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ of m linear equations in n unknowns can be written in linear combination form:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Note that the left side of this equation is simply a linear combination of the columns of A, with the scalars being the components of $\vec{\mathbf{x}}$. The system will have a solution if, and only if, $\vec{\mathbf{b}}$ can be written as a linear combination of the columns of A. Stated another way, we have the following:

> THEOREM 4.4.2: A system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution (meaning *at least* one solution) if, and only if, $\vec{\mathbf{b}}$ is in the column space of A.

Let's consider now only the case where $m = n$, so we have n linear equations in n unknowns. We have the following facts:

- If col(A) is all of $\mathbb{R}^n$, then $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ will have a solution for any vector $\vec{\mathbf{b}}$. What's more, *the solution will be unique*.
- If col(A) is a proper subspace of $\mathbb{R}^n$ (that is, it is not all of $\mathbb{R}^n$), then the equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ will have a solution if, and only if, $\vec{\mathbf{b}}$ is in col(A). If $\vec{\mathbf{b}}$ is in col(A) the system will have infinitely many solutions.

Next we define the **null space** of a matrix.

> DEFINITION 4.4.3: **Null Space of a Matrix**
>
> The **null space** of an $m \times n$ matrix A is the set of all solutions to $A\vec{\mathbf{x}} = \mathbf{0}$. It is a subspace of $\mathbb{R}^n$ and is denoted by null(A).

◇ **Example 4.4(b):** Determine whether $\vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ and $\vec{\mathbf{v}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ are in the null space of $A = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix}$.

Solution: A vector $\vec{\mathbf{x}}$ is in the null space of a matrix A if $A\vec{\mathbf{x}} = \mathbf{0}$. We see that

$$A\vec{\mathbf{u}} = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -21 \\ 11 \end{bmatrix} \quad \text{and} \quad A\vec{\mathbf{v}} = \begin{bmatrix} 2 & 5 & 1 \\ -1 & -7 & -5 \\ 3 & 4 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so $\vec{\mathbf{v}}$ is in null(A) and $\vec{\mathbf{u}}$ is not.

Again considering the case where $m = n$, we have the following fact about the null space:

- If null(A) is just the zero vector, A is invertible and $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution for any vector $\vec{\mathbf{b}}$.

We conclude by pointing out the important fact that for an $m \times n$ matrix A, the null space of A is a subspace of $\mathbb{R}^n$ and the column space of A is a subspace of $\mathbb{R}^m$.

Section 4.4 Exercises To Solutions

1. For each of the following, a matrix A and a vector $\vec{\mathbf{u}}$ are given. in each case, determine whether $\vec{\mathbf{u}}$ is in the column space of A. **You should be able to do all of these by inspection, meaning without doing any computations other than some mental arithmetic**.

   (a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \vec{\mathbf{u}} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{\mathbf{u}} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 2 & -3 \\ -5 & 1 \end{bmatrix}, \quad \vec{\mathbf{u}} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad \vec{\mathbf{u}} = \begin{bmatrix} 8 \\ -16 \end{bmatrix}$

   (e) $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}, \quad \vec{\mathbf{u}} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

   (f) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$

2. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}, \quad \vec{\mathbf{u}}_1 = \begin{bmatrix} 2 \\ 9 \\ -17 \end{bmatrix}, \quad \vec{\mathbf{u}}_2 = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}$.

   (a) Remember that the column space of A is simply the set of all vectors that are linear combinations of the columns of A. Determine whether the vector $\vec{\mathbf{u}}_1$ is in the column space of A by determining whether $\vec{\mathbf{u}}_1$ is a linear combination of the columns of A. Give the vector equation that you are trying to solve, and your row reduced augmented matrix. **Be sure to tell whether $\vec{\mathbf{u}}_1$ is in the column space of A or not! Do this with a brief sentence.**

   (b) If $\vec{\mathbf{u}}_1$ *IS* in the column space of A, give a *specific* linear combination of the columns of A that equals $\vec{\mathbf{u}}_1$.

   (c) Repeat parts (a) and (b) for the vector $\vec{\mathbf{u}}_2$.

3. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}, \quad \vec{\mathbf{v}}_1 = \begin{bmatrix} 8 \\ -8 \\ -4 \end{bmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}$. The null space of A is just all the vectors $\vec{\mathbf{x}}$ for which $A\vec{\mathbf{x}} = \mathbf{0}$, and it is denoted by null(A). This means that to check to see if a vector $\vec{\mathbf{x}}$ is in the null space we need only to compute $A\vec{\mathbf{x}}$ and see if it is the zero vector. Use this method to determine whether either of the vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ is in null(A). Give your answer as a brief sentence.

4. Let $A = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 3 & 4 \\ -1 & -4 & -5 \end{bmatrix}$.

   (a) Is $\vec{\mathbf{u}}_1 = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$ in $\mathrm{col}(A)$? Justify your answer.

   (b) Is $\vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $\mathrm{col}(A)$? Justify your answer.

   (c) Does the system $A\,\vec{\mathbf{x}} = \vec{\mathbf{b}}$ have a solution for all vectors $\vec{\mathbf{b}}$ in $\mathbb{R}^3$? Explain.

   (d) Give a non-zero vector $\vec{\mathbf{v}}_1$ that is in the null space of A.

   (e) Is A invertible? Explain.

5. (a) Give a 2×2 matrix A with no zero entries whose column space is all of $\mathbb{R}^2$.

   (b) Give a 2×2 matrix B with no zero entries whose column space is *NOT* all of $\mathbb{R}^2$.

   (c) For either of your matrices A and B from parts (a) and (b) that you can, find a nonzero vector in the null space of the matrix.

6. For non-square matrices, the column space and null space contain vectors in different $\mathbb{R}^n$s. Consider the matrix

$$A = \begin{bmatrix} 2 & -3 & 1 \\ -5 & 1 & 4 \end{bmatrix}.$$

   (a) Vectors in the column space are in what $\mathbb{R}^n$?

   (b) Vectors in the null space are in what $\mathbb{R}^n$?

   (c) Can you find a vector in the $\mathbb{R}^n$ of the column space that *IS NOT* in the column space of A? If so, give one. If not, explain.

   (d) Can you find a nonzero vector in the $\mathbb{R}^n$ of the null space that *IS* in the null space of A? If so, give one. If not, explain.

4.5 Least Squares Solutions to Inconsistent Systems

Performance Criterion:

4. (f) Find the least-squares approximation to the solution of an inconsistent system of equations. Solve a problem using least-squares approximation.

   (g) Give the least squares error and least squares error vector for a least squares approximation to a solution to a system of equations.

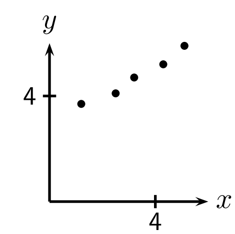
Let's begin with a motivating example.

◇ **Example 4.5(a):** Find the equation of the line containing the five points with coordinates

$$(1.2, 3.7) \quad (2.5, 4.1) \quad (3.2, 4.7) \quad (4.3, 5.2) \quad (5.1, 5.9).$$

Solution: We substitute each pair of points into the equation $y = mx + b$ of a line to get the system shown below and to the left.

$$\begin{aligned} 1.2m + b &= 3.7 \\ 2.5m + b &= 4.1 \\ 3.2m + b &= 4.7 \\ 4.3m + b &= 5.2 \\ 5.1m + b &= 5.9 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1.2 & 1 & 3.7 \\ 2.5 & 1 & 4.1 \\ 3.2 & 1 & 4.7 \\ 4.3 & 1 & 5.2 \\ 5.1 & 1 & 5.9 \end{bmatrix} \stackrel{rref}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



When then row-reduce the augmented matrix for this system in order to solve the system. When we do this, we get the last matrix above as our row-reduced matrix, indicating that the system has no solution. The five points are plotted above and to the right, and we can see that they are not on a line, which is why we were not able to solve the system. You may remember that such a system is sometimes referred to as **overdetermined**, meaning that there is too much information to allow a solution to the system. We conclude that there is no line through all of the given points.

Recall that an inconsistent system is one for which there is no solution. Often we wish to solve an inconsistent system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, and it is just not acceptable to have no solution. In those cases we can find some vector (whose components are the values we are trying to find when attempting to solve the system) that is "closer to being a solution" than all other vectors. The theory behind this process is part of the second term of this course, but we now have enough knowledge to find such a vector in a "cookbook" manner.

Suppose that we have a system of equations $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$. Pause for a moment to reflect on what we know and what we are trying to find when solving such a system: We have a system of linear equations, and the entries of A are the coefficients of all the equations. The vector $\vec{\mathbf{b}}$ is the vector whose components are the right sides of all the equations, and the vector $\vec{\mathbf{x}}$ is the vector whose components are the unknown values of the variables we are trying to find. So we know A and $\vec{\mathbf{b}}$ and we are trying to find $\vec{\mathbf{x}}$. If A is invertible, the solution vector $\vec{\mathbf{x}}$ is given by $\vec{\mathbf{x}} = A^{-1}\vec{\mathbf{b}}$. If A is not invertible there will be no solution vector $\vec{\mathbf{x}}$, but we can usually find a vector $\overline{\mathbf{x}}$ (usually spoken as "ex-bar") that comes "closest" to being a solution. Here is the formula telling us how to find that $\overline{\mathbf{x}}$:

THEOREM 4.5.1: **The Least Squares Theorem:** Let A be an $m \times n$ matrix and let $\vec{\mathbf{b}}$ be in $\mathbb{R}^m$. If $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a **least squares solution** $\overline{\mathbf{x}}$, it is given by

$$\overline{\mathbf{x}} = (A^T A)^{-1} A^T \vec{\mathbf{b}}$$

◇ **Example 4.5(b):** Find the least squares solution to

$$\begin{aligned} 1.3x_1 + 0.6x_2 &= 3.3 \\ 4.7x_1 + 1.5x_2 &= 13.5 \\ 3.1x_1 + 5.2x_2 &= -0.1 \end{aligned}.$$

