How do we solve $(1-x^2)y'' - 2xy' + n(n+1)y = 0$?

Let's assume the solution has a power series expansion

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots$$

around zero and try substituting, for $n = 2$. We'll need the first and second derivatives:

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots$$

From these we get the parts of $(1-x^2)y'' - 2xy' + n(n+1)y$:

$$\begin{aligned}(1-x^2)y'' &= (1-x^2)(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots) \\ &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots \\ &\qquad - 2a_2x^2 - 6a_3x^3 - 12a_4x^4 - 20a_5x^5 - 30a_6x^6 - 42a_7x^7 - \cdots\end{aligned}$$

and

$$\begin{aligned}-2xy' &= -2x(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots) \\ &= -2a_1x - 4a_2x^2 - 6a_3x^3 - 8a_4x^4 - 10a_5x^5 - 12a_6x^6 + \cdots\end{aligned}$$

and, for $n=2$,

$$\begin{aligned}n(n+1)y &= 6(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots) \\ &= 6a_0 + 6a_1x + 6a_2x^2 + 6a_3x^3 + 6a_4x^4 + 6a_5x^5 + 6a_6x^6 + \cdots\end{aligned}$$

Substituting all of these into the left side of the ODE gives us

$$
\begin{aligned}
(1-x^2)y'' - 2xy + 6y \;=\;& \quad 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots \\
& - 2a_2x^2 - 6a_3x^3 - 12a_4x^4 - 20a_5x^5 - 30a_6x^6 - 42a_7x^7 - \cdots \\
& - 2a_1x - 4a_2x^2 - 6a_3x^3 - 8a_4x^4 - 10a_5x^5 - 12a_6x^6 + \\
& + 6a_0 + 6a_1x + 6a_2x^2 + 6a_3x^3 + 6a_4x^4 + 6a_5x^5 + 6a_6x^6 + \cdots \\
=\;& (6a_0 + 2a_2) + (4a_1 + 6a_3)x + 12a_4x^2 + (-6a_3 + 20a_5)x^2 \\
& + (-14a_4 + 30a_6)x^4 + (-24a_5 + 42a_7)x^5 + (-36a_6 + 56a_7)x^6 + \cdots
\end{aligned}
$$

Because we are solving $(1-x^2)y'' - 2xy + 6y = 0$ the last expression above has to be equal to zero *for all* x, so every coefficient of the powers of x has to be zero. From that we get...

$$
\begin{aligned}
y &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \cdots \\
&= a_0 + a_1x - 3a_0x^2 - \tfrac{2}{3}a_1x^3 + 0x^4 - \tfrac{1}{5}a_1x^5 + 0x^6 - \tfrac{4}{35}a_1x^7 + 0x^8 + \cdots \\
&= a_0(1 - 3x^2) + a_1(x - \tfrac{2}{3}x^3 - \tfrac{1}{5}x^5 - \tfrac{4}{35}x^7 - \cdots)
\end{aligned}
$$