Math 341 HOMOGENEOUS COORDINATES **Not to turn in**

Let's begin with a couple simple, but useful, observations.

1. Let $A = \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

   (a) Find each of $A\mathbf{u}_1$, $A\mathbf{u}_2$, $A\mathbf{u}_3$.

   (b) Create a new matrix B whose columns are $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. Find the product AB.

   (c) What do you notice about the columns of AB?

We will soon wish to multiply a set of vectors by a matrix, and the above shows that we can do it in one computation rather than three. (Well, conceptually at least - it is really the three computations of part (a), packaged as one computation.)

2. Let $C = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$. Multiply $C\mathbf{w}$. Then compare C, $\mathbf{w}$ and $C\mathbf{w}$ with A, $\mathbf{u}_1$ and $A\mathbf{u}_1$ from the previous exercise.

For the remainder of the exercises we will be working with transformations in $\mathbb{R}^2$ that move points around in the plane. We will accomplish these transformations through multiplication by a matrix, so we need to think of a point like $(1,4)$ as the vector $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. However, we will see that it is far more useful to represent a point (x,y) as the vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, which we will refer to as the **homogeneous coordinates** of the point (x,y). Most, but not all, of the transformations we will be doing could be performed by acting on a vector in $\mathbb{R}^2$ with a 2×2 matrix. For one particular transformation that will not work, so we'll need to augment our 2×2 matrices in the way that the matrix A of Exercise 1 was augmented to get the matrix C of Exercise 2.

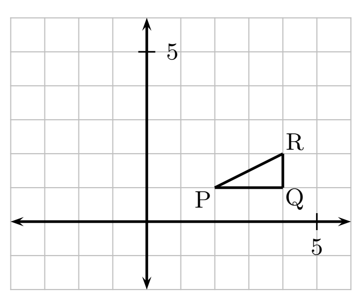
We will be performing **isometric** transformations on the triangle $\triangle PQR$ shown below in Exercise 2, with vertices at $P(2,1)$, $Q(4,1)$ and $R(4,2)$. Here's what we mean by an "isometric transformation:"

- The "transformation" part means that every point on $\triangle PQR$ will go to some other point in the plane. In particular, P will go to a point we'll call P', Q to Q' and R to R'.
- The word "isometric" means "same measure," which includes both lengths of sides and measures of angles. So the new triangle $\triangle P'Q'R'$ will be the same size and shape as $\triangle PQR$.

There are really only three kinds of isometric transformations: rotations, reflections and translations. The point of these exercises is to see how these transformations can be performed with matrices.

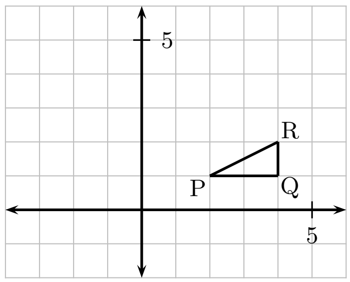
Put your work and any answers on another sheet of paper, but do the graphing on this sheet.

3. (a) Suppose that we wish to create a new triangle $\triangle P'Q'R'$ by multiplying each of P, Q and R by the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ to get the new points P', Q' and R'. Note that we can represent the triangle $\triangle PQR$ by the matrix $[PQR] = \begin{bmatrix} 2 & 4 & 4 \\ 1 & 1 & 2 \end{bmatrix}$. We can then use what we discovered in Exercise 1 to create the new triangle $\triangle P'Q'R'$ by simply multiplying $[PQR]$ by A to get a matrix $[P'Q'R']$ whose columns are the points P', Q' and R'. Find and draw $\triangle P'Q'R'$ - label its vertices with P', Q' and R'.

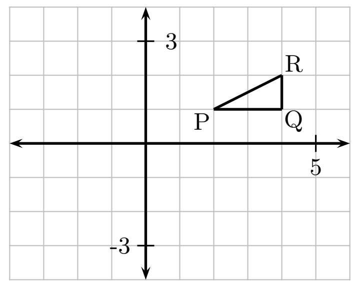


(b) You should be able to see that the matrix R rotated the triangle about the origin, so it must be a rotation matrix. This means it has to have the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. What is the angle θ in this case?

(c) On the drawing below and to the left, rotate $\triangle PQR$ through the same angle, **but about the point** $(2,3)$. (Do this by intuition - I'm not expecting you to use a matrix at this point.)



Exercise 3(c)



Exercise 4

4. Create a new triangle $\triangle P'Q'R'$ by multiplying $[PQR]$ by the matrix $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Plot P', Q' and R' on the graph above and to the right and connect them with segments to create $\triangle P'Q'R'$. Label its vertices with P', Q' and R'. What did B do to the triangle?

5. Now suppose that we wish to move the entire triangle left by three units and up by one, a type of transformation that we call a **translation**. Describe this transformation with an equation/rule of the form

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} \end{bmatrix}.$$

For the translation of the previous exercise we see that $T(\mathbf{0}) = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \neq \mathbf{0}$, so T is not a linear transformation. This is a problem, because *a transformation can only be accomplished by matrix multiplication if it is a linear transformation.* There is, however, a way around this - the trick is to use homogeneous coordinates. We represent each ordered pair in $\mathbb{R}^2$ by a vector in $\mathbb{R}^3$ that is simply the $\mathbb{R}^2$ vector with a third component of one. For a rotation or a reflection, we simply add another column and row to our transformation matrix, with all new entries being zeros *except the entry in the third column and third row, which is a one.* We can then multiply the matrix times the vector, resulting in another vector in $\mathbb{R}^3$ with third component one. Our new point in $\mathbb{R}^2$ is then the point whose coordinates are just the first two components of our new vector. For example, to do the rotation from Exercise 1 on the point $B(4,1)$ we would do the following computation to find that Q' is the point $(-1,4)$:

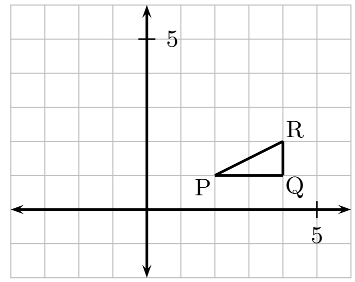
$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \tag{1}$$

So why move everything to $\mathbb{R}^3$ to do what we could do just fine in $\mathbb{R}^2$ before? Again, the reason is that *translation is not a linear transformation!* However, we can remedy this by the use of homogenous coordinates, as you are about to see.

