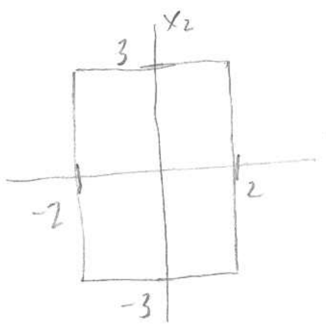
Math 342, Exam 4 Preparation

(1) $\|\vec{u}\| = \sqrt{\langle \vec{u},\vec{u}\rangle} = \sqrt{3(25)+4+2(9)} = \sqrt{97}$

(2) a) $\|\vec{x}\| = \max\{|\tfrac{1}{2}x_1|, |\tfrac{1}{3}x_2|\}$

$|\tfrac{1}{2}x_1| = 1$ when $x_1 = \pm 2$

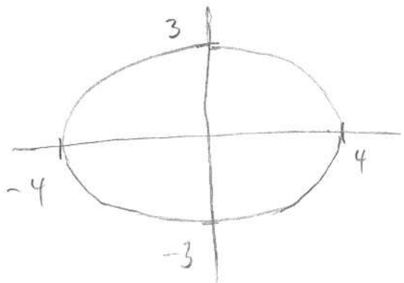
$|\tfrac{1}{3}x_2| = 1$ when $x_2 = \pm 3$



b) $\|\vec{x}\| = \sqrt{\tfrac{1}{16}x_1^2 + \tfrac{1}{9}x_2^2}$

Setting equal to 1 and squaring both sides gives $\dfrac{x_1^2}{16} + \dfrac{x_2^2}{9} = 1.$

x_1	x_2
0	3
0	-3
4	0
-4	0



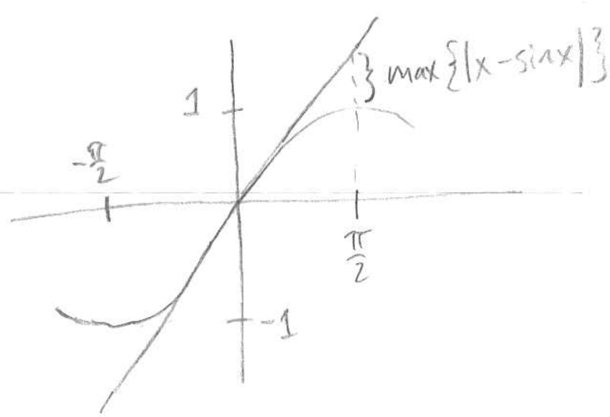
(3) $$\|x-\sin x\|_1 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |x-\sin x|\,dx \xrightarrow{\text{See graph below}} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (x-\sin x)\,dx \xrightarrow{\text{Wolfram}} = \frac{1}{4\pi}(\pi^2-8) \approx 0.1488$$

$$\|x-\sin x\|_2 = \left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}} (x-\sin x)^2\,dx\right)^{\frac{1}{2}} \xrightarrow{\text{Wolfram}} = \left(\frac{1}{12\pi}(-48+6\pi+\pi^3)\right)^{\frac{1}{2}} \approx 0.2219$$

To find $\|x-\sin x\|_\infty$ we need to find $\max\{|x-\sin x| : x \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$.

Let's look at the graph:

We can see that the maximum difference between $y=x$ and $y=\sin x$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is at $x = \pm\frac{\pi}{2}$. Thus



$$\|x-\sin x\|_\infty = \frac{\pi}{2} - \sin\frac{\pi}{2} = \boxed{\frac{\pi}{2} - 1} \approx 0.5708$$

(4) a) $\vec{w}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{w}_1}\vec{v}_2 = \begin{bmatrix}2\\1\\1\end{bmatrix} - \frac{4}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\-\frac{1}{3}\\-\frac{1}{3}\end{bmatrix}$ b) $\vec{w}_2 = \begin{bmatrix}2\\-1\\-1\end{bmatrix}$

c) $\vec{w}_3 = \vec{v}_3 - \text{proj}_{\vec{w}_1}\vec{v}_3 - \text{proj}_{\vec{w}_2}\vec{v}_3 = \begin{bmatrix}1\\2\\1\end{bmatrix} - \frac{4}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{-2}{4}\begin{bmatrix}2\\-1\\-1\end{bmatrix}$