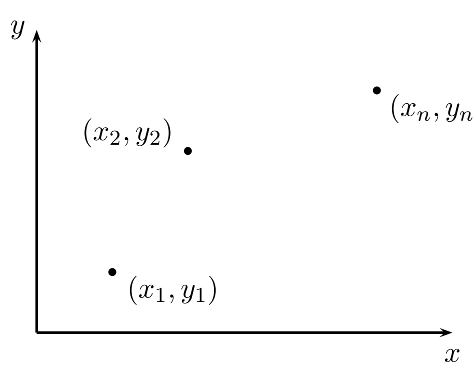
Solution: First we note that if we try to solve by row reduction we get no solution; this is an overdetermined system because there are more equations than unknowns. The matrix A and vector $\vec{\mathbf{b}}$ are

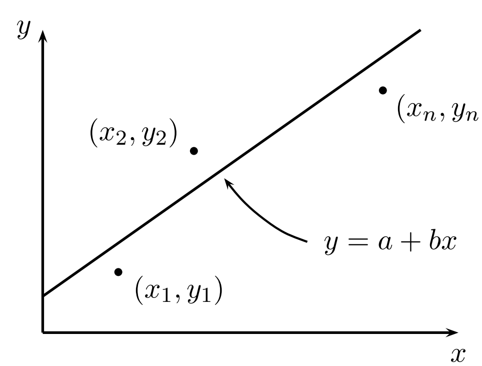
$$A = \begin{bmatrix} 1.3 & 0.6 \\ 4.7 & 1.5 \\ 3.1 & 5.2 \end{bmatrix}, \qquad \vec{\mathbf{b}} = \begin{bmatrix} 3.3 \\ 13.5 \\ -0.1 \end{bmatrix}$$

Using a calculator or *MATLAB*, we get

$$\overline{\mathbf{x}} = (A^T A)^{-1} A^T \vec{\mathbf{b}} = \begin{bmatrix} 3.5526 \\ -2.1374 \end{bmatrix}$$

Example 4.5(a) is a classic example of when we want to do something like this. We have a bunch of (x, y) data pairs from some experiment, and when we graph all the pairs they describe a trend. We then want to find a simple function $y = f(x)$ that best models that data. In some cases that function might be a line, in other cases maybe it is a parabola, and in yet other cases it might be an exponential function. Let's try to make the connection between this and linear algebra. Suppose that we have the data points (x_1, y_1), (x_2, y_2), ..., (x_n, y_n), and when we graph these points they arrange themselves in roughly a line, as shown to the left below. We then want to find an equation of the form $a + bx = y$ (note that this is just the familiar $y = mx + b$ rearranged and with different letters for the slope and y-intercept) such that $a + bx_i \approx y_i$ for $i = 1, 2, ..., n$, as shown to the right below.





If we substitute each data pair into $a + bx = y$ we get a system of equations which can be thought of in several different ways. Remember that all the x_i and y_i are known values - the unknowns are

a and b.

$$\begin{array}{c} a+x_1b=y_1 \\ a+x_2b=y_2 \\ \vdots \\ a+x_nb=y_n \end{array} \iff \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \iff A\vec{\mathbf{x}} = \vec{\mathbf{b}}$$

Above we first see the system that results from putting each of the (x_i, y_i) pairs into the equation $a + bx = y$. After that we see the $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ form of the system. We must be careful of the notation here. A is the matrix whose columns are a vector in $\mathbb{R}^n$ consisting of all ones and a vector whose components are the x_i values. It would be logical to call this last vector $\vec{\mathbf{x}}$, but instead $\vec{\mathbf{x}}$ is the vector $\begin{bmatrix} a \\ b \end{bmatrix}$. $\vec{\mathbf{b}}$ is the column vector whose components are the y_i values. Our task, as described by this interpretation, is to *find a vector* $\vec{\mathbf{x}}$ *in* $\mathbb{R}^2$ *that* A *transforms into the vector* $\vec{\mathbf{b}}$ *in* $\mathbb{R}^n$. Even if such a vector did exist, it couldn't be given as $\vec{\mathbf{x}} = A^{-1}\vec{\mathbf{b}}$ because A is not square, so can't be invertible. However, it is likely no such vector exists, but we *CAN* find the least-squares vector $\overline{\mathbf{x}} = \begin{bmatrix} a \\ b \end{bmatrix} = (A^TA)^{-1}A^T\vec{\mathbf{b}}$. When we do, its components a and b are the intercept and slope of our line.

◇ **Example 4.5(c):** Find the the least squares solution to the problem from Example 4.5(a).

Solution: Recall that we obtained the system shown below and to the left. The matrix form of the system is shown to the right of the system.

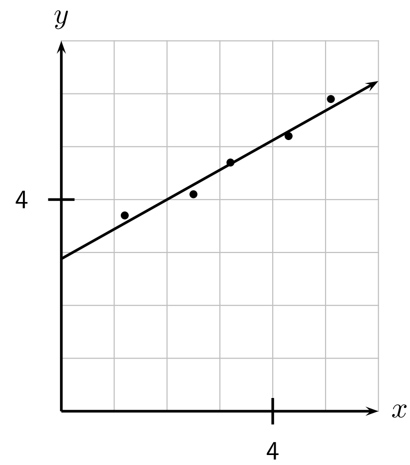
$$\begin{array}{rcl} 1.2m+b &=& 3.7 \\ 2.5m+b &=& 4.1 \\ 3.2m+b &=& 4.7 \\ 4.3m+b &=& 5.2 \\ 5.1m+b &=& 5.9 \end{array} \implies \begin{bmatrix} 1.2 & 1 \\ 2.5 & 1 \\ 3.2 & 1 \\ 4.3 & 1 \\ 5.1 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 3.7 \\ 4.1 \\ 4.7 \\ 5.2 \\ 5.9 \end{bmatrix}$$

From this we determine that $A = \begin{bmatrix} 1.2 & 1 \\ 2.5 & 1 \\ 3.2 & 1 \\ 4.3 & 1 \\ 5.1 & 1 \end{bmatrix}$, $\vec{\mathbf{x}} = \begin{bmatrix} m \\ b \end{bmatrix}$, $\vec{\mathbf{b}} = \begin{bmatrix} 3.7 \\ 4.1 \\ 4.7 \\ 5.2 \\ 5.9 \end{bmatrix}$. Note the difference between the y-intercept b of the line we are looking for and the vector $\vec{\mathbf{b}}$! We now apply the least squares theorem to obtain $\bar{x} = (A^TA)^{-1}A^T\vec{\mathbf{b}} = \begin{bmatrix} 0.56 \\ 2.88 \end{bmatrix}$. The equation of the line that is "closest" to containing the points

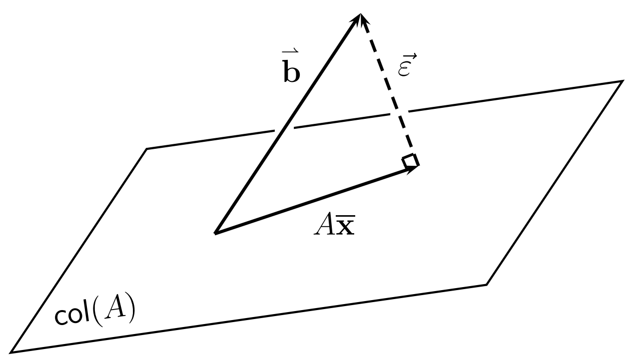
$$(1.2, 3.7) \quad (2.5, 4.1) \quad (3.2, 4.7) \quad (4.3, 5.2) \quad (5.1, 5.9).$$

is then $y = 0.56x + 2.88$.

To the right we see a plot of the five points and the line $y = 0.56x + 2.88$, showing that the line does a good job of coming close to going through all of the points.



Theoretically, here is what is happening when use least squares to solve an inconsistent system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$. The fact that the system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has no solution means that $\vec{\mathbf{b}}$ is not in the column space of A. The least squares solution to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ is simply the vector $\overline{\mathbf{x}}$ for which $A\overline{\mathbf{x}}$ is the closest vector to $\vec{\mathbf{b}}$ that is still in $\text{col}(A)$ - it is the projection of $\vec{\mathbf{b}}$ onto the column space of A. This is shown simplistically below, for the situation where the column space is a plane in $\mathbb{R}^3$.



To recap a bit, suppose we have a system of equations $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where there is no vector $\vec{\mathbf{x}}$ for which $A\vec{\mathbf{x}}$ equals $\vec{\mathbf{b}}$. What the least squares approximation allows us to do is to find a vector $\overline{\mathbf{x}}$ for which $A\overline{\mathbf{x}}$ is as "close" to $\vec{\mathbf{b}}$ as possible. We generally determine "closeness" of two objects by finding the difference between them. Because both $A\overline{\mathbf{x}}$ and $\vec{\mathbf{b}}$ are both vectors with the same number of components, we can subtract them to get a vector $\vec{\varepsilon}$ that we will call the **error vector**, shown above. The **least squares error** is then the magnitude of this vector:

<u>Definition 4.5.2</u>: If $\overline{\mathbf{x}}$ is the least-squares solution to the system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, the **least squares error vector** is

$$\vec{\varepsilon} = \vec{\mathbf{b}} - A\overline{\mathbf{x}}$$

and the **least squares error** is the magnitude of $\vec{\varepsilon}$.

◇ **Example 4.5(b):** Find the least squares error vector and least squares error vector for the solution obtained in Example 4.5(a).

Solution: The least squares error vector is

$$\vec{\varepsilon} = \vec{\mathbf{b}} - A\overline{\mathbf{x}} = \begin{bmatrix} 3.3 \\ 13.5 \\ -0.1 \end{bmatrix} - \begin{bmatrix} 1.3 & 0.6 \\ 4.7 & 1.5 \\ 3.1 & 5.2 \end{bmatrix} \begin{bmatrix} 3.5526 \\ -2.1374 \end{bmatrix} = \begin{bmatrix} -0.0359 \\ 0.0089 \\ 0.0016 \end{bmatrix}$$

The least squares error is $\|\vec{\varepsilon}\| = 0.0370$.