6. Multiply the homogenous coordinates of each vertex of our triangle (put into a 3×3 matrix representing the vertices of the triangle in homogeneous coordinates) by the matrix

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

to get the new points P', Q' and Plot P', Q' and R' on the graph to the right and draw the $\triangle P'Q'R'$. Label its vertices with P', Q' and R'.



7. (a) Give the matrix that translates all points 2 units to the right and 3 units down.

   (b) *Without doing any calculations*, give the matrix that should "undo" this translation.

   (c) Use your calculator to find the inverse of the matrix you gave in (a). It should agree with your answer to (b). Does it?

8. Now you will see the real power of these ideas. Suppose that we wish to find a matrix that will rotate all points $90°$ (counterclockwise, like usual) about the point $(2, 3)$, like you did in Exercise 3(c). The idea is to take advantage of the fact that we already know how to rotate about the origin, so we translate everything to put the point $(2, 3)$ at the origin, rotate and translate back. Make sure you see this as you work on the following. **All matrices and vectors are for homogenous coordinates from now on!**

   (a) Give the matrix that translates $(2, 3)$ to the origin, calling it $[T]$. Of course the matrix to translate back is just $[T]^{-1}$; give it also.

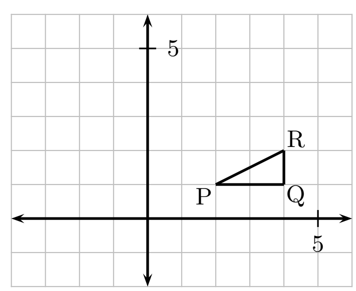
   (b) Given the homogeneous coordinate vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ for a point (x, y) in the plane, we want to multiply first by $[T]$, then by the homogeneous rotation matrix $[R_{90°}]$ and, finally, by $[T]^{-1}$. Recalling that the first transformation to act is the one that goes closest to the vector, we want to carry out the computation

   $$[T]^{-1}[R_{90°}][T]\mathbf{v} = [T^{-1}R_{90°}T]\mathbf{v}$$

   Compute the matrix product $[T^{-1}R_{90°}T] = [T]^{-1}[R_{90°}][T]$.

   (c) Use your answer to (b) to find the rotated coordinates P', Q' and R' of the points $P(2, 1)$, $Q(4, 1)$ and $R(4, 2)$.

   (d) Plot the rotated points and draw the rotated triangle $\triangle P'Q'R'$ on the grid to the right. If it doesn't look like what you predicted in 3(c), figure out which is wrong and correct it.



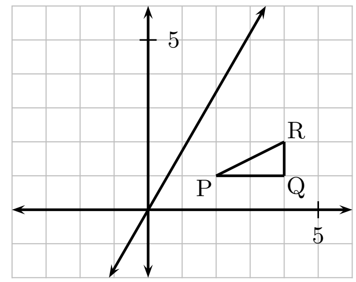
Exercise 6(d)

9. Suppose that we wanted to reflect $\triangle PQR$ across the line through the origin and at an angle of $60°$ to the positive x-axis, as shown in the picture above and to the right. We already know how to reflect across the x-axis (see Exercise 4), so let's take advantage of that fact. What we want to do is rotate the line to the x-axis, reflect, then rotate back.

   (a) Find the single matrix that does this by multiplying some other matrices. **Round the entries of the matrix to the nearest hundredth, or give them in exact form.**

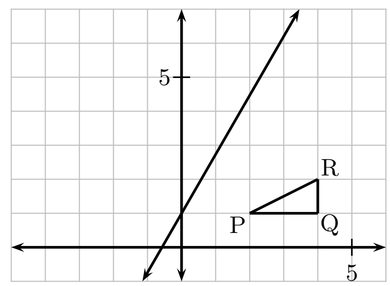
   (b) Apply the matrix to the homogenous coordinates of P, Q and R to get vertices of a new triangle $\triangle P'Q'R'$.

   (c) Draw $\triangle P'Q'R'$ on the graph above and to the right.



Exercise 7(c)

10. And now for the grand finale! Find a single matrix that will reflect $\triangle PQR$ across the line through the point $(0, 1)$ and at an angle of $60°$ to the x-axis, shown on the graph to the right. Test your result as you have been. **Round the entries of the matrix to the nearest hundredth, or give them in exact form.**



A Final Note: Pretty much every calculation we have done all term boils down to adding and multiplying. If one were to be writing code to do the things you've done in this assignment, you would simply do it as a bunch of multiplications and additions, rather than doing it with matrices. For example, to rotate the point (x, y) by an angle of θ we would simply compute the new coordinates (w, z) by

$$w = x\cos\theta - y\sin\theta, \qquad z = x\sin\theta + y\cos\theta$$

The advantage of linear algebra for tasks like this is not computational, but conceptual. Without the theory that we have developed, figuring out how to do transformations like the ones from the last three exercises would be far more difficult!