2.2 Kernel and Range of a Linear Transformation

Performance Criteria:

2. (c) Determine whether a given vector is in the kernel or range of a linear transformation. Describe the kernel and range of a linear transformation.
   (d) Determine whether a transformation is one-to-one; determine whether a transformation is onto.

When working with transformations $T : \mathbb{R}^m \to \mathbb{R}^n$ in Math 341, you found that any *linear* transformation can be represented by multiplication by a matrix. At some point after that you were introduced to the concepts of the *null space* and *column space* of a matrix. In this section we present the analogous ideas for general vector spaces.

Definition 2.4: Let V and W be vector spaces, and let $T : V \to W$ be a transformation. We will call V the **domain** of T, and W is the **codomain** of T.

Definition 2.5: Let V and W be vector spaces, and let $T : V \to W$ be a *linear* transformation.

- The set of all vectors $\mathbf{v} \in V$ for which $T\mathbf{v} = \mathbf{0}$ is a subspace of V. It is called the **kernel** of T, And we will denote it by $\ker(T)$.
- The set of all vectors $\mathbf{w} \in W$ such that $\mathbf{w} = T\mathbf{v}$ for some $\mathbf{v} \in V$ is called the **range** of T. It is a subspace of W, and is denoted $\text{ran}(T)$.

It is worth making a few comments about the above:

- The kernel and range "belong to" the transformation, not the vector spaces V and W. If we had another linear transformation $S : V \to W$, it would most likely have a different kernel and range.
- The kernel of T is a subspace of V, and the range of T is a subspace of W. The kernel and range "live in different places."
- The fact that T is linear is essential to the kernel and range being subspaces.

Time for some examples!

◇ **Example 2.2(a):** $T : M_{22} \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ c+d \end{bmatrix}$ is linear. Describe its kernel and range and give the dimension of each.

It should be clear that $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if, and only if, $b = -a$ and $d = -c$. The kernel of T is therefore all matrices of the form

$$\begin{bmatrix} a & -a \\ c & -c \end{bmatrix} = a\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

The two matrices $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$ are not scalar multiples of each other, so they must be linearly independent. Therefore the dimension of $\ker(T)$ is two.

Now suppose that we have any vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. Clearly $T\left(\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$, so the range of T is all of $\mathbb{R}^2$. Thus the dimension of $\text{ran}(T)$ is two. ♠

◇ **Example 2.2(b):** $T : \mathscr{P}_1 \to \mathscr{P}_1$ defined by $T(ax + b) = 2bx - a$ is linear. Describe its kernel and range and give the dimension of each.

$T(ax + b) = 2bx - a = 0$ if, and only if, both a and b are zero. Therefore the kernel of T is only the zero polynomial. By definition, the dimension of the subspace consisting of only the zero vector is zero, so $\ker(T)$ has dimension zero.

Suppose that we take a random polynomial $cx + d$ in the codomain. If we consider the polynomial $-dx + \frac{1}{2}c$ in the domain we see that

$$T(-dx + \frac{1}{2}c) = 2(\frac{1}{2}c)x - (-d) = cx + d.$$

This shows that the range of T is all of $\mathscr{P}_1$, and it has dimension two. ♠

◇ **Example 2.2(c):** $T : \mathscr{P}_2 \to \mathscr{P}_2$ defined by $T(ax^2 + bx + c) = ax^2 + (b + c)x + (a + b + c)$ is linear. Describe its kernel and range and give the dimension of each.

If $T(ax^2 + bx + c) = ax^2 + (b + c)x + (a + b + c) = 0$, then clearly $a = 0$ and $c = -b$. Thus the kernel of T is the set of all polynomials of the form $bx - b = b(x - 1)$. This set has dimension one ($x - 1$ is a basis).

The range of T is all polynomials of the form $ax^2 + (b + c)x + (a + b + c)$. If we let $b + c = d$, this is then the polynomials of the form $ax^2 + dx + (a + d) = a(x^2 + 1) + d(x + 1)$. The range of T therefore has dimension two. ♠

Definition 2.6: Let $T : V \to W$ be a linear transformation. The **nullity** of T is the dimension of the kernel of T, and the **rank** of T is the dimension of the range of T. They are denoted by $\text{nullity}(T)$ and $\text{rank}(T)$, respectively.

Examples 2.2(a),(b) and (c) illustrate the following important theorem, usually referred to as the **rank theorem**.

Theorem 2.7: Let $T : V \to W$ be a linear transformation. Then

$$\text{rank}(T) + \textbf{nullity}(T) = \dim(V),$$

where $\dim(V)$ is the dimension of V.

The last theorem of this section can be useful in determining the rank of a transformation.

Theorem 2.8: Let $T : V \to W$ be a linear transformation and let $\mathcal{B} = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be a spanning set for V. Then $T(\mathcal{B}) = \{T(\mathbf{v}_1), ..., T(\mathbf{v}_n)\}$ is a spanning set for the range of T.