Section 4.5 Exercises

To Solutions

1. Consider the points $(1, 8)$, $(2, 7)$, $(3, 5)$, $(4, 2)$.

   (a) Find the least squares approximating line $y = a + bx$ for the points. Give the system of equations to be solved (in any form), the matrix A and vector $\vec{\mathbf{b}}$, the solution vector $\vec{\mathbf{x}}$ and the equation of the line.

   (b) Plot the points and your line using something like *Desmos*. (To plot the points with *Desmos*, just list them exactly as they are given above.) Does the line seem to do a good job of coming close to all of the points?

   (c) Find the least squares approximating parabola $y = a + bx + cx^2$ for the points. Give the system of equations to be solved (in any form), the matrix A and vector $\vec{\mathbf{b}}$, the solution vector $\vec{\mathbf{x}}$ and the equation of the parabola.

   (d) Plot the points and your parabola using something like *Desmos*. Does the parabola seem to do a good job of coming close to all of the points?

2. We know that through any three *non-collinear* points in $\mathbb{R}^3$ there is exactly one plane. When we have more than three such points there is likely not a plane containing all the points, because this is an overdetermined situation. But we can use least-squares to determine the plane that best approximates a plane through all of the points. If the plane is not vertical, its equation can be written

   $$z = a + bx + cy,$$

   where each of a, b and c are constants. Find the equation of the plane closest to containing the points

   $$(1.1, 0.2, 4.7), \qquad (4.3, 6.5, 5.0), \qquad (3.2, 5.1, 4.5), \qquad (7.4, 2.8, 13.8),$$

   rounding each of a, b and c to the hundredth's place.

3. Consider the system $\begin{array}{rcl} 2x_1 + 3x_2 & = & 26 \\ -x_1 + 5x_2 & = & 13 \end{array}$.

   (a) Find the least-squared solution to the system.

   (b) Solve the system by some method you have used for consistent systems (those that have "ordinary" solutions).

   (c) How do your solutions from (a) and (b) compare?

4.6 Linear Independence

Performance Criterion:

4. (h) Determine whether a set $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_k$ of vectors is a linearly independent or linearly dependent. If the vectors are linearly dependent, (1) give a non-trivial linear combination of them that equals the zero vector, (2) give any one as a linear combination of the others, when possible.

In Exercise 6 of Section 4.1 we considered several sets of vectors in $\mathbb{R}^3$, including the sets

$$\mathcal{S}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \qquad \mathcal{S}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

$$\mathcal{S}_5 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

It should be clear that set $\mathcal{S}_1$ does not span $\mathbb{R}^3$, and that any vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ is in the span of $\mathcal{S}_2$:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Because $\mathcal{S}_2$ spans $\mathbb{R}^3$, including the additional vector as done in $\mathcal{S}_5$ does not increase the span, but of course it doesn't decrease it either. Therefore we can construct any vector in $\mathbb{R}^3$ as a linear combination of iether vectors from $\mathcal{S}_2$ or vectors from $\mathcal{S}_5$. We will find that $\mathcal{S}_5$ is undesirable for getting vectors in $\mathbb{R}^3$ for the following reason:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

This shows that the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ can be constructed as a linear combination of vectors in $\mathcal{S}_5$ in more than one way. In fact, there are infinitely many different linear combinations of the vectors in $\mathcal{S}_5$ that equal *any* vector in $\mathbb{R}^3$. On the other hand, it should be clear that, given any vector in $\mathbb{R}^3$, there is *only one way* to obtain that vector as a linear combination of vectors in $\mathcal{S}_2$. So even though both sets $\mathcal{S}_2$ and $\mathcal{S}_5$ span $\mathbb{R}^3$, We will find that a set like $\mathcal{S}_2$, with enough vectors to span, but not "too many," is more desirable.

Let's take a look at what is going on from a different perspective. Suppose that we are trying to create a set $\mathcal{S}$ of vectors that spans $\mathbb{R}^3$. We might begin with one vector, say $\vec{\mathbf{u}}_1 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$, in $\mathcal{S}$.

We know by now that the span of this single vector is all scalar multiples of it, which is a line in $\mathbb{R}^3$. If we wish to increase the span, we would add another vector to $\mathcal{S}$. If we were to add a vector like $\begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}$ to $\mathcal{S}$, we would not increase the span, because this new vector is a scalar multiple of $\vec{\mathbf{u}}_1$, so it is on the line we already have and would contribute nothing new to the span of $\mathcal{S}$. To increase the span, we need to add to $\mathcal{S}$ a second vector $\vec{\mathbf{u}}_2$ that is not a scalar multiple of the vector $\vec{\mathbf{u}}_1$ that we already have. It should be clear that the vector $\vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is not a scalar multiple of $\vec{\mathbf{u}}_1$, so adding it to $\mathcal{S}$ would increase it's span.

The span of $\mathcal{S} = \{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2\}$ is a plane. When $\mathcal{S}$ included only a single vector, it was relatively easy to determine a second vector that, when added to $\mathcal{S}$, would increase it's span. Now we wish to add a third vector to $\mathcal{S}$ to further increase its span. Geometrically it is clear that we need a third vector that is *not in the plane spanned by* $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2\}$. Probabilistically, just about any vector in $\mathbb{R}^3$ would do, but what we would like to do here is create an algebraic condition that needs to be met by a third vector so that adding it to $\mathcal{S}$ will increase the span of $\mathcal{S}$.

Let's begin with what we *DON'T* want: we don't want the new vector to be in the plane spanned by $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2\}$. Now every vector $\vec{\mathbf{v}}$ in that plane is of the form $\vec{\mathbf{v}} = c_1\vec{\mathbf{u}}_1 + c_2\vec{\mathbf{u}}_2$ for some scalars c_1 and c_2. We say the vector $\vec{\mathbf{v}}$ created this way is "dependent" on $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_2$, and that is what causes it to not be helpful in increasing the span of a set that already contains those two vectors. Assuming that neither of c_1 and c_2 is zero, we could also write

$$\vec{\mathbf{u}}_1 = \frac{c_2}{c_1}\vec{\mathbf{u}}_2 - \frac{1}{c_1}\vec{\mathbf{v}} \qquad \text{and} \qquad \vec{\mathbf{u}}_2 = \frac{c_1}{c_2}\vec{\mathbf{u}}_1 - \frac{1}{c_2}\vec{\mathbf{v}},$$

showing that $\vec{\mathbf{u}}_1$ is "dependent" on $\vec{\mathbf{u}}_2$ and $\vec{\mathbf{v}}$, and $\vec{\mathbf{u}}_2$ is "dependent" on $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{v}}$. So whatever "dependent" means (we'll define it more formally soon), all three vectors are dependent on each other. We can create another equation that is equivalent to all three of the ones given so far, and that does not "favor" any particular one of the three vectors:

$$c_1\vec{\mathbf{u}}_1 + c_2\vec{\mathbf{u}}_2 + c_3\vec{\mathbf{v}} = \vec{\mathbf{0}},$$

where $c_3 = -1$.

Of course, if we want a third vector $\vec{\mathbf{u}}_3$ to add to $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2\}$ to increase its span, we would not want to choose $\vec{\mathbf{u}}_3 = \vec{\mathbf{v}}$; instead we would want a third vector that is "independent" of the two we already have. Based on what we have been doing, we would suspect that we would want

$$c_1\vec{\mathbf{u}}_1 + c_2\vec{\mathbf{u}}_2 + c_3\vec{\mathbf{u}}_3 \neq \vec{\mathbf{0}}\ . \tag{1}$$

Of course even if $\vec{\mathbf{u}}_3$ was not in the plane spanned by $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_2$, (1) would be true if $c_1 = c_2 = c_3 = 0$, but we want that to be the only choice of scalars that makes (1) true.

We now make the following definition, based on our discussion:

Definition 4.6.1: **Linear Dependence and Independence**

A set $\mathcal{S} = \{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, ..., \vec{\mathbf{u}}_k\}$ of vectors is **linearly dependent** if there exist scalars $c_1, c_2, ..., c_k$, *not all equal to zero* such that

$$c_1\vec{\mathbf{u}}_1 + c_2\vec{\mathbf{u}}_2 + \cdots + c_k\vec{\mathbf{u}}_k = \mathbf{0}. \tag{2}$$

If (2) only holds for $c_1 = c_2 = \cdots = c_k = 0$ the set $\mathcal{S}$ is **linearly independent**.

We can state linear dependence (independence) in either of two ways. We can say that the set is linearly dependent, or the vectors are linearly dependent. Either way is acceptable. Often we will get lazy and leave off the "linear" of linear dependence or linear independence. This does no harm, as there is no other kind of dependence/independence that we will be interested in.

◇ **Example 4.6(a):** Determine whether the vectors $\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix}$ are linearly dependent, or linearly independent. If they are dependent, give a non-trivial linear combination of them that equals the zero vector. (Non-trivial means that not all of the scalars are zero!)

Solution: To make such a determination we always begin with the vector equation from the definition:

$$c_1 \begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

We recognize this as the linear combination form of a system of equations that has the augmented matrix shown below and to the left, which reduces to the matrix shown below and to the right.

$$\begin{bmatrix} -1 & 1 & 7 & 0 \\ -7 & -3 & -1 & 0 \\ 3 & 2 & 4 & 0 \\ 11 & 5 & 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we see that there are infinitely many solutions, so there are certainly values of c_1, c_2 and c_3, not all zero, that make (4) true, so the set of vectors is linearly dependent. To find a non-trivial linear combination of the vectors that equals the zero vector we let the free variable c_3 *be any value other than zero*. (You should try letting it be zero to see what happens.) If we take c_3 to be one, then $c_2 = -5$ and $c_1 = 2$. Then

$$2 \begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -14 \\ 6 \\ 22 \end{bmatrix} + \begin{bmatrix} -5 \\ 15 \\ -10 \\ -25 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

◇ **Example 4.6(b):** Determine whether the vectors $\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 5 \\ -1 \end{bmatrix}$ are linearly dependent, or linearly independent. If they are dependent, give a non-trivial linear combination of them that equals the zero vector. (Non-trivial means that not all of the scalars are zero!)

