Math 342 ASSIGNMENT 23, SPRING 2016 **Due at the start of class on 5/31**

1. Consider the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

   (a) Use the Wolfram Alpha 2×2 Eigenvalue and Eigenvector widget to find the eigenvalues and corresponding basis eigenvectors.

   (b) Sketch an accurate graph of the two basis vectors for the eigenspaces (both as position vectors).

   (c) Create a matrix P whose columns are the basis eigenvectors. Then find the product $P^{-1}AP$, which we'll refer to as D. Describe the matrix D as precisely and accurately as possible.

   (d) Find the product PDP^{-1} - what is it?

2. Now let $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$.

   (a) Give the eigenvalues and corresponding eigenvectors.

   (b) Sketch an accurate graph of the two basis vectors for the eigenspaces (both as position vectors).

   (c) What is special about the basis eigenvectors in this case? What is special about the matrix A that might be causing this?

3. Do this exercise "by hand." Portions of it will be a bit tedious, but it can't really be made any less so - try to hang with it! Here's the matrix this time:

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$$

   (a) What is special about A?

   (b) Use the Wolfram Alpha 3×3 Eigenvalue and Eigenvector widget to find the eigenvalues and corresponding basis eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

   (c) Normalize each of the eigenvectors, giving the resulting vectors **in exact form** and naming them $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$. I would advise that you *NOT* rationalize the denominators.

   (d) Let Q be the matrix whose columns are the normalized eigenvectors, and find the product Q^TQ. What does this tell us about the columns of Q?

   (e) Let D be the diagonal matrix with the eigenvalues on the diagonal, in order corresponding to the order of the eigenvectors in Q. Find the product QDQ^T. It looks bad to begin, but turns out nicely!

   (f) Note that if $\mathbf{v}$ is a standard "column" vector, then $\mathbf{v}^T$ is the corresponding row vector. Recall from our discussion of the outer product of two matrices that $\mathbf{v}$ times $\mathbf{v}^T$ is a 3×3 matrix when $\mathbf{v} \in \mathbb{R}^3$. Compute

$$\lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T + \lambda_3 \mathbf{q}_3 \mathbf{q}_3^T$$

   showing the steps clearly.

4. (a) Which theorem from Chapter 4 does Exercise 3(d) illustrate?

   (b) Which theorem from Chapter 4 does Exercise 3(e) illustrate?

   (c) Which theorem from Chapter 4 does Exercise 3(f) illustrate?

There's one more on the back!

5. Now consider the matrix $A = \begin{bmatrix} -4 & 2 & -2 \\ 2 & -7 & 4 \\ -2 & 4 & -7 \end{bmatrix}$.

(a) Again use Wolfram to find the eigenvalues and eigenvectors. Give the matrix Q, this time with its entries as **decimals rounded to the thousandth's place**. Give the matrix D as well.

(b) Compute QDQ^T, and note that it is not what you should have expected.

(c) Compute Q^TQ and give the result with entries rounded to the hundredth's place.

(d) Here's the problem: Q needs to have orthonormal columns, but the two eigenvectors that have the same eigenvalue are not orthogonal. (Check that!) **Using the eigenvectors given by Wolfram**, perform the Gram-Schmidt process with the two eigenvectors that share an eigenvalue. When you are done, normalize those two vectors. The first one should be the same as one of your original three.

(e) Build a new Q using the newest normalized vector obtained from Gram-Schmidt in place of what it was before. Check to see that QDQ^T now equals A. (It will not equal it exactly due to the rounding, but should be close.)