Math 341 EXAM 1, WINTER 2017 **Name:** 72 points

Unless stated otherwise, each numbered exercise is worth six points.

1. Consider the system $\begin{aligned} x - 2y - 3z &= -1 \\ 2x + \ y + \ z &= \ \ 6 \\ x + 3y - 2z &= 13 \end{aligned}$. Give the linear combination form and matrix form (*NOT* augmented matrix form) of the system in the spaces below.

$$x\begin{bmatrix}1\\2\\1\end{bmatrix} + y\begin{bmatrix}-2\\1\\3\end{bmatrix} + z\begin{bmatrix}-3\\1\\-2\end{bmatrix} = \begin{bmatrix}-1\\6\\13\end{bmatrix}$$

Linear Combination Form

$$\begin{bmatrix}1 & -2 & -3\\2 & 1 & 1\\1 & 3 & -2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-1\\6\\13\end{bmatrix}$$

Matrix Form

2. Find the determinant of the matrix below. **Combine like terms in your answer and put a box around your final answer.**

$2c \quad 3a \quad 4b$

$$\begin{bmatrix}a & b & c\\1 & 1 & 1\\2 & 3 & 4\end{bmatrix}\begin{matrix}a & b\\1 & 1\\2 & 3\end{matrix} = 4a+2b+3c-2c-3a-4b = a-2b+c$$

$4a \quad 2b \quad 3c$

3. (a) For the matrix $A = \begin{bmatrix}1 & 1 & 1\\2 & 3 & 4\end{bmatrix}$, $AA^T = \begin{bmatrix}1 & 1 & 1\\2 & 3 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 3\\1 & 4\end{bmatrix} = \begin{bmatrix}3 & 9\\9 & 29\end{bmatrix}$

   (b) Give two *words* for what kind of matrix AA^T is: square, symmetric

4. Matrix A is 4×3.

   (a) Give the dimensions of a matrix B for which AB would exist but BA would not: $3 \times \cancel{4}$

   (b) For your answer to part (a), the dimensions of AB would be $4 \times$ ____

   (c) Give the dimensions of a matrix C for which AC and CA would both exist: 3×4

5. Give a matrix A for which $A\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}3x_1 - x_2\\5x_1 + 4x_2\\-2x_1 + 3x_2\end{bmatrix}$: $\quad A = \begin{bmatrix}3 & -1\\5 & 4\\-2 & 3\end{bmatrix}$

6. Let $A = \begin{bmatrix} -9 & 4 \\ 7 & -3 \end{bmatrix}$. Use A to demonstrate the steps used to find the inverse of a matrix of any size. Indicate clearly what you are doing, and conclude by giving the inverse matrix A^{-1}. Use your calculator to do *rref*, but indicate where you are doing that by writing $\overset{rref}{\Longrightarrow}$.

$$\begin{bmatrix} -9 & 4 & 1 & 0 \\ 7 & -3 & 0 & 1 \end{bmatrix} \overset{rref}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 7 & 9 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix}$$

7. Determine whether $A = \begin{bmatrix} -3 & 2 \\ -5 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$ are inverses of each other, *without finding the inverse of either*. Show clearly the computation used to do this in the space below.

A and B (circle one) are are not inverses of each other.

$$AB = \begin{bmatrix} -3 & 2 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 7 & 2 \end{bmatrix}$$

8. Give **all** steps for how an inverse matrix is used to solve a matrix equation $A\mathbf{x} = \mathbf{b}$. Start with the given equation and end with $\mathbf{x} =$.

$$A\vec{x} = \vec{b}$$

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$

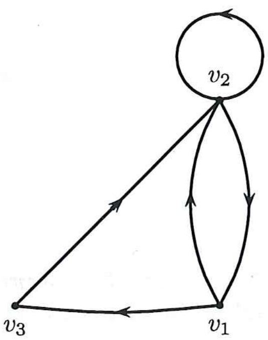
$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

12. Consider the directed graph shown below and to the left. There are $\underline{\ 2\ }$ 4-paths from v_2 to ~~v_4~~; indicate how you determined this in the space below.

V_3



$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A^{4} = \begin{bmatrix} 3 & 6 & 2 \\ 4 & 7 & 2 \\ 2 & 4 & 1 \end{bmatrix}$$

x2 x2 x2

13. For the directed graph in the previous exercise, give two 3-paths (there are more, but just give any two) from v_1 to v_2, using the standard notation for a path.

$V_1V_2V_1V_2$ $\quad$ $V_1V_3V_2V_2$

$V_1V_2V_2V_2$

14. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and suppose that A is a 2×2 matrix for which $A\mathbf{e}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $A\mathbf{e}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Find the entries of A using this information, **showing clearly how those facts are used.**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$a = -2$ $\qquad$ $b = 5$

$c = 1$ $\qquad$ $d = 3$

Too easy!

$$A = \begin{bmatrix} -2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A^2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 1 & 2 & 1 \end{bmatrix}}_{A^3}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 2 \\ 4 & 7 & 2 \\ 2 & 4 & 1 \end{bmatrix}$$

9. Recall that $\det(A) = 0$ if, and only if, A is *NOT* invertible. For each of the following scenarios, indicate *ALL possible* number of solutions to $A\mathbf{x} = \mathbf{b}$ by circling all correct choices for each.

   (a) $\mathbf{b} = 0$ (nothing known about $\det(A)$) none one infinitely many

   (b) $\det(A) = 0$ none one infinitely many

   (c) $\det(A) = 0$ and $\mathbf{b} = 0$ none one infinitely many

   (d) $\det(A) \neq 0$ and $\mathbf{b} = 0$ none one infinitely many

10. Suppose that, in $\mathbb{R}^2$, R is a rotation by any given angle that is not a multiple of $2\pi = 360°$, P is a projection onto any given line, and C is a reflection across any given line. **Use the space below to draw any pictures that might help you with this exercise and the next one.**

    (a) Give all of the matrices whose squares equal I: C

    (b) Give all of the matrices that equal their own squares: P

    (c) Give **all** the matrices M for which there is at least one nonzero vector $\mathbf{x}$ such that $M\mathbf{x} = \mathbf{x}$: C, P

    (d) Give all of the matrices that have inverses: C, R

11. Suppose that A is a 2×2 matrix. Finish each sentence by circling all transformations that would make the statement true. Assume that rotation means by any angle that is not a multiple of $2\pi = 360°$.

    (a) $A^3 = A^5$ if A is a rotation projection reflection

    (b) A changes the directions of all vectors if A is a rotation projection reflection

    (c) $A = A^{-1}$ if A is a rotation projection reflection

    (d) There are *nonzero* vectors $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{0}$ if A is a rotation projection reflection