Solution: To make such a determination we always begin with the vector equation from the definition:

$$c_1 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We recognize this as the linear combination form of a system of equations that has the augmented matrix shown below and to the left, which reduces to the matrix shown below and to the right.

$$\begin{bmatrix} 3 & 4 & -2 & 0 \\ -1 & 7 & 5 & 0 \\ 2 & 0 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We see that the only solution to the system is $c_1 = c_2 = c_3 = 0$, so the vectors are linearly independent.

A comment is in order at this point. The system $c_1 \vec{\mathbf{u}}_1 + c_2 \vec{\mathbf{u}}_2 + \cdots + c_k \vec{\mathbf{u}}_k = \mathbf{0}$ is homogeneous, so it will always have at least the zero vector as a solution. It is precisely *when the only solution is the zero vector that the vectors are linearly independent*.

Now suppose we have a set $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, ..., \vec{\mathbf{u}}_k\}$ of linearly dependent vectors in $\mathbb{R}^n$. By definitions, there are scalars $c_1, c_2, ..., c_k$, not all equal to zero, such that

$$c_1 \vec{\mathbf{u}}_1 + c_2 \vec{\mathbf{u}}_2 + \cdots + c_k \vec{\mathbf{u}}_k = \vec{\mathbf{0}}$$

Let c_j, for some j between 1 and k, be one of the non-zero scalars. (By definition there has to be at least one such scalar.) Then we can do the following:

$$\begin{aligned}
c_1 \vec{\mathbf{u}}_1 + c_2 \vec{\mathbf{u}}_2 + \cdots + c_j \vec{\mathbf{u}}_j + \cdots + c_k \vec{\mathbf{u}}_k &= \vec{\mathbf{0}} \\
c_j \vec{\mathbf{u}}_j &= -c_1 \vec{\mathbf{u}}_1 - c_2 \vec{\mathbf{u}}_2 - \cdots - c_k \vec{\mathbf{u}}_k \\
\vec{\mathbf{u}}_j &= -\frac{c_1}{c_j} \vec{\mathbf{u}}_1 - \frac{c_2}{c_j} \vec{\mathbf{u}}_2 - \cdots - \frac{c_k}{c_j} \vec{\mathbf{u}}_k \\
\vec{\mathbf{u}}_j &= d_1 \vec{\mathbf{u}}_1 + d_2 \vec{\mathbf{u}}_2 + \cdots + d_k \vec{\mathbf{u}}_k
\end{aligned}$$

This, along with the fact that the above computation can be reversed, gives us the following:

> <u>THEOREM 4.6.2</u>: A set $S = \{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, ..., \vec{\mathbf{u}}_k\}$ is linearly dependent if, and only if, at least one of these vectors can be written as a linear combination of the remaining vectors.

The importance of this, which we'll reiterate again later, is that *if we have a set of linearly dependent vectors with a certain span, we can eliminate at least one vector from our original set without reducing the span of the set*. If, on the other hand, we have a set of linearly *independent* vectors, eliminating any vector from the set will reduce the span of the set.

◇ **Example 4.6(c):** In Example 4.6(a) we determined that the vectors $\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix}$ are linearly dependent. Give one of them as a linear combination of the others.

Solution: In Example 4.6(a) we found that the vector equation

$$c_1 \begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

had infinitely many solutions, one of which was

$$2 \begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

We can easily solve that equation for the third vector to get

$$\begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix}$$

Note that we could also have solved (4) for either the first or second vectors. Solving for the first would give us

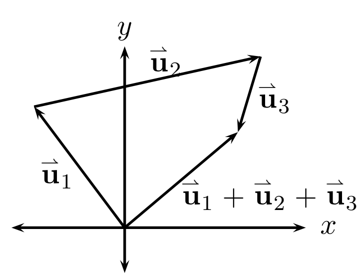
$$\begin{bmatrix} -1 \\ -7 \\ 3 \\ 11 \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 \\ -3 \\ 2 \\ 5 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 7 \\ -1 \\ 4 \\ 3 \end{bmatrix},$$

and we could solve for the second in a similar manner.

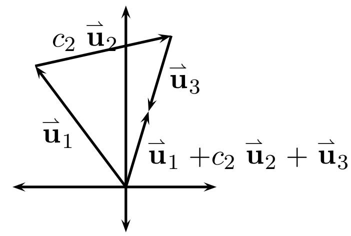
◇ **Example 4.6(d):** Determine whether the vectors $\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$ are linearly dependent.

Solution: We can easily observe that $\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$, so the first vector is a linear combination of the other two. By Theorem 4.6.2, the three vectors are linearly dependent.

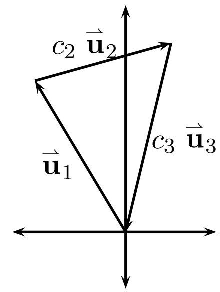
We now consider three vectors $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$ and $\vec{\mathbf{u}}_3$ in $\mathbb{R}^2$ whose sum is not the zero vector, and for which no two of the vectors are parallel. I have arranged these to show the tip-to-tail sum in the top diagram to the right; clearly their sum is not the zero vector.



At this point if we were to multiply $\vec{\mathbf{u}}_2$ by some scalar c_2 less than one we could shorten it to the point that after adding it to $\vec{\mathbf{u}}_1$ the tip of $c_2\,\vec{\mathbf{u}}_2$ would be in such a position as to line up $\vec{\mathbf{u}}_3$ with the origin. This is shown in the bottom diagram to the right.



Finally, we could then multiply $\vec{\mathbf{u}}_3$ by a scalar c_3 greater than one to lengthen it to the point of putting its tip at the origin. We would then have $\vec{\mathbf{u}}_1 + c_2\,\vec{\mathbf{u}}_2 + c_3\,\vec{\mathbf{u}}_3 = \mathbf{0}$. You should play around with a few pictures to convince yourself that this can always be done with three vectors in $\mathbb{R}^2$, as long as none of them are parallel (scalar multiples of each other). This shows us that *any three vectors in* $\mathbb{R}^2$ *are always linearly dependent*. In fact, we can say even more:



$\vec{\mathbf{u}}_1 + c_2\,\vec{\mathbf{u}}_2 + c_3\,\vec{\mathbf{u}}_3 = \mathbf{0}$

THEOREM 4.6.3: Any set of more than n vectors in $\mathbb{R}^n$ must be linearly dependent.

◇ **Example 4.6(e):** Determine whether the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$.

Solution: This is a set of three vectors in $\mathbb{R}^2$ ($\mathbb{R}^n$ with $n = 2$), so by the above theorem they must be dependent.

Note that Theorem 4.6.3 *doesn't* tell us that a set of n or fewer vectors in $\mathbb{R}^n$ must be linearly independent!

◇ **Example 4.6(f):** Determine whether the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ are independent.

Solution: Note that this is a set of two vectors in $\mathbb{R}^n$ with $n = 3$. The above theorem is therefore not helpful. However, we can see that the second vector is a linear combination of the first (two times it), so by Theorem 4.6.2 the vectors are linearly dependent.

Section 4.6 Exercises To Solutions

1. Determine whether each set of vectors is linearly independent or linearly dependent without applying the definition. That is, use Theorems 4.6.2 and 4.6.3. In each case, explain your reasoning.

(a) $S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} \right\}$,

(b) $S = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$

(c) $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(d) $S = \left\{ \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ 9 \end{bmatrix} \right\}$

(e) $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 7 \end{bmatrix} \right\}$

2. Consider the vectors $\vec{\mathbf{u}}_1 = \begin{bmatrix} -5 \\ 9 \\ 4 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$, and $\vec{\mathbf{u}}_3 = \begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix}$.

(a) Give the *VECTOR* equation that we must consider in order to determine whether the three vectors are linearly independent.

(b) Your equation has one solution for sure. What is it? What does it mean (in terms of linear dependence or independence) if that is the *ONLY* solution?

(c) Write your equation from (a) as a system of linear equations. Then give the augmented matrix for the system.

(d) Does the system have more solutions than the one you gave in (b)? If so, find one of them. (By "one" I mean one ordered triple of three numbers.)

(e) Find each of the three vectors as a linear combination of the other two.

3. Show that the vectors $\vec{\mathbf{u}} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} -4 \\ 4 \\ 9 \end{bmatrix}$ are linearly dependent. Then give one of the vectors as a linear combination of the others.

4. For the following, use the vectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$, $\vec{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$.

(a) Determine whether $\vec{\mathbf{u}} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix}$ are in $\text{span}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)$.

(b) Show that the vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_3$ are linearly dependent by the definition of linearly dependent. In other words, produce scalars c_1, c_2 and c_3 and demonstrate that they and the vectors satisfy the equation given in the definition.

(c) Since the vectors are linearly dependent, at least one the vectors can be expressed as a linear combination of the other two. Express $\vec{\mathbf{v}}_1$ as a linear combination of $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_3$.

4.7 Bases of Subspaces, Dimension

Performance Criterion:

4. (i) Determine whether a given set of vectors is a basis for a given subspace. Give a basis and the dimension of a subspace.

We have seen that the span of any set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. In a sense, the vectors whose span is being considered are the "building blocks" of the subspace. That is, every vector in the subspace is some linear combination of those vectors. Now, recall that if a set of vectors is linearly dependent, one of the vectors can be represented as a linear combination of the others. So if we are considering the span of a set of dependent vectors, we can throw out the one that can be obtained as a linear combination without affecting the span of the set of vectors.

So given a subspace, it is desirable to find what we might call a *minimal spanning set*, the smallest set of vectors whose linear combinations gives the entire subspace. Such a set is called a **basis**.

DEFINITION 4.7.1: **Basis of a Subspace**

For a subspace S, a **basis** is a set $\mathcal{B}$ of vectors such that

- the span of $\mathcal{B}$ is S,
- the vectors in $\mathcal{B}$ are linearly independent

The plural of basis is *bases* (pronounced "base-eez"). With a little thought, you should believe that *every subspace has infinitely many bases*. (This is a tiny lie - the trivial subspace consisting of just the zero has no basis vectors, which is a funny consequence of logic.)