Oops! This last vector is not orthogonal to the first two. I see my errors (two of them!)

$$= \begin{bmatrix}\frac{6}{6}\\\frac{12}{6}\\\frac{6}{6}\end{bmatrix} - \begin{bmatrix}\frac{8}{6}\\\frac{8}{6}\\\frac{8}{6}\end{bmatrix} + \begin{bmatrix}\frac{6}{6}\\-\frac{3}{6}\\-\frac{3}{6}\end{bmatrix} = \begin{bmatrix}\frac{4}{6}\\\frac{1}{6}\\-\frac{5}{6}\end{bmatrix} \Rightarrow \vec{w}_3 = \begin{bmatrix}4\\1\\-5\end{bmatrix}$$

Correction:

$$\vec{w}_2 = \begin{bmatrix}1\\2\\1\end{bmatrix} - \frac{4}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{-1}{6}\begin{bmatrix}2\\-1\\-1\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}-\frac{4}{3}\\-\frac{4}{3}\\-\frac{4}{3}\end{bmatrix} + \begin{bmatrix}\frac{2}{6}\\-\frac{1}{6}\\-\frac{1}{6}\end{bmatrix} = \begin{bmatrix}0\\\frac{1}{2}\\-\frac{1}{2}\end{bmatrix} \Longrightarrow \vec{w}_3 = \begin{bmatrix}0\\1\\-1\end{bmatrix}$$

A check will show that $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}4\\1\\-5\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$ is an orthogonal basis for

$\mathbb{R}^3$. We could now normalize these to get an orthonormal basis.

(5) Q^TQ is a diagonal matrix (zeros everywhere but on the diagonal) and the diagonal entries of Q^TQ are the squares of the norms of the columns of Q.

(6) a) Let $k_1c_1(x) + k_2c_2(x) + k_3c_3(x) + k_4c_4(x) = f(x)$. Then

$$\begin{aligned} k_1 + k_2 &= 4 \\ k_3 + k_4 &= 5 \\ k_3 - k_4 &= 2 \\ k_1 - k_2 &= 3 \end{aligned} \Rightarrow \begin{aligned} 2k_1 &= 7 \\ 2k_3 &= 7 \end{aligned} \Rightarrow k_1 = k_3 = \frac{7}{2}, \quad k_2 = \frac{1}{2}, \quad k_4 = \frac{3}{2} \qquad [f]_c = \begin{bmatrix}\frac{7}{2}\\\frac{1}{2}\\\frac{7}{2}\\\frac{3}{2}\end{bmatrix}$$

(6) (continued)

b)

$$[c_1]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \; [c_2]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \; [c_3]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \; [c_4]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$[P]_{\mathcal{C},\mathcal{E}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \Longrightarrow [P]_{\mathcal{E},\mathcal{C}} = [P]_{\mathcal{C},\mathcal{E}}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

c) $\langle f, c_2 \rangle = \int_0^1 f(x)\,c_2(x)\,dx = 4-3 = 1$, $\langle c_2, c_2 \rangle = \int_0^1 [c_2(x)]^2\,dx = 2$

d) $\dfrac{\langle f, c_2 \rangle}{\langle c_2, c_2 \rangle} = \dfrac{1}{2}$, the second component of $[f]_{\mathcal{C}}$.

(7) First polynomial is $\boxed{1}$

2nd polynomial: $x - \text{proj}_1 x = x - \dfrac{\int_0^\infty x e^{-x}\,dx}{\int_0^\infty 1 e^{-x}\,dx}\, 1 = x - \frac{1}{1} 1 = \boxed{x-1}$

3rd polynomial: $x^2 - \text{proj}_1 x^2 - \text{proj}_x x^2 = x^2 - \dfrac{\int_0^\infty x^2 e^{-x}\,dx}{\int_0^\infty 1 e^{-x}\,dx}\, 1 - \dfrac{\int_0^\infty x^2 (x-1) e^{-x}\,dx}{\int_0^\infty (x-1)^2 e^{-x}\,dx}(x-1)$

$$= x^2 - \frac{2}{1} 1 - \frac{4}{1}(x-1)$$

$$= x^2 - 2 - 4x + 4 = \boxed{x^2 - 4x + 2}$$