Exam 5

Legendre polynomial Handout

Assignment 20 - 23

* LU-factorization
* Least-squares
* Eigenvectors, eigenvalues, diagonalization

How do we find eigenvalues + eigenvectors?

$\boxed{A\vec{x} = \lambda\vec{x}}$

$A\vec{x} - \lambda\vec{x} = \vec{0}$

$A\vec{x} - \lambda I\vec{x} = \vec{0}$

$\underbrace{(A - \lambda I)}\vec{x} = \vec{0}$

There are more than $\vec{x} = 0$ if $A - \lambda I$ is not invertible

$B\vec{x} = \vec{0}$

$\vec{x} = \vec{0}$

Example: $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$

$$0 = \det(A - \lambda I)$$

$$= \det\left(\begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \det\begin{bmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{bmatrix}$$

$$= (-4-\lambda)(5-\lambda) - (3)(-6)$$

$$= \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$$

$\lambda = -1, 2$ are the eigenvalues.

$A-\lambda I$ not invertible $\Longleftrightarrow$ $\boxed{\det(A-\lambda I)=0}$

What about the eigenvectors?

$\lambda=-1$: $A-\lambda I = A+I = \begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix}$

Now solve $(A-\lambda I)\vec{x}=\vec{0}$

$$\begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \cancel{-3x_1-6x_2=0}$$

$3x_1+6x_2=0$

$x_1+2x_2=0$ $\quad x_1=2$

$x_2=-1$

$\vec{x}=\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\lambda = 2$ $A - \lambda I = \begin{bmatrix} -6 & -6 \\ 3 & 3 \end{bmatrix}$ $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$

$(A - \lambda I)\vec{x} = 0$ gives us $\cancel{-6x_1 - 6x_2 = 0}$

$3x_1 + 3x_2 = 0$

$x_1 + x_2 = 0$

$\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ $\lambda = -1, \vec{u}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\lambda = 2, \vec{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$P = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$ $\boxed{P^{-1}AP = D} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

$\boxed{A = PDP^{-1}}$ $A^5 = (PDP^{-1})^5$

$$A^5 = (PDP^{-1})^5$$

$$= (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})(\cancel{P}DP^{-1})$$

$$= PD^5P^{-1}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

$A=\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ $A-\lambda I=\begin{bmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{bmatrix}$

A cannot be diagonalized.

$$\det(A-\lambda I)=(2-\lambda)(4-\lambda)+1$$
$$=\lambda^2-6\lambda+9$$
$$=(\lambda-3)(\lambda-3)$$

Eigenvalue is $\lambda=3$

Eigenvector is $\vec{u}=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda=3$ is an eigenvalue with algebraic multiplicity 2, and geometric multiplicity 1.

A not symmetric, maybe $A = PDP^{-1}$
(if enough eigenvectors)

A is symmetric $A = QDQ^T$ Q is orthogonal
(orthonormal columns)