⋄ **Example 4.7(a):** Is the set $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

Solution: Clearly for any vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ we have $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so the

span of $\mathcal{B}$ is all of $\mathbb{R}^3$. The augmented matrix for testing for linear independence is simply the identity augmented with the zero vector, giving only the solution where all the scalars are zero, so the vectors are linearly independent. Therefore the set $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

The basis just given is called the **standard basis** for $\mathbb{R}^3$, and its vectors are often denoted by $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$ and $\vec{\mathbf{e}}_3$. There is a standard basis for every $\mathbb{R}^n$, and $\vec{\mathbf{e}}_1$ is always the vector whose first component is one and all others are zero, $\vec{\mathbf{e}}_2$ is the vector whose second component is one and all others are zero, and so on.

◇ **Example 4.7(b):** Let $S_1 = \left\{ \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix} \right\}$. In the previous section we saw that span(S_1) is a subspace of $\mathbb{R}^3$. Is S_1 a basis for span(S_1)?

Solution: Clearly S_1 meets the first condition for being a basis and, since we can see that neither of these vectors is a scalar multiple of the other, they are linearly independent. Therefore they are a basis for span(S_1).

◇ **Example 4.7(c):** Let $S_2 = \left\{ \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ -9 \end{bmatrix} \right\}$. Span$(S_2)$ is a subspace of $\mathbb{R}^3$; is S_2 a basis for span(S_2)?

Solution: Once again this set meets the first condition of being a subspace. We can also see that $(-1)\begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -9 \end{bmatrix}$, so the set S_2 is linearly dependent. Therefore it is NOT a basis for span(S).

◇ **Example 4.7(d):** The yz-plane in $\mathbb{R}^3$ is a subspace. Give a basis for this subspace.

Solution: We know that a set of two linearly independent vectors will span a plane, so we simply need two vectors in the yz-plane that are not scalar multiples of each other. The simplest choices are the two vectors $\vec{\mathbf{e}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{\mathbf{e}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, so they are a basis for the yz-plane.

Considering this last example, it is not hard to show that the set $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is also a basis for the yz-plane, and there are many other sets that are bases for that plane as well. All of those sets will contain two vectors, illustrating the fact that *every basis of a subspace has the same number of vectors*. This allows us to make the following definition:

> DEFINITION 4.7.2: **Dimension of a Subspace**
>
> The **dimension** of a subspace is the number of elements in a basis for that subspace.

Looking back at Examples 4.7(a), (b) and (d), we then see that $\mathbb{R}^3$ has dimension three, and span(S_1) has dimension two, and the yz-plane in $\mathbb{R}^3$ has dimension two.

Although its importance may not be obvious to you at this point, here's why we care about a basis rather than any set that spans a subspace:

THEOREM 4.7.3: Any vector in a subspace S with basis $\mathcal{B}$ is represented by one, and only one, linear combination of vectors in $\mathcal{B}$.

◇ **Example 4.7(e):** In Example 4.7(d) we determined that the set of all vectors of the form $\vec{\mathbf{x}} = \begin{bmatrix} a \\ a+b \\ b \\ a-b \end{bmatrix}$, for some real numbers a and b, is a subspace of $\mathbb{R}^4$. Give a basis for that subspace.

Solution: The key computation in Example 4.7(d) was

$$\vec{\mathbf{x}} = \begin{bmatrix} a \\ a+b \\ b \\ a-b \end{bmatrix} = \begin{bmatrix} a \\ a \\ 0 \\ a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ b \\ -b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

The set of vectors under consideration is spanned by $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$, and we can see that those two vectors are linearly independent (because they aren't scalar multiples of each other, which is sufficient for independence when considering just two vectors). Therefore they form a basis for the subspace of vectors of the given form.

Section 4.7 Exercises

To Solutions

1. Which of the following is a basis for $\mathbb{R}^2$? For those that aren't, tell why not.

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \qquad S_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \qquad S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

2. Which if the following is a basis for $\mathbb{R}^3$? For those that aren't, tell why not.

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \right\} \qquad S_4 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

3. Which of the following is a basis for the line in $\mathbb{R}^2$ containing the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$? For those that aren't, tell why not.

$$S_1 = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right\} \qquad S_2 = \left\{ \begin{bmatrix} 6 \\ 2 \end{bmatrix} \right\} \qquad S_3 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

4. Which of the following is a basis for the yz-plane in $\mathbb{R}^3$? For those that aren't, tell why not.

$$S_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\} \qquad S_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$S_3 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \qquad S_4 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

5. For each of the following sub*sets* of $\mathbb{R}^3$, think of each point as a position vector; each set then becomes a set of vectors rather than points. For each,

   - determine whether the set is a sub*space* and
   - if it is *NOT* a subspace, give a reason why it isn't by doing one of the following:
     - ◇ stating that the set does not contain the zero vector
     - ◇ giving a vector that is in the set and a scalar multiple that isn't (show that it isn't)
     - ◇ giving two vectors that in the set and showing that their sum is not in the set
   - if it *IS* a subspace, give a basis for the subspace.

   (a) All points on the horizontal plane at $z = 3$.

   (b) All points on the xz-plane.

   (c) All points on the line containing $\vec{\mathbf{u}} = [-3, 1, 4]$.

   (d) All points on the lines containing $\vec{\mathbf{u}} = [-3, 1, 4]$ and $\vec{\mathbf{v}} = [5, 0, 2]$.

   (e) All points for which $x \geq 0$, $y \geq 0$ and $z \geq 0$.

   (f) All points $\vec{\mathbf{x}}$ given by $\vec{\mathbf{x}} = \vec{\mathbf{w}} + s\,\vec{\mathbf{u}} + t\,\vec{\mathbf{v}}$, where $\vec{\mathbf{w}} = [1, 1, 1]$ and $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are as in (d).

   (g) All points $\vec{\mathbf{x}}$ given by $\vec{\mathbf{x}} = s\,\vec{\mathbf{u}} + t\,\vec{\mathbf{v}}$, where $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are as in (d).

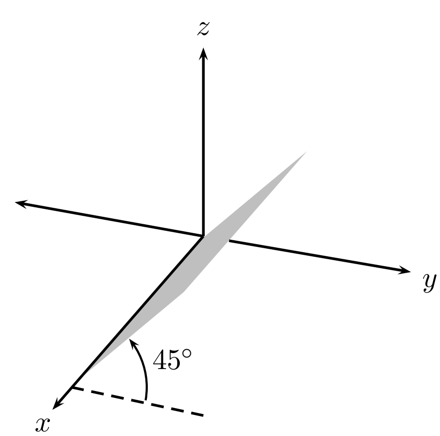
   (h) The vector **0**.

   (i) All of $\mathbb{R}^3$.

6. Determine whether each of the following is a subspace. If not, give an appropriate counterexample; if so, give a basis for the subspace.

   (a) The subset of $\mathbb{R}^2$ consisting of all vectors on or to the right of the y-axis.

   (b) The subset of $\mathbb{R}^3$ consisting of all vectors in a plane containing the x-axis and at a 45 degree angle to the xy-plane. See diagram to the right.



7. The xy-plane is a subspace of $\mathbb{R}^3$.

   (a) Give a set of at least two vectors in the xy-plane that is not a basis for that subspace, and tell why it isn't a basis.

   (b) Give a different set of at least two vectors in the xy-plane that is not a basis for that subspace *for a different reason*, and tell why it isn't a basis.

4.8 Bases for the Column Space and Null Space of a Matrix

Performance Criteria:

4. (j) Find the dimensions of, and bases for, the column space and null space of a given matrix.

   (k) Given the dimension of the column space and/or null space of the coefficient matrix for a system of equations, say as much as you can about how many solutions the system has.

In a previous section you learned about two special subspaces related to a matrix A, the column space of A and the null space of A. Remember the importance of those two spaces:

A system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution if, and only if, $\vec{\mathbf{b}}$ is in the column space of A.

If the null space of a square matrix A is just the zero vector, A is invertible and $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution for any vector $\vec{\mathbf{b}}$.

We would now like to be able to find bases for the column space and null space of a given vector A. The following describes how to do this:

Theorem 4.8.1: **Bases for Null Space and Column Space**

- A basis for the column space of a matrix A is the columns of A corresponding to columns of $rref(A)$ that contain leading ones.
- The solution to $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ (which can be easily obtained from $rref(A)$ by augmenting it with a column of zeros) will be an arbitrary linear combination of vectors. Those vectors form a basis for null(A).

◇ **Example 4.8(a):** Find bases for the null space and column space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix}.$$

Solution: The reduced row-echelon form of A is shown below and to the left. We can see that the first through third columns contain leading ones, so a basis for the column space of A is the set shown below and to the right.

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 4 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \right\}$$

If we were to augment A with a column of zeros to represent the system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ and row reduce we'd get the matrix shown above and to the left but with an additional column of zeros on the right. We'd then have x_4 as a free variable t, with $x_1 = -3t$, $x_2 = 7t$ and $x_3 = -4t$. The solution to $A\vec{\mathbf{x}} = \mathbf{0}$ is any scalar multiple of $\begin{bmatrix} -3 \\ 7 \\ -4 \\ 1 \end{bmatrix}$, so that vector is a basis for the null space of A.

◇ **Example 4.8(b):** Find a basis for the null space and column space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 7 & 1 \\ -2 & 1 & 7 \end{bmatrix}.$$

Solution: The reduced row-echelon form of this matrix is the identity, so a basis for the column space consists of all the columns of A. If we augment A with the zero vector and row reduce we get a solution of the zero vector, so the null space is just the zero vector (which is of course a basis for itself).

We should note in the last example that the column space is all of $\mathbb{R}^3$, so for any vector $\vec{\mathbf{b}}$ in $\mathbb{R}^3$ there is a vector $\vec{\mathbf{x}}$ for which $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$. Thus $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution for every choice of $\vec{\mathbf{b}}$.

There is an important distinction to be made between a subspace and a basis for a subspace:

- Other than the trivial subspace consisting of the zero vector, *a subspace is an infinite set of vectors*.
- A basis for a subspace *is a finite set of vectors*. In fact a basis consists of relatively few vectors; the basis for any subspace of $\mathbb{R}^n$ contains at most n vectors (and it only contains n vectors if the subspace is all of $\mathbb{R}^n$).

To illustrate, consider the matrix $A = \begin{bmatrix} 1 & 3 & -2 & -4 \\ 3 & 7 & 1 & 4 \\ -2 & 1 & 7 & 7 \end{bmatrix}$ from Example 4.8(a). The set $\left\{ \begin{bmatrix} -3 \\ 7 \\ -4 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space of A, whereas the set $\left\{ t \begin{bmatrix} -3 \\ 7 \\ -4 \\ 1 \end{bmatrix} \right\}$ *IS* the null space of A.

We finish this section with a couple definitions and a major theorem of linear algebra. The importance of these will be seen in the next section.

DEFINITION 4.8.2: **Rank and Nullity of a Matrix**

- The **rank** of a matrix A, denoted rank(A), is the dimension of its column space.
- The **nullity** of a matrix A, denoted nullity(A), is the dimension of its null space.

THEOREM 4.8.3: **The Rank Theorem**

For an $m \times n$ matrix A, rank(A) + nullity$(A) = n$.

Section 4.8 Exercises To Solutions

1. Consider the matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & 6 \\ 2 & 2 & -4 \end{bmatrix}$, which has row-reduced form $\begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

   In this exercise you will see how to find a basis for the null space of A. All this means is that *you are looking for a "minimal" set of vectors whose span (all possible linear combinations of them) give all the vectors $\vec{\mathbf{x}}$ for which $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$.*

   (a) Give the augmented matrix for the system of equations $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$, then give its row reduced form.

   (b) There are two free variables, x_3 and x_2. Let $x_3 = t$ and $x_2 = s$, then find x_1 (in terms of s and t). Give the vector $\vec{\mathbf{x}}$, in terms of s and t.

   (c) Write $\vec{\mathbf{x}}$ as the sum of two vectors, one containing only the parameter s and the other containing only the parameter t. Then factor s out of the first vector and t out of the second vector. You now have $\vec{\mathbf{x}}$ as all linear combinations of two vectors.

   (d) Those two vectors are linearly independent, since neither of them is a scalar multiple of the other, so both are essential in the linear combination you found in (c). They then form a basis for the null space of A. Write this out as a full sentence, "A basis for ...". *A basis is technically a set of vectors, so use the set brackets $\{\ \}$ appropriately.*

2. Consider the matrix $A = \begin{bmatrix} 1 & -1 & 5 \\ 3 & 1 & 11 \\ 2 & 5 & 3 \end{bmatrix}$

   (a) Solve the system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$. You should get infinitely many solutions containing one or more parameters. Give the general solution, in terms of the parameters. **Give all values in exact form**.

   (b) If you didn't already, you should be able to give the general solution as a linear combination of vectors, with the scalars multiplying them being the parameter(s). Do this.

   (c) The vector or vectors you see in (c) is (are) a basis for the null space of A. Give the basis.

3. Consider the matrix A from the previous exercise.

   (a) What is the nullity of A?

   (b) From the Rank Theorem, what is the rank of A?

   (c) When doing part (a) of the previous exercise you should have obtained the row reduced form of the matrix A (of course you augmented it). A basis for the column space of A is the columns of A (*NOT* the columns of the row reduced form of A!) corresponding to the leading variables in the row reduced form of A. Give the basis for the column space of A.

   (d) Does your result from part (c) agree with the rank that you determined in (b)? If not, find what's wrong and correct it.

4. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}$, $\vec{\mathbf{u}}_1 = \begin{bmatrix} 2 \\ 9 \\ -17 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}$

   (a) Determine whether each of $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_2$ is in the column space of A.

   (b) Find a basis for col(A). **Give your answer with a brief sentence, and indicate that the basis is a set of vectors.**

   (c) One of the vectors $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_2$ *IS* in the column space of A. Give a linear combination of the *basis vectors* that equals that vector.

   (d) What is the rank of A?

5. Again let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}$, and let $\vec{\mathbf{v}}_1 = \begin{bmatrix} 8 \\ -8 \\ -4 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}$.

   (a) Determine whether each of the vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ is in null(A). Give your answer as a brief sentence.

   (b) Determine a basis for null(A), giving your answer in a brief sentence.

   (c) Referring to your answer from part (d) of the previous exercise, what is the nullity of A?

   (d) Give the linear combinations of the basis vectors of the null space for either of the vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ that are in the null space.

6. Each of the following matrices is the row-reduced matrix of a given matrix A. For each, give $\text{rank}(A)$ and $\text{nullity}(A)$.

   (a) $\begin{bmatrix} 1 & 0 & 6 & -1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & -1 & 0 & -4 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4.9 Solutions to Systems of Equations

Performance Criterion:

4. (l) Determine, from given information about the coefficient matrix A and vector $\vec{\mathbf{b}}$ of a system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, whether the system has any solutions and, if it does, whether there is more than one solution.

You may have found the last section to be a bit overwhelming, and you are probably wondering why we bother with all of the definitions in that section. The reason is that those ideas form tools and language for discussing whether a system of equations

(a) has a solution (meaning at least one) and

(b) if it does have a solution, is there only one.

Item (a) above is what mathematicians often refer to as the *existence* question, and item (b) is the *uniqueness* question. Concerns with "existence and uniqueness" of solutions is not restricted to linear algebra; it is a big deal in the study of differential equations as well.

Consider a system of equations $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$. We saw previously that the product $A\vec{\mathbf{x}}$ is the linear combination of the columns of A with the components of $\vec{\mathbf{x}}$ as the scalars of the linear combination. This means that the system will only have a solution if $\vec{\mathbf{b}}$ is a linear combination of the columns of A. But all of the linear combinations of the columns of A is just the span of those columns - the column space! the conclusion of this is as follows:

A system of equations $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution (meaning *at least* one solution) if, and only if, $\vec{\mathbf{b}}$ is in the column space of A.

Let's look at some consequences of this.

◇ **Example 4.9(a):** Let $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ represent a system of five equations in five unknowns, and suppose that $\text{rank}(A) = 3$. Does the system have (for certain) a solution?

Solution: Since the system has five equations in five unknowns, $\vec{\mathbf{b}}$ is in $\mathbb{R}^5$. Because $\text{rank}(A) = 3$, the column space of A only has dimension three, so it is not all of $\mathbb{R}^5$ (which of course has dimension five). Therefore $\vec{\mathbf{b}}$ may or may not be in the column space of A, and we can't say for certain that the system has a solution.

◇ **Example 4.9(b):** Let $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ represent a system of three equations in five unknowns, and suppose that $\text{rank}(A) = 3$. Does the system have (for certain) a solution?

Solution: Because there are three equations and five unknowns, A is 3×5 and the columns of A are in $\mathbb{R}^3$. Because $\text{rank}(A) = 3$, the column space must then be all of $\mathbb{R}^3$. Therefore $\vec{\mathbf{b}}$ will be in the column space of A and the system has at least one solution.

Now suppose that we have a system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and a vector $\vec{\mathbf{x}}$ that *IS* a solution to the system. Suppose also that nullity$(A) \neq 0$. Then there is some $\vec{\mathbf{y}} \neq \vec{\mathbf{0}}$ such that $A\vec{\mathbf{y}} = \vec{\mathbf{0}}$ and

$$A(\vec{\mathbf{x}} + \vec{\mathbf{y}}) = A\vec{\mathbf{x}} + A\vec{\mathbf{y}} = \vec{\mathbf{b}} + \vec{\mathbf{0}} = \vec{\mathbf{b}}\,.$$

This shows that both $\vec{\mathbf{x}}$ and $\vec{\mathbf{x}} + \vec{\mathbf{y}}$ are solutions to the system, so the system does not have a unique solution. The thing that allows this to happen is the fact that the null space of A contains more than just the zero vector. This illustrates the following:

> A system of equations $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ can have a unique solution only if the nullity of A is zero (that is, the null space contains only the zero vector).

Note that this says nothing about whether a system has a solution to begin with; it simply says that if there is a solution and the nullity is zero, then that solution is unique.

◇ **Example 4.9(c):** Consider again a system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ of three equations in five unknowns, with rank$(A) = 3$, as in Example 4.9(b). We saw in that example that the system has at least one solution - is there a unique solution?

Solution: We note first of all that A is 3×5, so the n of the Rank Theorem is five. We know that $\text{rank}(A)$ is three so, by the Rank Theorem, the nullity is then two. Thus the null space contains more than just the zero vector, so the system does not have a unique solution.

◇ **Example 4.9(d):** Suppose we have a system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$, with nullity$(A) = 2$. Does the system have a solution and, if it does, is it unique?

Solution: Because the system is homogeneous, it has at least one solution, the $zero$ vector. But the null space contains more than just the zero vector, so the system has more than one solution, so there is not a unique solution.

Section 4.9 Exercises

To Solutions

1. Let $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ be a system of equations, with A an $m \times n$ matrix where $m = n$ unless specified otherwise. For each situation below, determine whether the system *COULD* have

   (*i*) no solution (*ii*) exactly one solution (*iii*) infinitely many solutions

   Give all possibilities for each.

   (a) $\det(A) = 0$ (b) $\det(A) \neq 0$ (c) $\vec{\mathbf{b}} = \vec{\mathbf{0}}$

   (d) $\vec{\mathbf{b}} = \vec{\mathbf{0}}$, A invertible (e) $m < n$ (f) $m > n$

   (g) columns of A linearly independent (h) columns of A linearly dependent

(i) $\vec{\mathbf{b}} = \vec{\mathbf{0}}$, columns of A linearly independent

(j) $\vec{\mathbf{b}} = \vec{\mathbf{0}}$, columns of A linearly dependent

2. Consider the system of equations
$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 4 \\ 2x_1 + x_2 - 4x_3 &= 3 \\ -3x_1 + 4x_2 - x_3 &= -2 \end{aligned}.$$

   (a) Give the $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ form of the system.

   (b) Find a basis for the column space of A.

   (c) What is $\text{rank}(A)$? Is the system guaranteed to have a solution?

   (d) What is $\text{nullity}(A)$? If the system has a solution, is it unique? That is, is there only one solution?

   (e) Find a basis for the null space of A.

   (f) Solve the system if possible. If it isn't, say so.

3. Now consider the system of equations
$$\begin{aligned} x_1 \qquad + 2x_3 &= -5 \\ -2x_1 + 5x_2 \qquad &= 11 \\ 2x_1 + 5x_2 + 8x_3 &= -7 \end{aligned}.$$
Repeat the steps of the previous exercise.

4. Compare the system
$$\begin{aligned} x_1 \qquad + 2x_3 &= -1 \\ -2x_1 + 5x_2 \qquad &= -1 \\ 2x_1 + 5x_2 + 8x_3 &= -5 \end{aligned}$$
with the one in Exercise 3.

   (a) Note that the system has the same A as that from Exercise 3, so the column space, null space rank and nullity are the same as they were in that exercise. Is the vector $\vec{\mathbf{b}}$ of this exercise in $\text{col}(A)$?

   (b) Solve the system.

4.10 Chapter 4 Exercises

1. (a) Give a set of three non-zero vectors in $\mathbb{R}^3$ whose span is a line.

   (b) Suppose that you have a set of two non-zero vectors in $\mathbb{R}^3$ that are not scalar multiples of each other. What is their span? How can you create a new vector that is not a scalar multiple of either of the other two vectors but, when added to the set, does not increase the span?

   (c) How many vectors need to be in a set for it to have a chance of spanning all of $\mathbb{R}^3$?

2. Give a set of nonzero vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ in $\mathbb{R}^2$ that **DOES NOT** span $\mathbb{R}^2$. Then give a third vector $\vec{\mathbf{v}}_3$ so that all three vectors **DO** span $\mathbb{R}^2$.

3. Give a set of three vectors, with no one being a scalar multiple of just one other, that span the xy-plane in $\mathbb{R}^3$.

4. The things in a set are called *elements*. The union of two sets A and B is a new set C consisting of every element of A along with every element of B and nothing else. (If something is an element of both A and B, it is only included in C once.) Every subspace of an $\mathbb{R}^n$ is a subset of that $\mathbb{R}^n$ that possesses some additional special properties. Show that the union of two subspaces is not generally a subspace by giving a specific $\mathbb{R}^n$ and two specific subspaces, then showing that the union is not a subspace.

5. Suppose that the column space of a 3×3 matrix A has dimension two. What does this tell us about the nature of the solutions to a system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$? Show that the vectors $\vec{\mathbf{u}}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\vec{\mathbf{u}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{\mathbf{u}}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are linearly dependent. Then give $\vec{\mathbf{u}}_2$ as a linear combination of $\vec{\mathbf{u}}_1$ and $\vec{\mathbf{u}}_3$.

6. (a) Give three non-zero linearly dependent vectors in $\mathbb{R}^3$ for which removing any one of the three leaves a linearly independent set.

   (b) Give three non-zero linearly dependent vectors in $\mathbb{R}^3$ for which removing one vector leaves a linearly independent set but removing a different one (of the original three) leaves a linearly dependent set.

7. Consider the vectors $\vec{\mathbf{u}} = \begin{bmatrix} 8 \\ -2 \\ 4 \end{bmatrix}$, $\vec{\mathbf{v}} = \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{\mathbf{w}} = \begin{bmatrix} -16 \\ 4 \\ -8 \end{bmatrix}$.

   (a) Is the set of all vectors $\vec{\mathbf{x}} = \vec{\mathbf{u}} + t\,\vec{\mathbf{v}}$, where t ranges over all real numbers, a subspace of $\mathbb{R}^3$? If so, give a basis for the subspace; if not, tell why not.

   (b) Is the set of all vectors $\vec{\mathbf{x}} = \vec{\mathbf{u}} + t\,\vec{\mathbf{w}}$, where t ranges over all real numbers, a subspace of $\mathbb{R}^3$? If so, give a basis for the subspace; if not, tell why not.

(c) Is the set of all vectors $\vec{\mathbf{x}} = \vec{\mathbf{u}} + s\vec{\mathbf{v}} + t\vec{\mathbf{w}}$, where t ranges over all real numbers, a subspace of $\mathbb{R}^3$? If so, give a basis for the subspace; if not, tell why not.

8. Consider the matrix $A = \begin{bmatrix} 2 & 2 & 2 \\ -2 & 5 & 2 \\ 8 & 1 & 4 \end{bmatrix}$.

(a) Find a basis for row(A), the row space of A. What is the dimension of row(A)?

(b) Find a basis for col(A), the column space of A. What is the dimension of col(A)?

9. Give bases for the null and column spaces of the matrix $A = \begin{bmatrix} -6 & 3 & 30 \\ 2 & -1 & -10 \\ -4 & 2 & 20 \end{bmatrix}$.

10. (a) Give a 3×3 matrix B for which the column space has dimension one. (**Hint:** What kind of subspace of $\mathbb{R}^3$ has dimension one?)

(b) Find a basis for the column space of B.

(c) What should the dimension of the null space of B be?

(d) Find a basis for the null space of B.

B Solutions to Exercises

B.4 Chapter 4 Solutions

Section 4.1 Solutions Back to 4.1 Exercises

1. (a) The span of the set is the x-axis.
   (b) The span of the set is the xz-plane, or the plane $y = 0$.
   (c) The span of the set is all of $\mathbb{R}^2$.
   (d) The span of the set is the origin.
   (e) The span of the set is a line through the origin and the point $(1, 2, 3)$.

2. (a) $\vec{\mathbf{w}}$ is not in the span of S. (b) $\vec{\mathbf{w}}$ is not in the span of S.

   (c) $\begin{bmatrix} 8 \\ 38 \\ -14 \\ 11 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -4 \\ -3 \\ 7 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 6 \\ -4 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix} = -4\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 11\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

3. (a) Because the vectors in $\mathcal{S}_1$ are not scalar multiples of each other, they span all of $\mathbb{R}^2$. The vectors in $\mathcal{S}_2$ are scalar multiples of each other, so their span is just a line in $\mathbb{R}^2$, and the spans of the two sets are not the same.
   (b) The span of each set is a line, but the lines spanned are not the same, so the spans of the two sets are different.
   (c) The two sets each span the same line. We can tell this because the vector in $\mathcal{S}_2$ is -3 times the vector in $\mathcal{S}_1$.
   (d) The two vectors in each set are not scalar multiples of each other, so both sets span all of $\mathbb{R}^2$ and their spans are equal.
   (e) We see that $\mathcal{S}_2$ is $\mathcal{S}_1$ with the additional vector $\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$ included. We can find that

   $$\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix},$$

   so the third vector in $\mathcal{S}_2$ is in the span of $\mathcal{S}_1$, so the spans of the two sets are the same.
   (f) This time $\mathcal{S}_2$ is $\mathcal{S}_1$ with the additional vector $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ included. We can find that this vector is not in the span of $\mathcal{S}_1$, so the spans of the two sets are not the same in this case.

Section 4.2 Solutions Back to 4.2 Exercises

1. (a) $\mathcal{S}$ is not closed under either addition or scalar multiplication.
   (b) $\mathcal{S}$ is closed under both addition and scalar multiplication.
   (c) $\mathcal{S}$ is not closed under either addition or scalar multiplication.
   (d) $\mathcal{S}$ is closed under both addition and scalar multiplication.

(e) $\mathcal{S}$ is not closed under either addition or scalar multiplication.

2. (a) The set is closed under both addition and scalar multiplication, and is spanned by $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

(b) The set is closed under both addition and scalar multiplication, and is spanned by $\mathcal{S} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(c) The set is not closed under either addition or scalar multiplication.

(d) The set is closed under both addition and scalar multiplication, and is spanned by $\mathcal{S} = \left\{ \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\}$.

(e) The set is closed under both addition and scalar multiplication, and is spanned by $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(f) The set is not closed under either addition or scalar multiplication.

Section 4.3 Solutions Back to 4.3 Exercises

1. (a) Not a subspace, doesn't contain the zero vector.

(b) Subspace. (c) Subspace.

(d) Not a subspace, the vector $\begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$ is not on either line because it is not a scalar multiple of either vector.

(e) Not a subspace, the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the set, but $-2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$ is not.

(f) The set is a plane not containing the zero vector, so it is not a subspace.

(g) This is a plane containing the origin, so it is a subspace.

(h) The vector $\mathbf{0}$ is a subspace. (i) Subspace.

2. (a) The set is not a subspace because it does not contain the zero vector. We can tell this because $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are not scalar multiples of each other.

(b) The set is a subspace, and either of the vectors $\vec{\mathbf{u}}$ or $\vec{\mathbf{w}}$ by itself is a basis, as is any scalar multiple of either of them.

Section 4.4 Solutions Back to 4.4 Exercises

1. (a) $\vec{\mathbf{u}}$ is in the column space of A.
   (b) $\vec{\mathbf{u}}$ is not in the column space of A.
   (c) $\vec{\mathbf{u}}$ is in the column space of A.
   (d) $\vec{\mathbf{u}}$ is in the column space of A.
   (e) $\vec{\mathbf{u}}$ is not in the column space of A.
   (f) $\vec{\mathbf{u}}$ is in the column space of A.

2. (a) $c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ -9 \\ 17 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 3 & -2 & -9 \\ -1 & -4 & 6 & 17 \end{bmatrix}$

   $\vec{\mathbf{u}}_1$ is in the column space.

   (b) $-3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ -9 \\ 17 \end{bmatrix}$.

   (c) $c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 3 & -2 & 15 \\ -1 & -4 & 6 & 2 \end{bmatrix}$

   $\vec{\mathbf{u}}_2$ is not in the column space.

3. $\vec{\mathbf{v}}_1$ is in null(A) and $\vec{\mathbf{v}}_2$ is not.

4. (a) $\vec{\mathbf{u}}_1$ is in $\text{col}(A)$. For example $\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$.

   (b) $\vec{\mathbf{u}}_2$ is not in $\text{col}(A)$ because there is no linear combination of the columns of A that equals $\vec{\mathbf{u}}_2$.

   (c) We know that $A\,\vec{\mathbf{x}} = \vec{\mathbf{u}}_2$ has no solution, because of Theorem 4.4.2.

   (d) There are many possibilities - one example is $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

   (e) A is not invertible. If it were, there would be a solution to $A\,\vec{\mathbf{x}} = \vec{\mathbf{b}}$ for all $\vec{\mathbf{b}}$ in $\mathbb{R}^3$, which we know is not the case.

6. (a) $\mathbb{R}^2$ (b) $\mathbb{R}^3$

   (c) It is clear that the first two columns of A span all of $\mathbb{R}^2$, so there are no vectors in $\mathbb{R}^2$ not in the column space of A.

   (d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is in the null space of A. There are, of course, other vectors in $\mathbb{R}^3$ that are in $\text{null}(A)$.

Section 4.5 Solutions Back to 4.5 Exercises

1. (a) $y = 10.5 - 2x$
   (b) The line seems to do a pretty good job of coming close to all of the points.
   (c) $y = 8 + 0.5x - 0.5x^2$
   (d) The parabola goes through all of the points.

2. $z = 2.95 + 1.79x - 0.85y$

Section 4.6 Solutions Back to 4.6 Exercises

1. (a) The set is linearly independent because the vectors are not scalar multiples of each other.
   (b) A set of one vector $\vec{\mathbf{v}}$ is linearly independent because the only solution to $c\,\vec{\mathbf{v}} = \vec{\mathbf{0}}$ is $c = 0$.
   (c) The set is linearly dependent because the second vector is the sum of the first and third.
   (d) The set is linearly dependent because the vectors are scalar multiples of each other. We can see this because both are scalar multiples of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.
   (e) The set is linearly dependent because it consists of more than three vectors in $\mathbb{R}^3$.

2. (a) $c_1\,\vec{\mathbf{u}}_1 + c_2\,\vec{\mathbf{u}}_2 + c_3\,\vec{\mathbf{u}}_3 = \vec{\mathbf{0}}$.
   (b) The zero vector (in other words, $c_1 = c_2 = c_3 = 0$) is a solution. If it is the *ONLY* solution, then the vectors are linearly independent.
   (c) $$\begin{aligned} -5c_1 + 5c_2 + 5c_3 &= 0 \\ 9c_1 + 0c_2 + 9c_3 &= 0 \\ 4c_1 + 6c_2 + 16c_3 &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} -5 & 5 & 5 & 0 \\ 9 & 0 & 9 & 0 \\ 4 & 6 & 16 & 0 \end{bmatrix}$$
   (d) $c_1 = -1,\ c_2 = -2,\ c_3 = 1$ OR $c_1 = 1,\ c_2 = 2,\ c_3 = -1$ OR any scalar multiple of these.
   (e) $\vec{\mathbf{u}}_1 = -2\,\vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3$, $\vec{\mathbf{u}}_2 = -\frac{1}{2}\,\vec{\mathbf{u}}_1 + \frac{1}{2}\,\vec{\mathbf{u}}_3$, $\vec{\mathbf{u}}_3 = \vec{\mathbf{u}}_1 + 2\,\vec{\mathbf{u}}_2$

3. Solving $c_1 \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix} + c_3 \begin{bmatrix} -4 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ gives the general solution $c_1 = -\frac{3}{2}t,\ c_2 = \frac{1}{2}t,\ c_3 = t$. Therefore $\vec{\mathbf{w}} = \frac{3}{2}\,\vec{\mathbf{u}} - \frac{1}{2}\,\vec{\mathbf{v}}$. (Also $\vec{\mathbf{u}} = \frac{1}{3}\,\vec{\mathbf{v}} + \frac{2}{3}\,\vec{\mathbf{w}}$ and $\vec{\mathbf{v}} = 3\,\vec{\mathbf{u}} - 2\,\vec{\mathbf{w}}$.)

4. (a) $\vec{\mathbf{u}}$ is not in $\text{span}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3)$, but $\vec{\mathbf{w}}$ is.
   (b) $-6\,\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2 + 5\,\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$.
   (c) $\vec{\mathbf{v}}_1 = \frac{1}{6}\,\vec{\mathbf{v}}_2 + \frac{5}{6}\,\vec{\mathbf{v}}_3$.

Section 4.7 Solutions Back to 4.7 Exercises

1. $\mathcal{S}_2$ is a basis for $\mathbb{R}^2$. $\mathcal{S}_1$ is not a basis because it is not linearly independent, $\mathcal{S}_3$ is not a basis because it doesn't span $\mathbb{R}^2$.

2. $\mathcal{S}_3$ is not a basis because the vectors are not independent. The otehr sets are all bases for $\mathbb{R}^3$.

3. $\mathcal{S}_2$ is a basis for the line in $\mathbb{R}^2$ containing the given vector. $\mathcal{S}_1$ is not a basis because the set is not independent, $\mathcal{S}_3$ is not a basis because it does not span the space because it isn't in it (it's not on the line)!

4. $\mathcal{S}_1$ is a basis for the yz-plane. $\mathcal{S}_2$ and $\mathcal{S}_4$ are not bases for the yz-plane because they doesn't span the space, and $\mathcal{S}_3$ isn't a basis because it is not a linearly independent set.

5. (a) Not a subspace, doesn't contain the zero vector.

   (b) Subspace, a basis is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ (c) Subspace, $\mathbf{u}$ is a basis.

   (d) Not a subspace, the vector $\begin{bmatrix}-3\\1\\4\end{bmatrix} + \begin{bmatrix}5\\0\\2\end{bmatrix} = \begin{bmatrix}2\\1\\6\end{bmatrix}$ is not on either line because it is not a scalar multiple of either vector.

   (e) Not a subspace, the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is in the set, but $-2\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-2\\-4\\-6\end{bmatrix}$ is not.

   (f) It can be shown that $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, so the set is a plane not containing the zero vector, so it is not a subspace.

   (g) This is a plane containing the origin, so it is a subspace. The set $\{\vec{\mathbf{u}}, \vec{\mathbf{v}}\}$ is a basis.

   (h) The vector $\mathbf{0}$ is a subspace. (i) All of $\mathbb{R}^3$ is a subspace.

6. (a) Not a subspace. $\begin{bmatrix}2\\3\end{bmatrix}$ is in the set, but $-1\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-2\\-3\end{bmatrix}$.

   (b) Subspace. A basis would be $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$.

Section 4.8 Solutions Back to 4.8 Exercises

1. (a) These are simply the matrices given in the exercise, each augmented with a column of zeros.

   (b) $\vec{\mathbf{x}} = \begin{bmatrix}-s+2t\\s\\t\end{bmatrix}$ (c) $\vec{\mathbf{x}} = s\begin{bmatrix}-1\\1\\0\end{bmatrix} + t\begin{bmatrix}2\\0\\1\end{bmatrix}$

   (d) A basis for the null space of A is $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$.

2. (a) $x_1 = -4t$, $x_2 = t$ and $x_3 = t$. (b) $\vec{\mathbf{x}} = t\begin{bmatrix}-4\\1\\1\end{bmatrix}$ (c) $\begin{bmatrix}-4\\1\\1\end{bmatrix}$

3. (a) $\text{nullity}(A) = 1$ (b) $\text{rank}(A) = 2$

(b) A basis for the column space of A is $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} \right\}$.

(c) The fact that the basis for the column space has two vectors agrees with the rank.

4. (a) $\vec{\mathbf{u}}_1$ is in the column space and $\vec{\mathbf{u}}_2$ is not.

(b) A basis for col(A) is $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} \right\}$.

(c) $\vec{\mathbf{u}}_1 = -3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$.

(d) $\text{rank}(A) = 2$

5. (a) $\vec{\mathbf{v}}_1$ is in null(A) and $\vec{\mathbf{v}}_2$ is not. (b) A basis for null(A) is $\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(b) $\text{nullity}(A) = 1$ (d) $\vec{\mathbf{v}}_1 = -2 \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

6. (a) $\text{rank}(A) = 2, \quad \text{nullity}(A) = 2$ (b) $\text{rank}(A) = 4, \quad \text{nullity}(A) = 1$

(c) $\text{rank}(A) = 3, \quad \text{nullity}(A) = 2$ (d) $\text{rank}(A) = 3, \quad \text{nullity}(A) = 1$

(e) $\text{rank}(A) = 2, \quad \text{nullity}(A) = 1$ (f) $\text{rank}(A) = 2, \quad \text{nullity}(A) = 3$

(g) $\text{rank}(A) = 3, \quad \text{nullity}(A) = 0$

Section 4.9 Solutions Back to 4.9 Exercises

1. (a) (i), (iii) (b) (ii) (c) (ii), (iii) (d) (ii)

(e) (i), (iii) (f) (i), (ii), (iii) (g) (ii) (h) (i), (iii)

(i) (ii) (j) (iii)

2. (a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -4 \\ -3 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$

(b) $\mathcal{B}_{\text{col}(A)} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix} \right\}$

(c) $\text{rank}(A) = 3$, so the column space spans $\mathbb{R}^3$ and so $\vec{\mathbf{b}}$ must be in the column space, no matter what it is. The system therefore has a solution.

(d) $\text{nullity}(A) = 0$ by the Rank Theorem. We already know that the system has a solution, and now we know it is unique.

(e) $\mathcal{B}_{\text{null}(A)} = \emptyset$, the empty set.

(f) The solution to the system is $\vec{\mathbf{x}} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$

3. (a) $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 5 & 0 \\ 2 & 5 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$

(b) $\mathcal{B}_{\text{col}(A)} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} \right\}$

(c) $\text{rank}(A) = 2$, so the column space does not span $\mathbb{R}^3$. The system may or may not have a solution, depending on whther $\vec{\mathbf{b}}$ is in $\text{col}(A)$.

(d) $\text{nullity}(A) = 1$ by the Rank Theorem. If the solution has a solution, it is not unique.

(e) $\mathcal{B}_{\text{null}(A)} = \left\{ \begin{bmatrix} -2 \\ -\frac{4}{5} \\ 1 \end{bmatrix} \right\}$, or any set containing just one scalar multiple of that vector.

(f) The system has no solution.